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AN UNIFIED APPROACH FOR THE CONTRACTIVE MAPPINGS

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ABSTRACT

In this paper we introduce a new definition of contractive type mappings called p-contraction. A selfmapping T of metric space (X, d) is called a p-contraction if there exists a map $p: R^3_* \to R_*$ where $R_* = \{t \in R : t \geqslant 0\}$, satisfying some suitable hypothesis, such that,

$$d(Tx, Ty) \leqslant p(\, d(x, y), d(x, Tx), d(y, Ty)\,).$$

for all pair $x, y \in X$.

The definition of p-contraction, which contains some of previous generalization of contraction mappings, implies all conclusions of Banach contraction principle.

1. INTRODUCTION

We recall Banach's contraction principle and its basic consequences. A selfmapping of a metric spac (X, d) is called a contraction if there exists h, $0 \leqslant h < 1$, such that for eaeh $x, y \in X$

$$d(Tx, Ty) \leqslant h\, d(x, y).$$

Any contraction map T of a complete metric space (X, d) has the following properties :

(A) T has a unique fixed point $z_* \in X$.

(B) Convergence of iterates $T^n x \to z_*$, as $n \to \infty$.

(C) Uniform convergence : there exists a neighbourhood U of z_* such that $T^n(\mathrm{U}) = \{T^n x : x \in \mathrm{U}\} \to \{z_*\}$, which means that for any neighbourhood V of z_* we can find an integer n_0 such that $T^n(\mathrm{U}) \subseteq V$ for all n greater than n_0.

(D) T is continuous.

On the other hand, Meyers [6] proved that every selfmapping T of complete metric space (X, d) having properties A, B, C and D is a contraction map under suitable topologically equivalent metric.

The purpose of this paper is to investigate p–contraction mappings (see precise definition below), mappings having some iterate which is a p–contraction and common fixed points of two mappings.

Applications of previous theorem of Meyers are also given.

2. Results on p–Contraction Mappings.

Let us introduce the set P of continuous $p : R^3_* \to R_*$, where $R_* = \{t \in R : t \geqslant 0\}$ satisfying the following properties :

(i) $p(1, 1, 1) = h < 1$;

(ii) Let $u > 0$. If $u \leqslant p(u, v, v)$ or $u \leqslant p(v, u, v)$ or $u \leqslant p(v, v, u)$ then $u \leqslant hv$.

Definition 1. A selfmapping of a metric space (X, d) is called a p-contraction if there exists a map $p \in P$, such that, for all $x, y \in X$

$$d(Tx, Ty) \leqslant p(\, d(x, y), d(x, Tx), d(y, Ty))$$

Now we indicate 3 examples of contractive type mappings, which can be regarded as a p-contraction, under suitable function p.

Indeed, If we put $p(u, v, z) = au + bv + cz$ $(a + b + c < 1)$, then we get the following well known :

Definition 2. (S. Reich [7], I. Rus [9]).

There exist non negative numbers a, b, c,. $a + b + c < 1$, such that for all pair $x, y \in X$

$d(Tx, Ty) \leqslant a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty)$.

Moreover if we put $p(u, v, z) = h \max\,(u, v, z)$ $(0 \leqslant h < 1)$, we have

Definition 3.

There exists a number h, $0 \leqslant h < 1$, such that for all pair $x, y \in X$

$$d(Tx, Ty) \leqslant h \max\,\{d(x, y), d(x, Tx), d(y, Ty)\},$$

At last if we choose $p(u, v, z) = (au^r + bv^r + cz^r)^{1/r}$, $r > 0$, $(a + b + c < 1)$ then we find the following :

Definition 4. (Delbosco) [2].

There exist non negative numbers a, b, c and there exists $r > 0$, such that for all pair $x, y \in X$

$$d(Tx, Ty)^r \leqslant a\, d(x, y)^r + b\, d(x, Tx)^r + c\, d(y, Ty)^r.$$

Theorem 2.1 Let T be a p-contraction mapping of a complete metric space (X, d). Then T has properties (A) and (B).

Proof. Fix $z \in X$. Suppose $z \neq Tz$ and set $z_n = T z_{n-1} = T^n z$. if we put $x = z_n$ and $y = z_{n-1}$, then inequality (1) becomes :

(2.1) $d(z_{n+1}, z_n) \leqslant p(\, d(z_n, z_{n-1}), d(z_n, z_{n+1}), d(z_{n-1}, z_n)\,).$

Hypothesis (ii) on function p implies

(2.2) $d(z_{n+1}, z_n) \leqslant h\, d(z_n, z_{n-1}).$

On the other hand, hypothesis (i) on p, proves that $(z_n)_{n \in N}$ is a Cauchy sequence and completeness of X assures that $z_n \to z_* \in X$.

To prove that z_* is fixed point of T, we take $x = z_*$ and $y = z_n$ in inequality (1) and we obtain

(2.3) $d(Tz_*, Tz_n) \leqslant p(\, d(z_*, z_n), d(z_*, Tz_*), d(z_n, Tz_n)\,).$

Since p is continuous in each variable, letting $n \to \infty$, we get

(2.4) $d(Tz_*, z_*) \leqslant p(\, 0, d(z_*, Tz_*), 0).$

$d(z_*, Tz_*)$ must be zero; hence $z_* = Tz_*$

Finally we prove that mapping T has a unique fixed point.

Indeed, let us suppose by contradiction, that $Tx = x$, $Ty = y$ and that $x \neq y$. Erom inequality (1) we get :

$d(x, y) = d(Tx, Ty) \leqslant p(\, d(x, y), 0, 0).$

It follows that $d(x, y) = 0$ *i. e.* $x = y$.

The proof is, thus, complete

Theorem 2.2 Let T be a continuous p-contraction selfmapping of a complete metric space (X, d). Then T has property (C).

Proof. For any $r > 0$, we define $V_r = \{x \in X : d(x, Tx) < r\}$. We show that V_r is an open subset of X, since it contains a neighbourhood for any point. Indeed, if $y \in V_r$, $d(y, Ty) = a < r$, then continuity of T implies that there exists r such that

(2.5) $d(z, y) < r'$ implies $d(z, Tz) < \frac{a+r}{2} < r$.

V_r is a neighbourhood of unique fixed point z_*, since $z_* \in V_r$ and V_r is an open subset of mearic space X.

Now we prove property (C) about T, i e.

(2.6) $\{T^n V_r\} \to \{z_*\}$, as n becomes infinite.

Let us consider $z = Ty \in T(V_r) = \{Tx : x \in V_r\}$. From inequality (1) we can write that

(2.7) $d(Ty, T^2y) \leqslant p(\, d(y, Ty), d(y, Ty), d(Ty, T^2y)\,)$

we use hypothesis (ii) on function to deduce that

(2.8) $d(Ty, T^2y) \leqslant h\, d(y, Ty)$.

Therefore it follows that

(2.9) $T(V_r) \subseteq V_{hr}$

and, by induction,

(2.10) $T^n(V_r) \subseteq V_{h^n r}$.

Thus, inclusion (2.10) proves (2.6) (uniform convergence of iterates - or - property (C)).

Theorem 2.3 Let T be a continuous p-contraction selfmapping of a complete metric space (X, d). Then T is a contraction map for suitable topologically equivalent metric.

Proof. It suffices to remark that T has properties (A), (B), (C) from previous Theorem 2.1 and 2.2 and T has also property (D) for hypothesis. Now applying Meyers's result [6], one proves this theorem.

3. MAPPINGS HAVING SOME ITERATE p-CONTRACTION.

Sometimes a selfmapping T is not a p-contraction, but it can have some iterate which is a p-contraction mapping, under suitable function $p \in P$.

From this point of view, we state and prove some results.

Theorem 3.1 Let T be a selfmapping of complete metric space (X, d) satisfying the following condition

(1*) there exists an integer n such that for all pair $x, y \in X$

$$d(T^n x, T^n y) \leqslant p(\, d(x, y), d(x, T^n x), d(y, T^n y)\,)$$

for some $p \in P$. Then T has properties (A) and (B).

Proof. If we put $S = T^n$,. then Theorem 2.1 shows that S has a unique fixed point that we call z_*.

Now we prove that z_* is also a fixed point of T. Indeed let us consider these equalities

$$(3.1)\quad T^n(Tz_*) = T(T^n z_*)$$

$$T^n(Tz_*) = Tz_*$$

$$S(Tz_*) = Tz_*$$

Uniqueness cf fixed point of S and (3.1) imply

$$(3.2)\quad Tz_* = z_*$$

Now we prove that z_* is the unique fixed point of T. Suppose $x \neq y$, $Tx = x$ and $Ty = y$ and apply (1*)

$$(3.3)\quad d(x, y) = d(T^n x, T^n y) \leqslant p(\, d(x, y), 0, 0)$$

$$d(x, y) \leqslant p(0, 0, d(x, y))$$

$$d(x, y) = 0$$

Hence $x = y$ and uniqueness of fixed point is proved.

To prove property (B) it suffices to remark that we can deduce the convergence of sequence $T^h z$ from the convergence of sequence $S^i z_0$, putting $z_0 = z, Tz, T^2 z \ldots, T^{n-1} z$.

Theorem 3.2 Let T a continuous selfmapping of a complete metric space (X, d) satisfying (1*). Then T has property (C).

Proof. For any $r > 0$ we define $V_r = \{x \in X : d(x, T^n x) < r\}$, where n is the integer of condition (1*).

We shall prove that V_r is an open subset of X containing the fixed point z_*, hence a neighbourhood of z_*.

Indeed if $y \in V_r$, $d(y, T^n y) = a < r$, then continuity of T implies that there exists r such that

(3.4) $d(z, y) < r'$ implies $d(z, Tz) < \dfrac{a + r}{2} < r$.

To prove property (C), we consider sequence $\{T^{kn}(V_r)\}$ $(k \in N)$ and deduce the following convergence result

(3.5) $\{T^{kn}(V_r)_{k \in N} \longrightarrow \{z_*\}$, as k becomes infinite.

Suppose $z \in T^n(V_r)$, then we can write $z = T^n y$, with $y \in V_r$. From condition (1*) it follows

(3.6) $d(z, T^n z) = d(T^n y, T^{2n} y) \leqslant p(\, d(y, T^n y), d(y, T^n y), d(T^n y, T^{2n} y))$

Now hypothesis (i) and (ii) on function p assure that

(3.7) $d(z, T^n z) \leqslant h\, d(y, T^n y) \leqslant hr.$

Therefore we have

(3.8) $T^n(V_r) \subseteq V_{hr}$

and, by induction,

(3.9) $T^{kn}(V_r) \subseteq V_{h^k r}$.

Thus uniform convergence (3.6) is proved as consequence of inclusion (3.9), letting $n \to \infty$.

4. Common Fixed Points Of Two Mappings.

The purpose of this section is to investigate common fixed points of two mappings f and g satisfying the following condition

(1**) There exists a function $p \in P$, such that, for all pair $x, y \in X$

$$d(fx, gy) \leqslant p(\, d(x, y), d(x, fx)\ d(y, gy)\,)$$

Theorem 4.1 Let f and g be two selfmapping of a complete metric space (X, d) satisfying (1**). Then f and g have a unique common fixed point.

Proof. Fix $x \in X$ and define the following sequence

(4.1) $x_1 = fx,\ x_2 = g\,x_1, \ldots, x_{2n-1} = fx_{2n-2},\ x_{2n} = g\,x_{2n-1}, \ldots$

If we put $y = x_1$ in (1**) we have

(4.2) $d(x_1, x_2) = d(fx, gx_1) \leqslant p(\, d(x, x_1), d(x, x_1).\ d(x_1, x_2)\,)$

$$d(x_1, x_2) \leqslant h\, d(x, x_1).$$

Similarly we get

(4.3) $d(x_2, x_3) \leqslant h\, d(x_1, x_2) \leqslant h^2\, d(x, x_1)$

and, by induction,

(4 4) $d(x_n, x_{n+1}) \leqslant h^n d(x, x_1)$.

Inequality (4.4) proves that sequence (4.1) is Cauchy.

Completeness of X assures that limit z_* of sequence (4.1) belongs to X.

To prove that z_* is a fixed point of f and g, we set $x = z_*$ and $y = x_{2n-1}$ in condition (1**) and deduce

(4.5) $d(fz_*, x_{2n}) \leqslant p(\, d(z_*, x_{2n-1}), d(z_*, fz_*), d(x_{2n-1}, x_{2n})\,)$

Letting $n \to \infty$, (4.5) shwos that z_* is a fixed point of g. On the other hand, z_* is the unique common fixed point of two mappings f and g. This has simple and standard proof.

Finally we make use of this Theorem 4.1, to study the class of selfmapping of a metric space (X, d) which satisfy the following condition

(1***) There exist two integers n, m, such that for all pair $x, y \in X$
$d(T^n x, T^m y) \leqslant p(\, d(x, y), d(x, T^n x), d(y, T^m y)\,)$
for some function $p \in P$.

Theorem 4.2 Let T be a selfmapping of a complete metric space (X, d) satisfying (1***). Then T has properties (A) and (B).

Proof. The proof of property (A) follows directly from Theorem 4 1 taking $f = T^n$ and $g = T^m$. Indeed there exists a unique fixed point such that $T^n z_* = z_* = T^m z_*$.

But we have

$$T^n(Tz_*) = T(T^n z_*) = Tz_* = T(T^m z_*) = T^m(Tz_*)$$

hence, uniqueness of common fixed point implies

$T(z_*) = z_*.$

To prove property (B), let us suppose that $n > m$. Theorem (4.1) implies that for any fixed $z \in X$ the sequence

(4 6) $z, T^n z, T^{n+m} z. T^{2n+m} z, T^{2n+2m} z, \ldots\ldots$

converges to fixed point z_*.

Putting $z = x, Tx, T^2x, \ldots, T^{n-1}x$, in (4 6) we obtain n sequences containing all elements of sequence $(T^k x)_{k \in N}$.

The property (B) is thus proved.

5. Comparison with some recent similar results.

Recently, some authors [3] [5] have established fixed point theorems involving function q: $R^5_* \to R_*$ satisfying :

(a) q is continuous;

(b) q is increasing, or nondecreasing; in each variable;

(c) $q\ (t, t, t, t, t) < t$ for each $t > 0$.

Now we present the corresponding contractive definition: A selfmapping T of a complete metric space is "q" contraction if there exists a function q such that for all pair $x, y \in X$

$$d(Tx, Ty) \leqslant p(\ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\ ).$$

Our purpose is to show that function p is not the restriction of some function q to its first three component.

Let us consider the following example :

$$p(u, v, z) = h\ max\ \{u\ exp\ [(u-v)\ (v-z)\ (z-u)]^2, v, z\},\ 0 \leqslant h < 1.$$

Since function p, as defined above, is not increasing or not decreasing in its first coordinate variable, p can not satisfy (b), and hence, can not be the restriction of some function q to its first three variable.

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