VOLUME 15

1985

Published by:
The Vijnana Parishad of India
DAYANAND VEDIC POSTGRADUATE COLLEGE
Bundelkhand University
ORAI, U. P., INDIA
TIME DEPENDENT SLOW FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN WAVY WALLS

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(Received: October 10, 1982; Revised: May 9, 1984)

ABSTRACT

Slow flow of a viscous incompressible fluid through rough parallel plates, under the application of periodic pressure gradient, has been considered. The expressions presented for the physical quantities involved are true for all values of the frequency of the pressure gradient. The roughness is assumed to be small. The effect of the frequency and time on the velocity profiles is studied numerically for the particular cases of sinusoidal roughness when the phase differences are zero and \( \pi \).

1. INTRODUCTION

The exact solutions of Navier–Stokes’ equations for a viscous incompressible fluid with axially parallel flow through a tube of circular cross-section, under the influence of periodic pressure gradient have been studied by Sexl [8] and Uchida [9], while those of co-axial circular cylinders by Verma [10]. Hepworth and Rice [4,5] have also studied the flow between parallel plates and circular rectangular tubes with arbitrary time varying pressure gradient. The slow viscous flow between rotating concentric infinite cylinders with axial roughness

In the present paper, the slow flow of a viscous incompressible fluid between wavy walls with roughness along their length has been investigated. The roughness of the walls is assumed to be small as compared to the distance between the walls. The expressions for the longitudinal and the transverse velocity and the pressure have been obtained by using Fourier transform technique. Two particular cases of sinusoidal roughness have been studied numerically.

2. Formulation of Problem

Under the assumption of slow motion, the Navier–Stokes' equations and the equation of continuity, in rectangular co–ordinate system, for a viscous incompressible fluid, in the non–dimensional form, are given by

\[
\begin{align*}
\frac{\partial U}{\partial T} &= -\frac{\partial P}{\partial X} + \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} \right), \\
\frac{\partial W}{\partial T} &= -\frac{\partial P}{\partial Z} + \left( \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2} \right),
\end{align*}
\]

and

\[
\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0,
\]

where
The plates are symmetrically placed on the two sides of the \( x \)-axis, with a distance \( 2a \) between them and \( z \) is measured at right angles to the plates. \( U, V, P \) and \( T \) are non–dimensional longitudinal velocity, transverse velocity, pressure and time respectively.

Under the assumption of slow motion,

(4) \( \nabla^2 P = 0 \).

Using (4) in (1), we have

(5) \[
\begin{align*}
\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} \right)^2 U &= \frac{\partial}{\partial T} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} \right).
\end{align*}
\]

The boundary conditions are

(6) \[
\begin{align*}
U &= 0 = W \text{ at } Z = 1 + \varepsilon N_1 (X) \\
\text{and } Z &= -1 + \varepsilon N_2 (X), \; T > 0, \; X > 0,
\end{align*}
\]

where \( \varepsilon << \) is the roughness parameter and \( N_1(X) \) and \( N_2(X) \) are arbitrary functions of \( X \).

Let

(7) \[
\begin{align*}
P(X, Z, T) &= P_0(X, T) + P_1(X, Z, T), \\
W(X, Z, T) &= W_1(X, Z, T),
\end{align*}
\]

and

\[
\begin{align*}
U(X, Z, T) &= U_0(Z; T) + U_1(X, Z, T),
\end{align*}
\]

where \( P_1, U_1 \) and \( W_1 \) are the variations caused by the roughness and
$P_0$ and $U_0$ are the quantities for the case of smooth plates.

Let

\[ (8) \quad -\frac{\partial P_0}{\partial X} = K \cos nT = \text{Re} \left[ Ke^{inT} \right], \]

and

\[ (9) \quad U_0 = \text{Re} \left[ f(Z) e^{inT} \right], \]

where $\text{Re}$ means 'the real part of' and $K$ is a constant.

The solution for $U_0$ is given by

\[ (10) \quad U_0(Z, T) = \text{Re} \left[ \frac{K}{m^2} \left( 1 - \frac{\cosh mZ}{\cosh Z} \right) e^{inT} \right], \]

where $m = \sqrt{\text{in}}$.

Form (1), (2), (3), (5) and (7), we have

\[ (11) \quad \frac{\partial U_1}{\partial T} = -\frac{\partial P_1}{\partial T} + \left( \frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2} \right), \]

\[ (12) \quad \frac{\partial W_1}{\partial T} = -\frac{\partial P_1}{\partial Z} + \left( \frac{\partial^2 W_1}{\partial X^2} + \frac{\partial^2 W_1}{\partial Z^2} \right), \]

\[ (13) \quad \frac{\partial U_1}{\partial X} + \frac{\partial W_1}{\partial Z} = 0, \]

and

\[ (14) \quad \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} \right)^2 U_1 = \frac{\partial}{\partial T} \left( \frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2} \right), \]

under the boundary conditions.
(15) \( U_1 = -U_0, \, W_1 = 0 \) at \( Z = 1 + \varepsilon N_1(X) \) and
\[
Z = -1 + \varepsilon N_2(X), \, T > 0, \, X > 0,
\]
and
\[
U_1 = 0 \text{ at } -1 + \varepsilon N_2(X) < Z < 1 + \varepsilon N_1(X), \, T > 0, \, X > 0.
\]

3. METHOD OF SOLUTION

Following (9), we assume

(16) \[ U_1(X, Z, T) = \text{Re} \left[ U_1(X, Z) e^{i n T} \right] \]

and therefore equation (14) reduces to

(17) \[
\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} \right)^2 U_1 = \text{in} \left( \frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2} \right).
\]

Let us suppose that

(18) \[
\left( \frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2} \right) = f(X, Z)
\]

and

(19) \[
F(\xi, Z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(X, Z) \sin (\xi X) \, dX,
\]

where \( F(\xi, Z) \) is the Fourier sine transform of \( f(X, Z) \).

Using equation (18) in (17), taking the Fourier sine transform and solving the resulting equation, we get

(20) \[
U(\xi, Z) = A(\xi) e^{-bZ} + B(\xi) e^{bZ} + C(\xi) e^{-\xi Z} + D(\xi) e^{\xi Z},
\]

where \( U(\xi, Z) \) is the Fourier sine transform of \( U_1(X, Z) \) and \( A(\xi) \).
$B(\xi), C(\xi)$ and $D(\xi)$ are constants of integration and $b = (\xi^2 + \text{in})^{1/2}$.

Taking inverse Fourier sine transform of equation (20), we get

\begin{align*}
(21) \quad U_1(X, Z, T) &= \Re \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{0}^{\infty} \left[ A(\xi) e^{-bZ} + B(\xi) e^{bZ} + C(\xi) e^{-\xi Z} + D(\xi) e^{\xi Z} \right] \sin(\xi X) e^{i\pi T} d\xi.
\end{align*}

Using (21), in (13), we get

\begin{align*}
(22) \quad W_1(X, Z, T) &= \Re - \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{0}^{\infty} \left( - \frac{\xi}{b} A(\xi) e^{-bZ} + \frac{\xi}{b} B(\xi) e^{bZ} - C(\xi) e^{-\xi Z} + D(\xi) e^{\xi Z} \cos(\xi X) e^{i\pi T} \right) d\xi.
\end{align*}

Using equation (21) and (22) in (11) and (12), we get

\begin{align*}
(23) \quad P_1(X, Z, T) &= \Re \left( \frac{2\pi}{2} \right)^{\frac{1}{2}} \ln \int_{0}^{\infty} \left[ C(\xi) e^{-\xi Z} + D(\xi) e^{\xi Z} \cos(\xi X) \right] e^{i\pi T} d\xi + C,
\end{align*}

where $C$ is a constant of integration.

Let us assume

\begin{align*}
(24) \quad A(\xi) &= A_0(\xi) + \epsilon A_1(\xi) + ... \quad \text{and similar expansions for } B(\xi), \ C(\xi) \text{ and } D(\xi).
\end{align*}

Using the boundary conditions (15) in equations (21) and (22) and equating the co-efficients of $e^{0}$ and $\epsilon$ and inverting the resulting equations by Fourier sine and cosine integral theorems, we get

\begin{align*}
(25) \quad A_0(X) = B_0(X) = C_0(X) = D_0(X) = 0,
\end{align*}
\[(26) \quad A_1(X) = \frac{Kb'}{4} g(n) \left[ \frac{1}{G(X)} \{ N_1(X) - N_2(X) \} \times \right. \]
\[
sinh X - \frac{1}{H(X)} \{ \tilde{N}_1(X) + \tilde{N}_2(X) \} \cosh X \}
\]
\[
B_1(X) = \frac{Kb'}{4} g(n) \left[ \frac{1}{G(X)} \{ N_1(X) - N_2(X) \} \times \right.
\[
sinh X + \frac{1}{H(X)} \{ \tilde{N}_1(X) + \tilde{N}_2(X) \} \cosh X \}
\]
\[
C_1(X) = \frac{KXg(n)}{4} \left[ - \frac{1}{G(X)} \{ N_1(X) - N_2(X) \} \times \right.
\[
sinh b' + \frac{1}{H(X)} \{ \tilde{N}_1(X) + \tilde{N}_2(X) \} \cosh b' \}
\]

and
\[
D_1(X) = \frac{KXg(n)}{4} \left[ - \frac{1}{G(X)} \{ N_1(X) - N_2(X) \} \times \right.
\[
sinh b' - \frac{1}{H(X)} \{ \tilde{N}_1(X) + \tilde{N}_2(X) \} \cosh b' \}
\]

where \(G(X) = b' \cosh b' \sinh X - X \cosh X \sinh b',\)

\(H(X) = b' \sinh b' \cosh X - X \cosh b' \sinh X,\)

\(b' = (X^2 + in)^{1/2}\) and \(g(n)\) is a known function of \(n\), given by

\(g(n) = (1/\sqrt{in}). \tanh \left( i^{1/2} n^{1/2} \right). \tilde{N}_1\) and \(\tilde{N}_2\) are Fourier sine transforms of \(N_1\) and \(N_2\), respectively.

Thus the complete expressions for the velocities and pressure are given by

\[(27) \quad U(X, Z, T) = U_0(Z, T) + U_1(X, Z, T)\]
\[ \begin{align*}
&= \text{Re} \left[ \frac{K}{\ln} \left( 1 - \frac{\cosh (i^{1/2} n^{1/2} Z)}{\cosh (i^{1/2} n^{1/2})} e^{inT} \right) \right] \\
&+ \text{Re} \left[ \frac{K/2 g(n) \epsilon}{(2/\pi)^{1/2}} \right] \\
&\int_0^\infty \left[ \frac{b \sinh \xi \cosh bZ - \xi \sinh b \cosh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \right] \{ \tilde{N}_1 (\xi) + \tilde{N}_2 (\xi) \} \\
&+ \frac{b \cosh \xi \sinh bZ - \xi \cosh b \sinh \xi Z}{b \cosh \xi \sinh b - \xi \cosh b \sinh \xi} \cdot \{ \tilde{N}_1 (\xi) + \tilde{N}_2 (\xi) \} \\
&\sin (\xi X) e^{inT} d\xi, \\
\end{align*} \]

(28) \quad W (X, Z, T) = W_1 (X, Z, T) = \text{Re} \left[ - \frac{K/2 g(n) \epsilon}{(2/\pi)^{1/2}} \right] \\
\int_0^\infty \xi \left[ \frac{\sinh \xi \sinh bZ - \sinh b \sinh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \right] \{ \tilde{N}_1 (\xi) + \tilde{N}_2 (\xi) \} \\
+ \frac{\cosh \xi \cosh bZ - \cosh b \cosh \xi Z}{b \cosh \xi \sinh b - \xi \cosh b \sinh \xi} \cdot \{ \tilde{N}_1 (\xi) + \tilde{N}_2 (\xi) \} \\
\cosh (\xi X) e^{inT} d\xi, \\
\text{and} \\
\]

(29) \quad P(X, Z, T) = P_0 (X, T) + P_1 (X, Z, T) \\
= C - K X \cos nT - \text{Re} \left[ \frac{K/2 g(n) \epsilon}{(2/\pi)^{1/2}} \right] in. \\
\int_0^\infty \left[ \frac{\sinh b \cosh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \right] \{ \tilde{N}_1 (\xi) - \tilde{N}_2 (\xi) \} \]
\[ \frac{\cosh b \sinh \xi Z}{b \cosh \xi \sinh b - \xi \cosh b \sinh \xi} \{ \hat{N}_1 (\xi) + \hat{N}_2 (\xi) \} \]

\[ \cosh (\xi X) e^{i nT} \{ \hat{N}_1 (\xi) + \hat{N}_2 (\xi) \} \]

where \( C \) is a constant of integration.

4. Particular Cases (Sinusoidal Roughness)

Case (i) Let

\[ N_1 (X) = \sin X/l = - N_2 (X) \]

where \( 2\pi l \) is the wavelength of the roughness at the walls.

i.e. the phase difference in the roughness of the walls is \( \pi \).

We may form write

\[ \hat{N}_1 (\xi) = - \hat{N}_2 (\xi) = (\pi/2)^{1/2} \, \delta (\xi - 1/l), \]

where \( \hat{N}_1 (\xi) \) and \( \hat{N}_2 (\xi) \) are the Fourier sine transforms of \( N_1 (X) \) and \( N_2 (X) \), respectively, and \( \delta \) is the Dirac delta function.

Substituting (31) in (27) and (28) and using the property of Dirac delta function (Sneddon [7]), the longitudinal and the transverse velocities for \( l = 1 \) and for different values of \( n \) are obtained as follows:

For \( n = 1 \)

\[ U(X, Z, T) = \{ U_0 (Z, T) \}_n -1 - Re \{ K_0 [ (1.07626 + 2.46087i) \cosh (Z(1+i)^{1/2}) - (0.68334 + 2.61039i) \cosh Z] \sin X \cdot e^{it} \}, \]

\[ W(X, Z, T) = Re \{ K_0 [ (1.07626 + 2.46087i) \]
For $n = 2$

\[ U(X, Z, T) = [U_0(Z, T)]_{n=2} - Re [K \epsilon [(0.68334 + 2.61039 i) \sinh Z \cos X e^{iT}]]. \]

\[ \cosh \{Z (1+2i)^{1/2}\} - (0.47817 + 1.05276 i) \cosh Z \sin X e^{2iT}, \]

\[ W(X, Z, T) = Re [K \epsilon [(0.76328 + 0.84048 i) \sinh \{Z (1+2i)^{1/2}\} \sin X e^{2iT}]]. \]

For $n = 4$

\[ U(X, Z, T) = [U_0(Z, T)]_{n=4} - Re [K \epsilon [(0.35991 + 0.16441 i) \cosh \{Z (1+4i)^{1/2}\} \cosh Z \sin X e^{4iT}]]. \]

\[ W(X, Z, T) = Re [K \epsilon [(0.35991 + 0.16441 i) \sinh \{Z (1+4i)^{1/2}\} \cosh Z \cos X e^{4iT}]]. \]

**Case (ii)**

Let

\[ N_1(X) = N_2(X) = \sin (X/l), \]

i.e. the phase difference in the roughness of the two walls is zero.

The expression for $U$ and $W$ for $l = 1$, now are

For $n = 1$,
\[(39)\] \( U(X, Z, T) = [U_0(Z, T)]_{n-1} \)

\[\quad - \text{Re} \left[ K e \left[ (0. 11265 + 1. 20146 i) \sinh \{Z(1+i)^{1/2}\} \right. \right. \]
\[\quad \left. \left. - (0. 11925 + 1. 05053 i) \sinh Z \right] \sin X. e^{i\tau} \right], \]

\[(40)\] \( W(X, Z, T) = \text{Re} \left[ K e \left[ (0. 11265 + 1. 20146 i) \frac{\cosh \{Z(1+i)^{1/2}\}}{(1+i)^{1/2}} \right. \right. \]
\[\quad \left. \left. - (0. 11925 + 1. 05053 i) \cosh Z \right] \cos X. e^{i\tau} \right]. \]

For \( n = 2 \),

\[(41)\] \( U(X, Z, T) = [U_0(Z, T)]_{n-2} \)

\[\quad - \text{Re} \left[ K e \left[ (0. 20451 + 0. 59854 i) \sinh \{Z(1+2i)^{1/2}\} \right. \right. \]
\[\quad \left. \left. - (0. 08542 + 0. 48298 i) \sinh Z \right] \sin X. e^{2i\tau} \right], \]

\[(42)\] \( W(X, Z, T) = \text{Re} \left[ K e \left[ (0. 20451 + 0. 59854 i) \frac{\cosh \{Z(1+2i)^{1/2}\}}{(1+2i)^{1/2}} \right. \right. \]
\[\quad \left. \left. - (0. 08542 + 0. 48298 i) \cosh Z \right] \cos X. e^{2i\tau} \right]. \]

For \( n = 4 \)

\[(43)\] \( U(X, Z, T) = [U_0(Z, T)]_{n-4} \)

\[\quad - \text{Re} \left[ K e \left[ (0. 19472 + 0. 20719 i) \sinh \{Z(1+4i)^{1/2}\} \right. \right. \]
\[\quad \left. \left. - (0. 04171 + 0. 21354 i) \sinh Z \right] \sin X. e^{4i\tau} \right], \]

\[(44)\] \( W(X, Z, T) = \text{Re} \left[ K e \left[ (0. 19472 + 0. 20719 i) \right. \right. \]
\[\quad \left. \left. \frac{\cosh \{Z(1+4i)^{1/2}\}}{(1+4i)^{1/2}} \right. \right. \]
\[\quad \left. \left. - (0. 04171 + 0. 21354 i) \cosh Z \right] \cos X. e^{4i\tau} \right]. \]
5. NUMERICAL DISCUSSIONS

The longitudinal velocity profiles for the wavy walls (phase difference of the sinusoidal roughness of the walls being \( \pi \)) at different cross-sections for \( \varepsilon = 0, 1, l = 1 \) and for different values of \( n \) and \( T \) are shown by Figures 1, 2 and 3. It is observed that the longitudinal velocity decreases as the distance between the walls increases and vice versa. For \( T = 0 \) and \( T = 1 \), the increase in the frequency of the applied pressure gradient decreases the magnitude of the longitudinal velocity. For \( T = 2 \), the magnitude of the longitudinal velocity is quite small for \( n = 1 \), which further decreases when \( n \) becomes 2, but again increases as it becomes 4.

The transverse velocity profiles for the same values of parameters as above, at the plane \( X = 0 \) are depicted in Figures 4, 5 and 6. For \( X = \pi/2 \) and \( 3\pi/2 \), the transverse velocity vanishes identically for all values of \( Z \) and for \( X = \pi \), it is the image of the curves drawn for \( X = 0 \). For \( T = 0 \) and \( T = 1 \), the increase in the frequency of the applied pressure gradient decreases the transverse velocity. For the forward pressure gradient, the resulting effect of the longitudinal and the transverse velocities is that the direction of the flow is towards the boundary when the width between the plates increases and vice versa.

Figure 7 depicts the transverse velocity profiles for the same values of \( \varepsilon, l, n \) and \( T \), at the plane \( X = 0 \) when the phase difference of the sinusoidal roughness of the walls is zero. It is noted that the transverse velocity is optimum at the mid plane and for \( T = 0 \) and \( T = 1 \), it decreases with the increase in the frequency of applied pressure gradient. The disturbances in the flow become more rapid with the increase in the roughness parameter \( \varepsilon \).
Fig. 1: The longitudinal velocity profiles at different sections of a roughness wave taking $l = 1$, $\varepsilon = 0.1$, and $T = 0$ for $n = 1 (-)$, $2(-.-.-.-)$ and $4(- - - -)$.

Fig. 2: The longitudinal velocity profiles at different sections of the roughness wave taking $l = 1$, $\varepsilon = 0.1$, and $T = 1$ for $n = 1 (-)$, $2(-.-.-.-)$ and $4(- - - -)$. 
Fig. 3: The longitudinal velocity profiles at different sections of a roughness wave taking $l = 1$, $e = 0.1$ and $T = 2$ for $n = 1$ \((-\)) and \(2(-\cdots-\cdot-\cdots-\)) and \(4(-\cdots-\cdot-\cdots-\cdot-\cdot-\cdot-\)).

Fig. 4: The transverse velocity profiles of a roughness wave taking $l = 1$, $e = 0.1$ and $T = 0$ at $X = 0$ plane for $n = 1$ \((-\)) and \(2(-\cdots-\cdot-\cdots-\) and \(4(-\cdots-\cdot-\cdot-\cdot-\cdot-\cdot-\).
Fig. 5: The transverse velocity profiles of a roughness wave taking $l = 1, \varepsilon = 0.1$ and $T = 1$ at $X = 0$ plane for $n = 1 (-)$, $2(\ldots -)$ and $4(\ldots \ldots \ldots)$.

Fig. 6: The transverse velocity profiles of a roughness wave taking $l = 1, \varepsilon = 0.1$ and $T = 2$ at $X = 0$ plane for $n = 1 (-)$, $2(\ldots - -)$ and $4(\ldots \ldots \ldots)$.
Fig. 7: The transverse velocity profiles of a roughness wave taking $l = 1, \varepsilon = 0.1$ at $X = 0$ plane for $n = 1 (-)$, $2(\ldots \ldots)$ and $4(\ldots \ldots)$ at (a) $T = 0$, (b) $T = 1$ and (c) $T = 2$.

Acknowledgements

The authors are thankful to the University Grants Commission, New Delhi, for providing the financial assistance.

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RELATIONS BETWEEN THE HANKEL AND
MULTIVARIABLE $H$-FUNCTION OPERATORS

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(Received: March 27, 1983; Revised: May 17, 1984)

ABSTRACT

The object of this paper is to obtain relations between the Hankel operator and the multivariable $H$–function operator. Several results of S.L.Kalla and R.K. Saxena[4], and of R.K. Saxena and R.K. Kumbhat [5], become special cases of our main relations.

1. Introduction

The multivariable $H$–function due to H. M. Srivastava and R. Panda [8, p. 130, Eq. (1. 1)] is defined and represented as follows (See also Srivastava, Gupta and Goyal [7, p 251, Eq. (C. 1) ]:

$$H \left[ \begin{array}{c} Z_1 \\ Z_r \end{array} \right] = \begin{array}{c} o, n; m_1, n_1; \ldots ; m_r, n_r \\ p, q; p_1, q_1; \ldots ; p_r, q_r \end{array} \left[ \begin{array}{c} Z_1 \\ Z_r \end{array} \right] (a_{ij}; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}; p; b_{ij}; \beta_j^{(1)}, \ldots, \beta_j^{(r)}; q)
$$

$$(c_j^{(1)}, \gamma_j)_1, p_1; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_1, p_r$$

$$(d_j^{(1)}, \delta_j)_1, q_1; \ldots; (d_j^{(r)}, \delta_j^{(r)})_1, q_r$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(s_1) \cdots \phi_r(s_r) \Psi (s_1, \ldots, s_r)$$
where \( \psi = \sqrt{-1} \),

\[
\phi_{i}(s_{i}) = \frac{m_{i}}{\prod_{j=1}^{m_{i}} \Gamma(d_{i}^{(i)} - \delta^{(i)} s_{i})} \frac{n_{i}}{\prod_{j=1}^{n_{i}} \Gamma(1 - c_{i}^{(i)} + \gamma^{(i)} s_{i})} \frac{p_{i}}{\prod_{j=1}^{p_{i}} \Gamma(c_{i}^{(i)} - \gamma^{(i)} s_{i})} \frac{q_{i}}{\prod_{j=m_{i}+1}^{q_{i}} \Gamma(1 - d_{i}^{(i)} + \delta^{(i)} s_{i})} \frac{r_{i}}{\prod_{j=n_{i}+1}^{r_{i}} \Gamma(1 - b_{i} + \delta^{(i)} s_{i})} (i = 1, \ldots, r),
\]

and

\[
\Psi(s_{1}, \ldots, s_{r}) = \frac{n}{\prod_{j=1}^{n} \Gamma(1 - a_{j} + \sum_{i=1}^{r} a_{i}^{(i)} s_{i})} \frac{p}{\prod_{j=1}^{p} \Gamma(a_{j} - \sum_{i=1}^{r} a_{i}^{(i)} s_{i})} \frac{q}{\prod_{j=1}^{q} \Gamma(1 - b_{j} + \sum_{i=1}^{r} b_{i}^{(i)} s_{i})} \frac{r}{\prod_{j=n+1}^{r} \Gamma(1 - c_{j} + \sum_{i=1}^{r} c_{i}^{(i)} s_{i})} \frac{m}{\prod_{j=m+1}^{m} \Gamma(d_{j} - \delta s_{j})} \frac{m_{i}}{\prod_{j=1}^{m_{i}} \Gamma(d_{i}^{(i)} - \delta^{(i)} s_{i})} (i = 1, \ldots, r).
\]

For further details and asymptotic expansions of the above function (1.1), refer to [7] and [8].

Banerji and Sethi [1] have defined the multivariable \( H \)-function operator as follows:

\[
\delta_{\lambda_{n}} f(x) = x^{\delta_{\lambda_{n}}} \int_{0}^{x} t^{\delta} (x^{\delta} - t^{\delta})^{\beta} f(t) \left[ \begin{array}{c} \lambda_{1} U \\ \vdots \\ \lambda_{n} U \end{array} \right] dt \quad \ldots (1.2)
\]

\[
\alpha_{\lambda_{n}} f(x) = x^{\alpha} \int_{0}^{x} t^{\alpha} (x^{\alpha} - t^{\alpha})^{\beta} f(t) \left[ \begin{array}{c} \lambda_{1} V \\ \vdots \\ \lambda_{n} V \end{array} \right] dt \quad \ldots (1.3)
\]

where
\[ U = \left( \frac{t^\xi}{x^\xi} \right)^{mi} \left( 1 - \frac{t^\xi}{x^\xi} \right)^{ni}, \]

\[ V = \left( \frac{x^\xi}{t^\xi} \right)^{mi} \left( 1 - \frac{x^\xi}{t^\xi} \right)^{ni}, \]

and \( \xi, m_i, n_i \) are positive numbers. These operators exist under the following set of conditions:

(i) \( 1 \leq p, q < \infty, p^{-1} + q^{-1} = 1 \)

(ii) \( \text{Re} \left( \delta + \xi \sum_{i=1}^{r} m_i \frac{b_j(i)}{\beta_i(i)} \right) > - \frac{1}{q_i} \)

(iii) \( \text{Re} \left( \beta + \xi \sum_{i=1}^{r} n_i \frac{b_j(i)}{\beta_i(i)} \right) > - \frac{1}{q_i} \)

(iv) \( \text{Re} \left( \alpha + \xi \sum_{i=1}^{r} m_i \frac{b_j(i)}{\beta_i(i)} \right) > - \frac{1}{p_i} \)

(v) \( f(x) \in L_{p_i} (0, \infty) \)

\( (j = 1, 2, \ldots, m_r; \ i, \ldots, r) \)

The conditions (v) ensure that both \( P \) and \( S \) exist and also that both belong to \( L_{p_i} (0, \infty) \).

The operator of the Hankel transformation is given by

(see Sneddon [6])

\[ S'_{n, \alpha} f = 2^\alpha x^{-\alpha} \int_{0}^{\infty} y^{1-\alpha} f(y) J_{2n+\alpha} (xy)dy, \quad \ldots (1.4) \]
where \( J_n(z) \) denotes the Bessel function defined by

\[
J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+n}}{2^{2k+n} k! \Gamma(1+n+k)} \quad \cdots (1.5)
\]

We also need [2, p. 10, Eq. (17)]

\[
\int_0^x t^{m-1} (x^\xi - t^\xi)^{n-1} \, dt = \frac{x^{m+n\xi+1} \Gamma\left(\frac{m+1}{\xi}\right) \Gamma(n+1)}{\Gamma\left(\frac{m+1}{\xi} + n+1\right)} \quad \cdots (1.6)
\]

\((\text{Re} \, \xi > 0, \text{Re} \, n > 0, \text{Re} \, m > 0)\)

and [3, p. 295, Eq. (3)]

\[
\int_0^\infty t^{\mu-1} (t^\nu - x^\nu)^{\nu-1} \, dt = x^{\nu-p\nu-p} B(1 - \nu - \frac{\mu}{p}, \nu), \quad \cdots (1.7)
\]

where \( B(\alpha, \beta) \) is the Beta function.

2. The Main Results

The following results will be established here:

\[
\begin{align*}
\delta, \beta & \quad P_x S_n, \alpha f = x^2 H \
(\lambda_n) & \quad p+2, q+1: p_1, q_1 ; \ldots ; p_r, q_r \left[ \lambda_r \right] \\
(a_1; \alpha_1^{(1)}, \ldots, \alpha_1^{(r)}), & \quad (\frac{\delta + 2k + 2\eta + 2}{\xi}; m_1, \ldots, m_r), \\
(b_j; \beta_j^{(1)}, \ldots, \beta_j^{(r)}), & \quad (\frac{\delta \xi + 2\xi + 2k + 2\eta + 2}{\xi}; m_1 + n_1, \ldots, m_r + n_r), \\
(\beta + 2; n_1, \ldots, n_r): & \quad m_1 + n_1, \ldots, m_r + n_r)
\end{align*}
\]
\[ \begin{aligned}
\left( c_j^{(1)}, \gamma_j^{(1)} \right)_1 p_1; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_1 p_r \\
\left( \delta_j^{(1)}, \gamma_j^{(1)} \right)_1 q_1; \ldots; (\delta_j^{(r)}, \gamma_j^{(r)})_1 q_r \end{aligned} \]
\[ S'_{n, \alpha} f \]
\[ \alpha, \beta \]
\[ S_{\alpha} \cdot S'_{\beta}, \quad f = x^{2\varepsilon} H \]
\[ p + 2, q + 1: p_1, q_1; \ldots; p_r, q_r \]
\[ \left( \lambda_1 \right) \]
\[ \left( \lambda_n \right) \]
\[ (a_j; a_j^{(1)}, \ldots, a_j^{(r)}), \left( \frac{a + 2k + 2\beta + 2\varepsilon}{\xi} ; m_1, \ldots, m_r \right), (\beta + 1; m_1, \ldots, m_r): \]
\[ (b_j; b_j^{(1)}, \ldots, b_j^{(r)}), \left( \frac{a + \xi + \varepsilon - 2k - 2\beta}{\xi} ; m_1 + n_1, \ldots, m_r + n_r \right): \]
\[ \left( c_j^{(1)}, \gamma_j^{(1)} \right)_1 p_1; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_1 p_r \]
\[ \left( \delta_j^{(1)}, \gamma_j^{(1)} \right)_1 q_1; \ldots; (\delta_j^{(r)}, \gamma_j^{(r)})_1 q_r \]
\[ S'_{\beta}, \quad f \]
\[ \text{Proof of (2.1). Consider} \]
\[ I \equiv \delta, \beta \]
\[ P_z S'_{n, \alpha} f = x^{-\varepsilon - \xi \beta - 1} \int_0^x t^{\xi} (x^{\xi} - t^{\xi})^\beta H \left[ \begin{array}{c} \lambda_1 U \\ \lambda_n U \end{array} \right] dt \]
\[ \sum_{k=0}^{\infty} \frac{(-1)^k (t y)^{2k + 2n + \alpha}}{2^{2k + 2n + \alpha} k! F(1 + 2n + \alpha, k)} \frac{1}{(2\pi y)^r} \]

which, on replacing the \( H \)-function by (1.1) and the Bessel function by (1.5), becomes

\[ I = 2^\alpha \xi x^{-\varepsilon - \xi \beta - 1} \int_0^\infty y^{1-\alpha} f(y) J_{2n+\alpha} (ty) dy \int_0^x t^{\xi - \alpha} (x^{\xi} - t^{\xi})^\beta \]

\[ \sum_{k=0}^{\infty} \frac{(-1)^k (t y)^{2k + 2n + \alpha}}{2^{2k + 2n + \alpha} k! F(1 + 2n + \alpha, k)} \frac{1}{(2\pi y)^r} \]
\[
\int_{L_1}^{\ldots} \int_{L_r}^{\ldots} \Pi \phi_t(s_i) \Psi(s_1, \ldots, s_r) \\
\cdot \left( \frac{t^\xi}{x^\xi} \right) m_i \sum s_i (\lambda_i)^{s_i} (1 - t^\xi) m_i \sum s_i \ dt. \\
\]

On changing the order of integration (which is justified under the conditions stated with the definitions of the operators), we get

\[
I = 2^\alpha \xi x^{-\xi} x_{\beta-1} \int_0^\infty y^{1-\alpha} f(y) dy \left( \frac{1}{(2\pi)^r} \right) \\
\int_{L_1}^{\ldots} \int_{L_r}^{\ldots} \Pi \phi_t(s_i) \Psi(s_1, \ldots, s_r) \\
\cdot \left( \frac{1}{x^\xi \sum s_i (m_i + m_s)} \right) ds_1 \ldots ds_r \sum_{k=0}^\infty \frac{(-1)^k y^{2k+2\eta+\alpha}}{2^{2k+2\eta+\alpha} k! \Gamma(1+2\eta+\alpha+k)} \\
\cdot \int_0^\infty (\delta+2k+2\eta+\xi m_i \sum s_i + 1)^{-1} (x^\xi - t^\xi) \left( \beta+\eta \sum s_i + 1 \right)^{-1} dt. \\
\]

Evaluating the \(t\)-integral, using (1.6), we get

\[
I = 2^\alpha \xi x^{-\xi} x_{\beta-1} \int_0^\infty y^{1-\alpha} f(y) dy \left( \frac{1}{(2\pi)^r} \right) \\
\int_{L_1}^{\ldots} \int_{L_r}^{\ldots} \Pi \phi_t(s_i) \Psi(s_1, \ldots, s_r) \\
\cdot \left( \frac{1}{x^\xi \sum s_i (m_i + m_s)} \right) \sum_{k=0}^\infty \frac{(-1)^k y^{2k+2\eta+\alpha}}{2^{2k+2\eta+\alpha} k! \Gamma(1+2\eta+\alpha+k)} \\
\cdot \int_0^\infty \left( (\beta+\eta \sum s_i + 1)^{-1} \right) dt. \\
\]
\[ \zeta - 1 \sum m_i \sum s_i + 1 \]

\[ \frac{\Gamma(\frac{8 + 2k + 2\eta + \xi}{\xi})}{\Gamma(\frac{8 + 2k + 2\eta + \xi}{\xi})} \frac{\Gamma(\beta + \eta + \xi)}{\xi} \sum s_i + 1 + 1 \]

\[ x^2 H^{(r)} \quad m_1, n_1; \ldots; m_r, n_r \quad \lambda \]

\[ \begin{array}{c}
(a_j; a_j^{(1)}, \ldots, a_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; m_1, \ldots, m_r, \frac{8 + 2k + 2\eta + 2}{\xi}; m_1 + n_1, \ldots, m_r + n_r)
\end{array} \]

\[ \quad \lambda \quad \begin{array}{c}
(b_j; b_j^{(1)}, \ldots, b_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; m_1 + n_1, \ldots, m_r + n_r)
\end{array} \]

\[ \begin{array}{c}
(c_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; b_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; c_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; d_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; e_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; f_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; g_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; h_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; i_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; j_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; k_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; l_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; m_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; n_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; o_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; p_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; q_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; r_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; s_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; t_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; u_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; v_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; w_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; x_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; y_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi}; z_j^{(1)}, \gamma_j^{(r)}), \quad \frac{8 + 2k + 2\eta + 2}{\xi};
\end{array} \]

This completes the proof of (2.1).

Similarly, by using (1.7), we get the result (2.2).

3. SPECIAL CASES

Setting \( n = 2 \) in (2.1) and (2.2) we obtain the relationships between the operators studied in [5]. The results of [4] can be also obtained by further specializing our results.

Acknowledgements

Our thanks are due to Dr. R. K. Saxena, Professor of Mathematics, University of Jodhpur, and Dr. B. L. Mathur, Defence Laboratory, Jodhpur, for giving all help. We are also grateful to Professor
H. M. Srivastava, University of Victoria, Canada, for giving suggestions for revising of the paper.

REFERENCES


FIXED POINT THEOREM FOR MAPPING WITH A GENERALIZED CONTRACTIVE ITERATE IN 2-METRIC SPACES

By

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(Received: July 22, 1983; Revised: June 1, 1984)

The concept of 2 metric spaces has been investigated by S. Gähler in a series of papers ([3] to [6]) and other related works have been done by himself and others for extensive bibliography, we mention the work of K. Iseki [8].

The notion of contraction type mapping and fixed point theorem in 2-metric spaces has been introduced by K. Iseki [7]. Das and Gupta [1] generalized Banach contraction principle for mapping of $T$ satisfying

$$d(T_x, T_y) \leq \alpha \frac{d(y, Ty) [1 + d(x, T_x)]}{[1 + d(x, y)]} + \beta d(x, y)$$

for all $x, y \in X$, $\alpha, \beta > 0$, $\alpha + \beta < 1$.

The object of this paper is to establish a fixed point theorem for functions satisfying generalized contractive type condition which generalized the result of Guseman [2].

**Lemma.** Let $X$ be 2-metric space, and $T : X \rightarrow X$ be a mapping. Let $B \subseteq X$ with $T(B) \subseteq B$. If there exist a $u \in B$ and positive integer $n(u)$
such that $T^n(u)(u) = u$, and (for some non-negative constants $a$, $\beta$ with $\alpha + \beta < 1$) let

$$d(T^n(u)(x), T^n(u)(u), a) \leq a \frac{d(u, T^n(u)(u), a) [1 + d(x, T^n(u)(x), a)]}{[1 + d(x, u, a)]} + \beta d(x, u, a)$$

for all $x, a \in X$. Then $x$ is the unique fixed point of $T$ in $B$, and $T^n(u)(y_0) \to u$ for each $y_0 \in B$.

**Proof** By (1), $u$ is the unique fixed point of $T^n(u)$ in $B$ then $T(u) = T(T^n(u)(u)) = T^n(u)(T(u)) \Rightarrow Tu = u$. But $u$ is the unique fixed point of $T$ in $B$. Let $y_0 \in B$, and note that $T(B) \subset B \Rightarrow \{T^n(y_0); n \geq 1\} \subset B$.

Let $r(y_0) = \max \{d(T^n(y_0), u, a); 1 \leq m < n(u) - 1\}$ and for all $n$ sufficiently large with $n = r \cdot n(u) + s$, where $n(u) = p$, $n = rp + s$, with $r > 0$, $0 \leq s \leq p - 1$, such that $T^n(u) = (u)$.

Then

$$d(T^n(y_0), u, a) \leq d(T^n(y_0), u, T^n(u)) + d(T^n(y_0), T^n(u), a)$$

$$+ d(T^n(u), u, a).$$

This is can be written as

$$d(T^n(y_0), u, a) = d(T^n(y_0), T^n(u), a)$$

$$= d(T^{r \cdot n + s}(y_0), T^n(u), a)$$

$$= d(T^n T^{(r-1) \cdot n + s}(y_0), T^n(u), u)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$

$$d(u, T^n(u), a) [1 + d(T^{(r-1) \cdot n + s} (y_0), u), T^n(u), a)$$
\[ \leq \frac{\alpha \cdot d(T^{r+1}w(y_0), x, a)}{[1 + d(x, y_0, a)]} + \beta \cdot d(T^{r+1}w(y_0), u, a) \]

\[ = \beta \cdot d(T^{r+1}w(y_0), u, a) < \beta^2 \cdot d(T^{r+1}w(y_0), u, a) < \ldots \]

\[ < \beta^r \cdot d(T^w(y_0), u, a) \]

\[ < \beta^r \cdot r(y_0) \]

and it follows that \( T^w(y_0) \to u \) is finite.

**Theorem.** Let \((X, d)\) be a complete 2-metric space and let \( T \) be a mapping of \( X \) into itself. Suppose there exists a \( B \subset X \), such that

(a) \( T(B) \subset B \).

(b) \( \alpha + \beta < 1 \) and each \( y \in B \), there is an integer \( n(y) \geq 1 \), with

\[ d(T^{n(y)}(x), T^{n(y)}(y), a) \leq \frac{\alpha \cdot d(y, T^{n(y)}(y), a) \cdot [1 + d(x, T^{n(y)}(x), a)]}{[1 + d(x, y, a)]} + \beta \cdot d(x, y, a) \]

for all \( x \in B \).

(c) For some \( x_0 \in B \), \( Cl\{T^n(x_0) ; n \geq 1\} \subset B \).

Then there is a unique \( u \in B \), such that \( T(u) = u \). Furthermore, if

\[ d(T^{n(w)}(x), T^{n(w)}(y), u) \leq \frac{\alpha \cdot d(y, T^{n(w)}(y), a) \cdot [1 + d(x, T^{n(w)}(x), a)]}{[1 + d(x, y, a)]} + \beta \cdot d(x, y, a) \]

for all \( a, x \in X \), then \( y \) is the unique fixed point in \( X \), and \( T^n(x_0) \to u \) for each \( x_0 \in X \).
Similarly,

\[ d(x_3, x_2, a) = d(T^{m_1}(T^{m_2}(x_1), x_1, a) ) \]

\[ \leq \beta d(T^{m_2}(x_1), (x_1), a) \]

\[ \leq \beta \beta d(T^{m_2}(x_0), x_0, a). \]

Proceeding in this way we can get

\[ d(x_{i+1}, x_i, a) \leq \beta^i d(T^{m_1}(x_0), x_0, a), \quad i \geq 1 \]

\[ \leq \beta^i r(x_0), \quad i \geq 1. \]

By routine calculation one can show that the following inequality holds

\[ d(x_i, x_{i+1}, a) \leq \sum_{l=1}^{j-1} d(x_{l+1}, x_l, a), \quad j > i \]

\[ \leq \frac{\beta^i}{1-\beta} r(x_0), \quad j > i. \]

It follows that the sequence \( \{x_i\} \) is a Cauchy, using completeness and (c), we have \( x_i \rightarrow u \in B \). There is an integer \( n(u) \geq 1 \), such that

\[ d(T^n(u)(y), T^n(u)(u), a) \leq a \frac{d(u, T^n(u)(u), a) [1 + d(y, T^n(u)(u), a)]}{[1 + d(y, u, a)]} \]

\[ + \beta d(y, u, a) \]

for each \( y \in B \).

It implies that \( T^n(u)(x_i) \rightarrow T^n(u)(u) \).
Then \( d(T^{n(u)}(u), a) = \lim_{i \to \infty} d(T^{n(u)}(x_i), u, a) \).

However,

\[ d(T^{n(u)}(x_i), x_i, a) = d(T^{n(u)}(x_{i-1}), T^{n-1}(x_{i-1}), a) \leq \beta d(T^{n(u)}(x_{i-1}), x_{i-1}, a) \text{ from (A)} \]

\[ \leq \beta^i d(T^{n(u)}(x_0), (x_0), a) \to 0 \text{ as } i \to \infty. \]

Hence \( T^{n(u)}(u) = u \). By the lemma, \( u \) is the unique fixed point of \( T \) in \( B \), and \( T^n(y_0) \to u \) for each \( y_0 \in B \).

The last assertion of the theorem follows directly from the lemma.

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A FIXED POINT THEOREM FOR $L$-CONTRACTIONS IN GENERALIZED METRIC SPACES

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(Received: August 6, 1984)

ABSTRACT

The aim of this note is to extend a result concerning the existence of fixed-point for contractive mappings in generalized metric spaces to a more general class of mappings.

Before stating our result, we introduce, for sake of completeness, some standard notions [1].

Let $E$ be a Banach space. A subset $K$ of $E$ is called a cone if it is closed, convex, $tK \subseteq K$ for $t \in \mathbb{R}^+$ and $K \cap (-K) = 0$.

Given a cone $K$ in $E$ we define a partial ordering in $E$ by writing $x < y$ if and only if $y - x \in K$.

Moreover, $K$ is called normal if there exists $\sigma > 0$ such that $0 < x < y$ implies $\|x\| \leq \sigma \|y\|$.

We shall assume in the following that $K$ is a normal cone.

Definition. A topological space $X$ is said to be a generalized metric space if there exists a function $d : X \times X \to K$ such that:

(i) $d(x, y) = 0 \in K \iff x = y$;
(ii) \(d(x, y) = d(y, x)\);

(iii) \(d(x, y) < d(x, z) + d(z, y)\)

("<" denotes the partial ordering induced by \(K\)).

Let \(l: K \to R^+\) be a sublinear positively homogenous functional (that is, if \(u, v \in R^+\) then

\[l(u + v) \leq l(u) + l(v)\]

and \(l(tu) = tl(u)\) for \(t \geq 0\) such that \(l^{-1}(0) = 0\). Then, if \(d^*: X \times X \to R^+\) is the function defined by

\[d^*(x, y) = l(d(x, y)),\]  \hfill (1)

we have that \((x, d^*)\) is a metric space.

We shall say that a generalized metric space is a complete generalized metric space if it is complete in the metric defined by (1).

In a complete generalized metric space \((x, d)\) the following result was proved in [1, Theorem 6.2].

Let \(T: X \to X\) be a map such that

\[d(T(x), T(y)) < Ed(x, y),\]  \hfill (2)

where \(L\) is a positive \((L(K) \subseteq K)\) and bounded linear operator in \(E\) with spectral radius \(r(L)\) less than 1. If \(x\) is complete generalized metric space with \(l(u) = \| u \|\), then \(T\) has a unique fixed point which is the limit of the successive approximations

\[x_{n+1} = Tx_n,\]  \hfill n = 0, 1, 2

for any initial value \(x' = x_0 \in X\).

In the Theorem below we shall show that if \(T: X \to X\) is not a
necessarily continuous map which satisfies a relaxed condition than (2) then the same result holds.

More precisely we have the following

**Theorem.** Let \( X \) be a generalized metric space which is complete in the metric (1) with \( l(u) = \| u \| \) (\( \| u \| \) stands for the norm in \( E \)).

If \( T: \to X \) satisfies

\[
d(T(x), T(y)) < L[ d(T(x), x) + d(T(y), y) ],
\]

where \( L \) is a bounded positive linear operator in \( E \) with spectral radius smaller than \( \frac{1}{2} \), then the equation

\[
x = Tx
\]

has a unique solution in \( x \), which is the limit of the successive approximations

\[
x_{n+1} = Tx_n, \quad n = 0, 1, 2
\]

for any initial approximation \( x' \in X \).

**Proof.** Since \( L \) is sublinear, it follows from (3) for \( x = Tx' \) and \( y = x' \)

\[
d(T(Tx'), T(x')) < Ld(T^2(x'), T(x')) + Ld(T(x'), x').
\]

That is

\[
(I - L) d(T^2(x'), T(x')) < Ld(T(x'), x').
\]

Being \( r(L) < \frac{1}{2} < 1 \), we have that \( (I - L) \) is invertible, hence

\[
d(T^2(x'), T(x')) < (I - L)^{-1} Ld(Tx', x').
\]

Clearly we have that
\[ d(T^{n+1}(x'), T^n(x')) < (I - L)^{-n} L^n d(T(x'), x'). \] (5)

Indeed, for \( n = 1 \), it has been already proved.

Assume that (5) is true for \( n = 1 \), then

\[
d(T^{n+1}(x'), T^n(x')) = d(T(T^n(x')), T(T^{n-1}(x'))) < \]

\[
< Ld(T^{n+1}(x'), T^n(x')) + Ld(T^n(x'), T^{n-1}(x')).
\]

That is

\[
(I - L) d(T^{n+1}(x'), T^n(x')) < Ld(T^n(x'), T^{n-1}(x')).
\]

By the inductive assumption

\[
(I - L) d(T^{n+1}(x'), T^n(x')) < L(I - L)^{-(n-1)} L^{n-1} d(T(x'), x').
\]

Since

\[
L(I - L)^{-(n-1)} = (I - L)^{-(n-1)} I,
\]

we obtain

\[
(I - L) d(T^{n+1}(x'), T^n(x')) < (I - L)^{-(n-1)} L^n d(T(x'), x').
\]

Thus (5) is proved.

Furthermore, we have

\[
d(T^{n+m}(x'), T^n(x')) < d(T^{n+m}(x'), T^{n+m-1}(x')) + d(T^{n+m-1}(x'), T^{n+m-2}(x')) +
\]

\[
+ ... + d(T^{n+1}(x'), T^n(x')) <
\]

\[
< \{(I - L)^{-(n+m-1)} L^{n+m-1} + (I - L)^{-(n+m-2)} L^{n+m-2} + ... +
\]

\[
+ (I - L)^{-n} L^n\} d(T(x'), x') <
\]
\[ < (I-L)^{-n} L^n \left( \sum_{m=1}^{\infty} (I-L)^{-(m-1)} L^{m-1} (d(T(x'), x')) \right) = (I-L)^{-1} L^n u' \]

where \( u' \) is the unique solution of

\[ u = (I-L)^{-1} Lu + d(T(x'), x'). \]

Indeed, from the spectral mapping theorem, we have that the spectral operator \((I-L)^{-1} L\) is such that \( \rho((I-L)^{-1} L) < 1 \).

Hence

\[ d(T^{n+m}(x'), T^n(x')) < ((I-L)^{-1} L)^n u'. \]

Being \( K \) normal, we have

\[ \| d(T^{n+m}(x'), T^n(x')) \| \leq \sigma \| (I-L)^{-1} L^n u' \|. \tag{6} \]

Since the right-hand side of (6) is going to zero when \( n \to +\infty \), we obtain that \( \{T^n(x')\} \) is a Cauchy-sequence with respect to the metric \( d^* \). Being \((X, d^*)\) complete, we denote by \( x* \) the limit of \( \{T^n(x')\} \). Then the following inequalities hold

\[ d(T(x*), x*) < d(T(x*), T^n(x')) + d(T^n(x'), x*) < \]

\[ < (I-L)^{-1} L d(x*, T^{n-1}(x')) + d(T^n(x'), x*). \]

Finally, using the normality of \( K \),

\[ \| d(T(x*), x*) \| \leq \sigma (\| (I-L)^{-1} L \| + \| d(x*, T^{n-1}(x')) \| + \]

\[ + \| d(T^n(x'), x*) \| ). \tag{7} \]

Letting \( n \to +\infty \) in (7), we obtain

\[ T(x*) = x*. \]
At last, we would like to add in passing that the Theorem would be still true if the map $T$ is a "generalized contraction" map (cfr. [2 Ch. 1] for an extensive bibliography).

REFERENCES


INTEGRALS INVOLVING A GENERAL CLASS OF POLYNOMIALS

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(Received: August 17, 1984)

ABSTRACT

H. M. Srivastava and N. P. Singh [4] recently established integrals involving certain products of a general class of polynomials $S_{n}^{m}(x)$ and the multivariable $H$-function. The aim of this paper is to further extend these results and establish certain integrals (both single and multiple) involving the products of $S_{n}^{m}(x)$ and the multivariable $H$-functions.

1. INTRODUCTION

Srivastava [2, p. 1, eqn. (1)] introduced a general class of polynomials $S_{n}^{m}(x)$ defined by means of the following equation:

$$S_{n}^{m}[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^{k}, n = 0, 1, 2, \ldots,$$

where $m$ is an arbitrary positive integer and the coefficients $A_{n, k}$ ($n, k \geq 0$) are arbitrary constants.

On specializing these arbitrary coefficients $A_{n, k}$, $S_{n}^{m}(x)$ yields a number of known polynomials as special cases. These include, among
others, the Jacobi polynomials, the Hermite polynomials, the Laguerre polynomials, the Gould–Hopper polynomials, Brafman polynomials, etc. (see [4]). Srivastava and Singh [4] recently established integrals involving the products of $S_n^m(x)$ and certain multivariable $H$–functions. In this paper we further extend these results and establish integrals (both, single and multiple) involving the products of $S_n^m(x)$ and multivariable $H$–functions. Our results are thus capable of yielding a number of integrals involving products of various polynomials with the multivariable $H$ functions and other special functions to which the multivariable $H$–functions reduce.

In the sequel, we also require the following general class of polynomials of $r$ variables:

$$\mathcal{R}^{m_1, \ldots, m_r}_n (x_1, \ldots, x_r)$$

where

$$J = \sum_{k_1, \ldots, k_r = 0}^n (-n)_j A(n; k_1, \ldots, k_r) \prod_{i=1}^r \{x_i^{k_i} / k_i!\},$$

where $J = \sum_{i=1}^r (m_i, k_i)$, $m_i \geqslant 1$, $i = 1, \ldots, r$, and the coefficients $A(n; k_1, \ldots, k_r)$, are arbitrary constants.

The multivariable $H$–function is represented here in the following manner (see [3, p. 251, eqn. (C.1)]):

$$H \left[ \begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \right] = H \left[ \begin{array}{c} 0, N : M_1, N_1; \ldots; M_r, N_r \\ P, Q : P_1, Q_1; \ldots; P_r, Q_r \end{array} \right] \left[ \begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \right]$$
For the definition of $H\left[\frac{z_1}{z_r}\right]$ and the various conditions on the parameters as also the conditions of convergence, etc., we refer to [3, p. 252, eqns. (C 1) to (C. 8)]. These conditions are assumed to be satisfied throughout this paper. Also, the appearance of an asterisk (* ) at a particular place indicates that parameters at that place are the same as on the right side of (1.3) at the corresponding place.

2. THE SINGLE INTEGRALS

For $i = 1, \ldots, r$ and with

$$X_i(x) = (x-a)^{p_i} (\beta-x)^{q_i}, \quad p_i, q_i > 0$$

$$Y_i(x) = (x-\gamma)^{-p_i-q_i} X_i(x)$$

and

$$f(x) = (x-a)^{-1} (\beta-x)^{-1} (x-\gamma)^{-\mu}$$

$$g(x) = (x-a)^{\mu} (\beta-x)^{\lambda} (x-\gamma)^{-\mu-\lambda}, \quad \mu, \lambda \geq 0$$

(both not zero simultaneously), we have:

\[
(2.1) \quad \int_{\alpha}^{\beta} (x-a)^{-1} (\beta-x)^{-1} S_{n}^{m} [a(x-a)^{\mu} (\beta-x)^{\lambda}] \left[\frac{z_1 X_1(x)}{z_r X_r(x)}\right] dx
\]

\[
= (\beta-a)^{\mu+\lambda-1} \sum_{k=0}^{[n/m]} \frac{(-n)^{mk}}{k!} A_n, \quad \alpha \leq (\beta-a)^{\mu+\lambda}
\]
\[ H_{N+2, \, Q+1} = \begin{bmatrix} \ldots \ \{ \begin{array}{c} z_1 \ (\beta-a)^{p_1+q_1} \\ \vdots \\ z_r \ (\beta-a)^{p_r+q_r} \end{array} \end{bmatrix} \begin{bmatrix} E : * , \ldots , * \\ \vdots \\ F : * , \ldots , * \end{bmatrix} \] (2.2)

\[
\left( \begin{array}{c} \beta f(x) S_n^m [ g(x) ] H \begin{bmatrix} z_1 Y_1(x) \\ \vdots \\ z_r Y_r(x) \end{bmatrix} \end{array} \right) \ dx
\]

\[
= R \sum_{k=0}^{[n/m]} \left\{ \frac{(-n)^{m_k}}{k!} A_{n,k} \begin{bmatrix} a(\beta-a)^{u+\lambda} (\beta-\gamma)^{-v} (\alpha-\gamma)^{-\lambda} \end{bmatrix} \right\}
\]

\[ H_{N+2, \, Q+1} = \begin{bmatrix} \ldots \ \{ \begin{array}{c} Z_1 \\ \vdots \\ Z_r \end{array} \end{bmatrix} \begin{bmatrix} E : * , \ldots , * \\ \vdots \\ F : * , \ldots , * \end{bmatrix} \] , \gamma < a < \beta

with

\[
R = (\beta-a)^{u+v-1} (\beta-\gamma)^{-v} (\alpha-\gamma)^{-v}
\]

and

\[ Z_i = z_i (\beta-a)^{p_i+q_i} (\beta-\gamma)^{-p_i} (\alpha-\gamma)^{-q_i} , \ i = 1 , \ldots , r ; \]

\[ f(x) S_n^m [ g(x) ] H \begin{bmatrix} (-1)^{p_1+q_1} z_1 Y_1(x) \\ \vdots \\ (-1)^{p_r+q_r} z_r Y_r(x) \end{bmatrix} \ dx
\]

\[
= R \sum_{k=0}^{[n/m]} \left\{ \frac{(-n)^{m_k}}{k!} A_{n,k} \begin{bmatrix} a(\beta-a)^{u+\lambda} (\gamma-\beta)^{-v} (\gamma-a)^{-\lambda} \end{bmatrix} \right\}
\]

\[ H_{N+2, \, Q+1} = \begin{bmatrix} \ldots \ \{ \begin{array}{c} (-1)^{p_1+q_1} Z_1 \\ \vdots \\ (-1)^{p_r+q_r} Z_r \end{array} \end{bmatrix} \begin{bmatrix} E : * , \ldots , * \\ \vdots \\ F : * , \ldots , * \end{bmatrix} \] , \gamma < a < \beta

where \( R \) and \( Z_i \) are defined with (2.2) above.
In equations (2.1) to (2.3),

\[ E : (1-u-\mu k; p_1, \ldots, p_r), (1-v-\lambda k; q_1, \ldots, q_r), (a_j, \alpha_{j'}, \ldots, \alpha_{j'^{(r)}})_{1,\Omega} \]

\[ F : (1-u-v-(\mu+\lambda) k; p_1+q_1, \ldots, p_r+q_r), (b_j, \beta_{j'}, \ldots, \beta_{j'^{(r)}})_{1,\Omega} \]

and (2.1)–(2.3) are valid if

\[ \min \{ Re \left( u + \sum_{i=1}^{r} p_i D_i^{(i)} , v + \sum_{i=1}^{r} q_i D_j^{(j)} \right) \} > 0 \]

where

\[ (2.4) \ D_j^{(i)} = d_j^{(i)}/\delta_j^{(i)}, \ j = 1, \ldots, M_s, \ i = 1, \ldots, r. \]

3. THE MULTIPLE INTEGRALS

For \( i = 1, \ldots, r \) and with

\[ U_i (x_i) = (x_i - x_i)^{p_i} (x_i - x_i)^{q_i}, p_i, q_i > 0 \]

\[ W_i (x_i) = (x_i - \gamma_i)^{-p_i + q_i} U_i (x_i) \]

\[ V_i(x_i) = (\gamma_i - a_i)^{\mu_i} (x_i - x_i)^{\lambda_i}, \mu_i, \lambda_i \geq 0 \text{ (not all zero simultaneously)} \]

\[ T_i (x_i) = (x_i - \gamma_i)^{-\mu_i - \lambda_i} V_i (x_i), \]

and

\[ f(x_i) = (x_i - a_i)^{u_i-1} (x_i - a_i)^{v_i-1} (x_i - \gamma_i)^{-\mu_i - \nu_i}, \]

we have:
(3.1) \[
\begin{align*}
&\int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} \left[ (x_i - a_i) u_i - 1 \right] (\beta_i - x_i)^{\nu_i - 1} \, dx_1 \ldots dx_r \\
&= \prod_{i=1}^{r} \left( B_i \right) \sum_{\substack{k_1, \ldots, k_r = 1 \to n}} \left\{ (-n) \prod_{i=1}^{r} A(n; k_i) \right\} \prod_{i=1}^{r} \left\{ \frac{a_i k_i^i (\beta_i - a_i)^{(\mu_i + \lambda_i) k_i}}{k_i !} \right\} \\
&\times R_{n_1}^{m_1}, \ldots, m_r \ (a_1 V_1(x_1), \ldots, a_r V_r(x_r)) H \left[ \begin{array}{c} z_1 U_1 (x_1) \\ \vdots \\ z_r U_r (x_r) \end{array} \right] dx_1 \ldots dx_r
\end{align*}
\]

(3.2) \[
\begin{align*}
&\int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} \left[ f(x_i) \right] R_{n_1}^{m_1}, \ldots, m_r (a_1 T_1(x_1), \ldots, a_r T_r(x_r)) \\
&= \sum_{i=1}^{r} \left( \gamma_i \right) \prod_{k_1, \ldots, k_r = 0}^{\infty} \left\{ (-n) \prod_{i=1}^{r} A(n; k_i) \right\} \prod_{i=1}^{r} \left\{ \frac{a_i k_i^i h_i k_i}{k_i !} \right\} \\
&\times R_{n_1}^{m_1}, \ldots, m_r \ (a_1 V_1(x_1), \ldots, a_r V_r(x_r)) H \left[ \begin{array}{c} z_1 U_1 (x_1) \\ \vdots \\ z_r U_r (x_r) \end{array} \right] dx_1 \ldots dx_r
\end{align*}
\]

\[0, N: M_1, N_1 + 2; \ldots; M_r, N_r + 2\]

\[P, Q: P_1 + 2, Q_1 + 1; \ldots; P_r + 2, Q_r + 1\]

\[\frac{z_1 (\beta_1 - z_1)^{p_1 + q_1}}{z_r (\beta_r - a_r)^{p_r + q_r}} \quad * : C\]

\[\frac{z_r}{z_r'} \quad * : D\]

with \(B_i = (\beta_i - a_i)^{u_i + v_i - 1}, i = 1, \ldots, r\); \(a_i < \beta_i, i = 1, \ldots, r\), where \(\gamma_i < a_i < \beta_i, i = 1, \ldots, r\), with
\[ \eta_i = (\beta_i - a_i) \mu_i + \nu_i - 1 \quad (\beta_i - \gamma_i) - \mu_i \quad (a_i - \gamma_i) - \nu_i \]

\[ h_i = (\beta_i - a_i) \mu_i + \lambda_i \quad (\beta_i - \gamma_i) - \mu_i \quad (a_i - \gamma_i) - \lambda_i \]

\[ Z_i' = z_i (\beta_i - a_i) p_i + q_i' \quad (\beta_i - \gamma_i) - p_i \quad (a_i - \gamma_i) - q_i', \quad i = 1, \ldots, r; \]

(3.3) \[ \int_{a_1}^{\beta_1} \ldots \int_{a_r}^{\beta_r} \Pi_{i=1}^{r} f(x_i) \]

\[ R_{n}^{m_1, \ldots, m_r} \left( (-1)^{\mu_1 + \lambda_1} a_1 T_1(x_1), \ldots, (-1)^{\mu_r + \lambda_r} a_r T_r(x_r) \right) \]

\[ H \left[ (-1)^{p_1 + q_1} z_1 W_1(x_1) \right] dx_1 \ldots dx_r \]

\[ (-1)^{p_r + q_r} z_r W_r(x_r) \]

\[ = \prod_{i=1}^{r} (\eta_i) \sum_{k_1, \ldots, k_r=0}^{f \leq n} \left\{ (-1)^{j} A(n; k_1, \ldots, k_r) \prod_{i=1}^{r} \left[ (-1)^{\mu_i + \lambda_i} a_i^{k_i} \frac{\mu_i}{k_i!} \right] \right\} \]

\[ 0, \quad N: M_1, N_1 + 2; \ldots; M_r, N_r + 2 \]

\[ P, Q: P_1 + 2, Q_1 + 1; \ldots; P_r + 2, Q_r + 1 \]

H \[ \left[ (-1)^{p_1 + q_1} z_1' \right]^{*: C} \]

\[ \left[ (-1)^{p_r + q_r} z_r' \right]^{*: D} \]

for \( a_i < \beta_i < \gamma_i \), \( i = 1, \ldots, r \),

where \( \eta_i, h_i \) and \( Z_i' \) are given above with (3.2).

In equations (3.1) to (3.3), we have

\[ C : (1 - u_1 - \mu_1 k_1, p_1), (1 - v_1 - \lambda_1 k_1, q_1), (c_i', \gamma_i'), 1_{P_1} ; \ldots ; \]

\[ (1 - u_r - \mu_r k_r, p_r), (1 - v_r - \lambda_r k_r, q_r), (c_i^{(r)}, \gamma_i^{(r)}), 1_{P_r} \]

\[ D : (d_i', \delta_i), 1_{Q_1} , (1 - u_1 - v_1 - (\mu_1 + \lambda_1) k_1, p_1 + q_1) ; \ldots ; \]
and the conditions of validity of (3.1) to (3.3) are:

\[
\min \{ \Re (u_i + q_i) \} > 0, \quad j = 2, \ldots, M_i; \quad i = 1, \ldots, r,
\]

where \( D_i^{(c)} \) being given by (2.4);

\[
(3.4) \quad \frac{1}{2\pi r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( \sum_{i=1}^{r} s_i t_i \right) \left[ (t_1 + p_1)^{-\alpha_i} \right] \left[ z_1(t_1 + p_1)^{-\mu_1} \right] dt_1 \ldots dt_r
\]

\[
= \prod_{i=0}^{r} (s_i a_i, -1) \exp \left( - \sum_{i=1}^{r} s_i p_i \right) \sum_{k_1, \ldots, k_r = 0}^{J \leq n} \left( -n \right) J A(n; k_1, \ldots, k_r)
\]

\[
= \prod_{i=1}^{r} \left[ \frac{a_i k_i s_i \lambda_i k_i}{k_i!} \right]
\]

\[
(\omega = \sqrt{-1}) \quad \text{with}
\]

\[
D_1 : (d_j^{(r)}, \delta_j^{(r)})_1, Q_1, (1 - z_1 - \lambda_1 k_1, \mu_1), \ldots, (d_j^{(r)}, \delta_j^{(r)}, \delta_j^{(r)})_1, Q_r
\]

\[
(1 - \alpha_r - \lambda_r k_r, \mu_r)
\]

provided that \( \lambda_i, \mu_i, s_i > 0, \eta_i + \Re (p_i) > 0, \Re (a_i) > 0,
\]

\[
\Re (a_i) + \eta_i \sum_{j=1}^{r} \mu_j > 0, \quad i = 1, \ldots, r
\]
\[(3.5) \frac{1}{(2\pi\omega)^r} \int_{\eta_1-\omega}^{\eta_1+\omega} \cdots \int_{\eta_r-\omega}^{\eta_r+\omega} \exp \left( \sum_{i=1}^{r} s_i t_i \right) \prod_{i=1}^{r} \left[ (t_i+\omega)^{-\alpha_i} \right] \]

\[R_{n}^{m_1, \ldots, m_r} \left( a_1(t_1+p_1)^{-\lambda_1}, \ldots, a_r(t_r+p_r)^{-\lambda_r} \right) H \left[ \frac{z_1(t_1+p_1)^{\mu_1}}{z_r(t_r+p_r)^{\mu_r}} \right] dt_1 \cdots dt_r \]

\[= \exp \left( - \sum_{i=1}^{r} s_i p_i \right) \prod_{i=1}^{r} \left( s_i^{\alpha_i-1} \right) \sum_{k_1, \ldots, k_r = 0}^{J \leq n} \left\{ (-n)^J A(n; k_1, \ldots, k_r) \right\} \]

\[\Pi_{i=1}^{r} \left\{ \frac{a_i k_i s_i \lambda_i k_i}{k_i !} \right\} \left\{ 0, N: M_1, N_1; \ldots; M_r, N_r \right\} \]

\[P, Q: P_1 + 1, Q_1, \ldots, P_r + 1, Q_r \]

\[\left[ \begin{array}{c} z_1 s_1 - \mu_1 \\ \vdots \\ z_r s_r - \mu_r \end{array} \right] \left\{ \begin{array}{c} *: C_1 \\ *: *; \ldots; * \end{array} \right\} \]

(\(\omega = \sqrt{-1}\)) with

\[C_1 : (c_1', \gamma_1')_1, P_1, \left( \alpha_1 + \lambda_1 k_1, \mu_1 \right); \ldots; (c_r', \gamma_r')_1, P_r, \left( \alpha_r + \lambda_r k_r, \mu_r \right)\]

provided that \(\lambda_i, \mu_i > 0, \gamma_i + \text{Re} (p_i) > 0, \text{Re} (\alpha_i) > 0, \)

and \(\text{Re} (\alpha_i) > \gamma_i \sum_{j=1}^{r} \mu_i, i = \ldots, r.\)

**Method of proof:** To establish (2.1), we substitute for \(S_n^m \left[ x \right]\) occurring in the integrand on the left of (2.1), from (1.1), and then integrate term-by-term to get

\[\sum_{k=0}^{[n/m]} \frac{(-n)^{nk}}{k!} A_{n,k} a^k \int_{\beta}^{\alpha} (\chi-\alpha)^{\mu k - 1} (\beta-\chi)^{\nu + \lambda k - 1} \]
We next substitute for the multivariable \( H \)-function occurring in the integrand of the above integral in terms of the multiple contour integral from [3, p. 251, eqn. (C. 1)], change the order of integrations, evaluate the \( x \)-integral by means of [1, p. 10, eqn. (13)] and then interpret the resulting integral with the help of [3, p. 251, eqn. (C. 1)] to get the desired result.

Formulas (2.2) and (2.3) are established similarly with the help of [1, p. 10, eqns. (14), (15)].

The multiple integrals (3.1) to (3.3) are established with the help of (1.2) in a similar manner, while as the multiple contour integrals (3.4) to (3.5) are established with the help of the following result:

\[
\frac{1}{2\pi i} \int_{\gamma - \infty}^{\gamma + \infty} e^{s t} (t + \rho)^{-\beta} \, dt = \frac{s^{\alpha-1} e^{-s\rho}}{\Gamma(\alpha)}, \quad s > 0,
\]

\((\omega = \sqrt{-1})\), where, for convergence \( \eta + \text{Re} (\rho) > 0 \) and \( \text{Re} (\alpha) > 0 \).

4. Particular Cases

(i) If, in (2.1), we take \( \mu = 0 \) and \( \lambda = 1 \), it reduces to the known result [4, p. 166, eqn. (2.2)].

(ii) If, in (3.1), we replace the general class of polynomials of \( r \) variables, viz.

\[
R_{m_1,\ldots,m_r}^{m_1,\ldots,m_r}(x_1,\ldots,x_r)
\]

by the product of \( r \) polynomials of the type \( S_{m_i}^{m_i}(x_i) \) it yields:
\[
(4.1) \left[ \beta_1 \ldots \beta_r \right]_{\alpha_r} \prod_{i=1}^{r} \left( \frac{x_i-a_i}{\alpha_i} \right)^{m_i-1} \frac{(\beta_i-x_i)^{m_i-1}}{n_i!} S_{n_i}^{m_i} [a_i V_i(x_i)] \times
\]

\[
H \left[ \begin{array}{c} z_1 U_1(x_1) \\ \vdots \\ z_r U_r(x_r) \end{array} \right] dx_1 \ldots dx_r
\]

\[
= \prod_{i=1}^{r} \left( B_i \right) \left\{ \frac{(-ni)^m k_i}{k_i!} \right\}_{i=1}^{r} A_{n_i, k_i} a_i^{k_i} (\beta_i-a_i)(\mu_i+\lambda_i)k_i
\]

\[
0, N: M_1, N_1+2; \ldots ; M_r, N_r+2
\]

\[
P, Q: P_1+2, Q_1+1; \ldots ; P_r+2, Q_r+1
\]

where \( U_i(x_i), V_i(x_i) \) and \( B_i, C, D \) are given above. The conditions of validity of (4.1) are:

\[
\min \{ \text{Re} \ (ui+p_i D_j^{(i)}, vi+q_i D_j^{(i)}) \} > 0, \ j=1, \ldots , M_i; \ i=1, \ldots , r,
\]

where \( D_j^{(i)} \) being given by (2.4).

ACKNOWLEDGEMENTS

The authors are grateful to Prof. H. M. Srivastava for his kind encouragement during the preparation of this paper. The authors are also thankful to the U. G. C. for providing financial assistance.

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SOME RESULTS ON FIXED POINTS FOR THREE MAPS

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(Received: March 15, 1984)

In this note we present some results on fixed point theorem for three maps which generalize the results of Park and Rhoades [4] and Park [3] respectively. First we give the following definitions and notations.

Throughout the paper \( w \) stands for the set of all nonnegative integers and \( R_+ \) for the set of all non-negative real numbers.

A point \( x_0 \in X \) is called regular for \( f, g \) and \( h \), or, simply regular, if there exists a sequence \( \{ x_n \} \subset X \) satisfying, \( hx_{2n+1} = fx_{2n} \) and \( hx_{2n+2} = gx_{2n+1} \) for each \( n \in w \), and \( \sup \{ d (hx_i, hx_j) \mid i, j \in w \} < \infty \). It should be noted that for \( f(X) \cup g(X) \subseteq h(X) \), such a \( \{x_n\} \) always exists.

Let \( O(x_0) = \{ hx_n \mid n \in w \} \), and \( \delta [ O(x_0) ] \) denote the diameter of \( O(x_0) \).

Let \( \delta (x,y) = \text{diameter} \{ O(x) \cup O(y) \} \) where \( x \) and \( y \) are regular.

We have the following results:

**Theorem 1.** Let \( f, g, h \) be self-maps of a metric space \( (X,d) \) such that \( fh = hf \), \( gh = hg \), \( \phi : R_+ \longrightarrow R_+ \), \( \phi \) non-decreasing, continuous from the right, and satisfying \( \phi (t) < t \) for each \( t > 0 \). Suppose there
exists a regular point $x_0 \in X$ such that $\{hx_n\}$ has a cluster point $a_0 \in X$, which is regular.

If

1. $d(fx, gy) \leq \phi(\delta(O(x) \cup O(y)))$ for each $x, y \in \{x_n\} \cup \{a_n\} \cup \{a_0\}$, where $\{a_n\}$ is defined by $ha_{2n+1} = fa_{2n}$ and $ha_{2n+2} = ga_{2n+1}$ for $n \in \mathbb{N}$, and
2. $h$ is continuous at $a_0$,

then $ha_0$ is a common fixed point for $f, g, h$ and $\{hx_n\}$ converges to $a_0$. If (1) is satisfied for all regular points $x, y \in X$, then $ha_0$ is the unique common fixed point of $f, g$ and $h$.

The result can be easily proved by coupling the proof for two maps by Park and Rhoades [4] with the usual technique for the proof of fixed point for three maps, given by Ganguly([21]) and Singh[5].

**Theorem 2.** Let $f, g, h$ be self-maps of a complete metric space $(X, d)$ such that $fh=hf$, $gh=hg$, $h$ is continuous, and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\phi(t) < t$ for each $t > 0$. If every point of $X$ is a regular point and (1) is satisfied for all $x, y \in X$, then $f, g$ and $h$ have a unique common fixed point and $\{hx_n\}$ converges to the fixed point for each $x \in X$.

The result follows from Theorem 1.

**Remark 1.** For $f=g$, we have Theorems 1 and 2 of Park and Rhoades [4].

**Remark 2.** From the logic of Park and Rhoades ([4], pp. 116–117) it follows that Theorem 1 of Ganguly [1] is a special case of Theorems 2 and 1, respectively.

Now we state a different result which can also be easily proved.

**Theorem 3.** Let $(X, d)$ be a metric space, $f, g$ and $h$ be the same as above. Suppose there exists a regular point $x_0 \in X$ such that
(1) \( O(x_0) \) has a cluster point \( a_0 \in X \) which is regular;

(2) for any \( \epsilon > 0 \), there exist \( \epsilon_0 < \epsilon \) and \( \delta_0 > 0 \) such that for any \( x, y \in \{x_n\} \cup \{a_n\} \cup \{h_{a_0}\} \),

\[
\epsilon \leq \delta(x, y) < \epsilon + \delta_0 \implies d(fx, gy) \leq \epsilon_0, \quad \text{where} \quad \{a_n\} \quad \text{is}
\]

the same as in Theorem 1, and

(3) if \( h_{a_0} \) is continuous at \( a_0 \),

then \( h_{a_0} \) is a common fixed point of \( f, g \) and \( h \), and \( \{h x_n\} \) converges to \( a_0 \). If (2) is satisfied for all regular points \( x, y \in X \), then \( h_{a_0} \) is the

unique common fixed point of \( f, g \) and \( h \).

**Remark 3.** For \( f=g \), we have Theorem 2 (C \( \delta' \)) of Park [3].

**ACKNOWLEDGEMENT**

The author is extremely grateful to Prof. S. Park, Seoul University, Korea, for his valuable help in the preparation of this paper. He also thanks Dr. S. L. Singh for sending the reprints of his papers mentioned below. Finally, he feels indebted to Prof. H. M. Srivastava, Canada, for his kind encouragement and suggestions.

**REFERENCES**


A NOTE ON ENTIRE FUNCTIONS OF IRREGULAR
(p, q)-GROWTH

By

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(Received: June 20, 1983; Revised: August 23, 1984)

1. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a nonconstant entire function. Set
\[
M(r) = M(r, f) = \max_{|z| = r} |f(z)|, \quad \mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| r^n,
\]
where \( M(r) \) and \( \mu(r) \) are called the maximum modulus and the
maximum term of \( f(z) \), respectively. The concept of \( (p, q) \)-order,
lower \( (p, q) \)-order, \( (p, q) \)-type and lower \( (p, q) \)-type of an entire
function \( f(z) \) having index-pair \( (p, q) \), \( p > q \geq 1 \) has recently been
introduced by Juneja et al. ([2], [3]). Thus \( f(z) \) is said to be of
\( (p, q) \)-order \( \rho \) and lower \( (p, q) \)-order \( \lambda \) if

\[
(1.1) \quad \lim_{r \to \infty} \sup_{\inf} \frac{\log[p] M(r)}{\log[q] r} = \varphi(p, q) \equiv \rho
\]

and the function \( f(z) \) having \( (p, q) \)-order \( \varphi(b < \rho < \infty) \) is said to be
of \( (p, q) \)-type \( T \) and lower \( (p, q) \)-type \( t \) if

\[
(1.2) \quad \lim_{r \to \infty} \sup_{\inf} \frac{\log[p-1] M(r)}{(log[q^{-1}] r) ^ \rho} = T(p, q) \equiv T
\]

where \( b = 0 \) if \( p > q \) and \( b = 1 \) if \( p = q \) and \( \log[p] x \) stands for the
Definition. An entire function for which \((p, q)\)-order and lower \((p, q)\)-order are equal is said to be of regular \((p, q)\)-growth. Functions which are not of regular \((p, q)\)-growth are said to be of irregular \((p, q)\)-growth.

It is easy to see that the lower \((p, q)\)-type of an entire function of irregular \((p, q)\)-growth is always zero. Hence the study of growth properties of entire functions with nonzero lower \((p, q)\)-type is limited to functions of regular \((p, q)\)-growth. Thus we need to define a new constant to study the growth of such functions.

Let \(f(z)\) be an entire function of lower \((p, q)\)-order \(\lambda(b < \lambda < \infty)\).

A real valued positive function \(\lambda(r)\) defined on \((0, \infty)\) is said to be a lower proximate order of an entire function \(f(z)\) with index-pair \((p, q)\) if

\[
\begin{align*}
(i) \quad & \lambda(r) \to \lambda \text{ as } r \to \infty \text{ and} \\
(ii) \quad & \Lambda_{[q]}(r) \lambda(r) \to 0 \text{ as } r \to \infty
\end{align*}
\]

where \(\Lambda_{[q]}(r) = \log[q]_r \log[q-1]_r \ldots \log r \cdot r\).

If such a function exists and

\[
\lim_{r \to \infty} \inf \frac{\log[p-1] M(r)}{(\log[q-1]_r \lambda(r)} = t_{\lambda}, \quad 0 < t_{\lambda} < \infty,
\]

then \(t_{\lambda}\) is termed as generalized \(\lambda(p,q)\)-type of \(f(z)\) with respect to the comparison function \(\lambda(r)\).

The object of this note is to prove the following theorem which generalizes a result due to Kumar [4].
2. Theorem. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of lower \((p, q)\) order \( \lambda(b < \lambda < \infty) \). Then a necessary and sufficient condition for \( f(z) \) to be of generalized \( \lambda(p, q)\)-type \( t_\lambda \) is that

\[
(2.1) \quad \lim_{n \to \infty} \inf \left[ \frac{\phi(\log[p-2]n)}{\log[q-1] (-1/n \log |a_n|)} \right]^{\lambda-A} = t_\lambda/M,
\]

provided \( \Psi(n) = |a_n/a_{n+1}| \) forms a nondecreasing function of \( n \) for \( n > n_0 \), where

\[
M = \begin{cases} 
\frac{1}{\lambda} \left( \frac{\lambda-1}{\lambda} \right)^{\lambda-1} & \text{if } (p, q) = (2,2) \\
\frac{1}{e\lambda} & \text{if } (p, q) = (2,1) \\
1 & \text{for all other index-pairs } (p, q),
\end{cases}
\]

\( A = 1 \) if \((p, q) = (2,2)\) and 0 otherwise and \( \Phi(t) \) is a real valued function defined by

\[
(2.2) \quad t = (\log[q-1]r)^{\lambda(r)-A} \Rightarrow \Phi(t) \log[q-1]r.
\]

3. To prove this theorem we need the following lemma:

Lemma 1. \([1]\). \((\log[q-1]r)^{\lambda(r)-A}\) is a monotone increasing function for sufficiently large values of \( r \), where \( 0 \leq A \leq \lambda \).

Lemma 2. Let \( \lambda(r) \) be a lower proximate order of an entire function with index-pair \( (p, q) \). Then

\[
(3.1) \quad \lim_{t \to \infty} \frac{\Phi(ht)}{\Phi(t)} = h^{1/\lambda-A}, \quad 0 < h < \infty,
\]

Proof. In view of (2.2), we have
\[
\frac{d[\log \Phi(t)]}{d[\log t]} = \frac{d[\log \frac{q}{T}]_t}{d[(\lambda (r) - A) \log \frac{q}{T}]} = \frac{1}{(\lambda (r) - A) + \Lambda \frac{q}{T}(r)}.
\]

Hence,

\[
\lim_{t \to \infty} \frac{d[\log \Phi(t)]}{d[\log t]} = \frac{1}{\lambda - A}.
\]

It follows that for given \( \varepsilon > 0 \) and \( t > t_0 \),

\[
\left( \frac{1}{\lambda - A} - \varepsilon \right) d[\log t] < \frac{d[\log \Phi(t)]}{d[\log t]} < \left( \frac{1}{\lambda - A} + \varepsilon \right) d[\log t],
\]

which gives on integrating between the limits \( t \) and \( ht \)

\[
(3.2) \quad \left( \frac{1}{\lambda - A} - \varepsilon \right) \log h < \log \frac{\Phi(ht)}{\Phi(t)} < \left( \frac{1}{\lambda - A} + \varepsilon \right) \log h.
\]

Since \( \varepsilon > 0 \) is arbitrary, on passing to limits in (3.2) we get the required result.

4. Proof of the Theorem. Let \( 0 < t_\lambda < \infty \). Then from (1.3), we have for any \( \varepsilon > 0 \) and \( r > r_0 (\varepsilon) \),

\[
(4.1) \quad \log \frac{p-1}{M(r)} > (t_\lambda - \varepsilon) \left( \log \frac{q-1}{r} \right)^{\lambda(r)}
\]

Since \( f(z) \) is of finite \((p,q)\)-order, following Valiron [5], it can easily be proved that

\[
\log \frac{p-1}{M(r)} \sim \log \frac{p-1}{\mu(r)} \text{ as } r \to \infty.
\]

Thus, from (4.1), we have
\[
\log \mu(r) > \exp \left\{ (t_\lambda - \varepsilon) \left( \log q - 1 \right)_r \lambda(r) \right\}.
\]

Since \( \Psi(n) \) is a nondecreasing function of \( n \), we have for \( r \) satisfying
\( \Psi(n-1) \leq r < \Psi(n) \),

\[
\log |a_n| + n \log r > \exp \left\{ (t_\lambda - \varepsilon) \left( \log q - 1 \right)_r \lambda(r) \right\}
\]
or,

\[
X = \frac{\Phi(\log \left[ \frac{p-2}{n} \right]) \lambda - \Lambda}{\exp \left\{ (\lambda - \Lambda) \log q - 1 \left\{ -1/n \log |a_n| \right\} \right\}} > \frac{\Phi(\log \left[ \frac{p-2}{n} \right]) \lambda - \Lambda}{\exp[\lambda - \Lambda] \log q - 1 \left\{ \log r - 1/n \exp \left\{ (t_\lambda - \varepsilon) \left( \log q - 1 \right)_r \lambda(r) \right\} \right\}}.
\]

With little calculation and the use of properties of \( \Psi(t) \) and \( \lambda(r) \), we can see that the minimum value of the expression on the right hand side of the above inequality is attained for a value of \( r = r(n) \) given by

\[
(4.3) \quad \frac{E \left[ (p-2) \left( \log q - 1 \right)_r \lambda(r) \right]}{\Lambda \left[ q - 1 \right]^{(r)}} = \frac{n}{\lambda r},
\]

where \( E \left[ q \right](x) = \prod_{i=0}^{q} \exp[i] x \).

For the index-pair \((p,q) = (2,1)\), (4.3) reduces to

\[
r^{\lambda(r)} = \frac{n}{\lambda(t_\lambda - \varepsilon)} \Leftrightarrow r = \Phi \left( \frac{n}{\lambda(t_\lambda - \varepsilon)} \right).
\]

Therefore,

\[
= \frac{[\Phi(n)]^\lambda}{\exp \left\{ \lambda \left\{ \log r - r^{\lambda(r)} (t_\lambda - \varepsilon)/n \right\} \right\}}.
\]
or,

\[(4.4) \quad X > e^\lambda (t_\lambda - \epsilon).\]

For \((p, q) = (2, 2)\), (4.3) reduces to

\[(log r)^\lambda (r-1) = \frac{n}{\lambda(t_\lambda - \epsilon)} \iff log \ r = \Phi\left( \frac{n}{\lambda(t_\lambda - \epsilon)} \right).\]

Therefore, we have in this case

\[X > \left[ \frac{\Phi(n)}{\Phi(\frac{n}{\lambda(t_\lambda - \epsilon)})} \right]^{\lambda-1} \]

i.e.,

\[(4.5) \quad X > \frac{\lambda^\lambda}{(\lambda-1)^{\lambda-1}} (t_\lambda - \epsilon).\]

For \((p, q) \neq (2, 1)\) and \((2, 2)\), (4.3) gives

\[log^{[q-1]} r)^\lambda(r) = \frac{1}{t_\lambda - \epsilon} \quad log^{[p-2]} n/\lambda \iff log^{[q-1]} r = \Phi\left( \frac{1}{(t_\lambda - \epsilon)} \log^{[p-2]} n/\lambda \right).\]

Thus

\[X > \left[ \Phi(\log^{[p-2]} n) \right]^\lambda \exp \left\{ \lambda \log^{[q]} (r^{e-1}/\lambda) \right\}, \quad \text{or, from Lemma 2,}\]

\[\infty \left[ \frac{\Phi \left( \log^{[p-2]} n \right)}{\Phi \left( \frac{1}{t_\lambda - \epsilon} \log^{[p-2]} n/\lambda \right)} \right]^\lambda.\]
(4.6) \( X > (t_\lambda - \varepsilon) \)

(4.2), (4.4), (4.5) and (4.6) combine into

(4.7) \( \lim \inf_{n \to \infty} X \geq t_\lambda / M. \)

The inequality in (4.7) obviously holds when \( t_\lambda = 0 \) and for \( t_\lambda = \infty \), the same arguments with an arbitrary large number in place of \( (t_\lambda - \varepsilon) \) lead to the result \( \lim \inf_{n \to \infty} X = \infty. \)

To obtain (2.1), we show that the strict inequality in (4.7) cannot hold. Hence, let a number \( t_\alpha > t_\lambda \) be given such that

\[
\lim \inf_{n \to \infty} \left[ \frac{\Phi(\log [p-2]n)}{\log [q-2](-1/n \log \left| a_n \right|)} \right]^{\lambda - A} = \frac{t_\alpha}{M}.
\]

Let us choose a number \( t_\beta \) such that \( t_\lambda < t_\beta < t_\alpha \). Then we have for all large values of \( n > n_0 \),

\[
\log \left| a_n \right| - n \exp[q-2] \left[ \frac{\Phi(\log [q-2]n)}{(t_\beta / M)^{1/\lambda - A}} \right].
\]

Hence, by Cauchy's inequality \( |a_n| r^n \leq M(r) \), we, again have for \( r > r_0 \) and \( n > n_0 \),

(4.8) \( \log M(r) > -n \exp[q-2] \left[ \frac{\Phi(\log [p-2]n)}{(t_\beta / M)^{1/\lambda - A}} \right] + n \log r. \)

For \( (p, q) = (2, 1) \), choose \( r \) so that \( n = \lambda t_\beta r^{\lambda(r)} \). Then, in view of Lemma 2, we have

\[
\log M(r) > - n \log \left[ \frac{\Phi(n)}{(e \lambda t_\beta)^{1/\lambda}} + n \log \Phi \left( \frac{n}{\lambda t_\beta} \right) \right].
\]
\[ n \log \left[ \left( e^\lambda \ t_\beta \right)^{1/\lambda} \ \frac{\Phi(n/\lambda \ t_\beta)}{\Phi(n)} \right] \]

(4.5) \( \cong t_\beta \ r^{\lambda(r)} \).

In case \((p, q) = (2,2)\), choose

\[
\left( \log r \right)^{\lambda(r)-1} = \frac{M n}{t_\beta \left( \frac{\lambda-1}{\lambda} \right)^{\lambda-1}} \Rightarrow \log r = \Phi \left( \frac{M n}{t_\beta \left( \frac{\lambda-1}{\lambda} \right)^{\lambda-1}} \right),
\]

then (4.8) gives

\[
\log M(r) > n \left[ \log r - (n) \left( M/t_\beta \right)^{1/\lambda - 1} \right]
\]

\[
\cong n/\lambda \log r
\]

(4.10)

\[
\cong t_\beta \left( \log r \right)^{\lambda(r)}.
\]

Lastly let us take \((p, q) \neq (2,1)\) and \((2,2)\). Then we choose \(r\) such that

\[
t_\beta \left( \log \left[ q-1 \right] r/e^\varepsilon \right)^{\lambda(r/e^\varepsilon)} = \log \left[ p-2 \right] n/\lambda \Rightarrow \log \left[ q-1 \right] (r/e^\varepsilon) =
\]

\[
= \Phi \left( \frac{\log \left[ p-2 \right] n/\lambda}{t_\beta} \right), \quad \varepsilon > 0.
\]

Using (3.1), (4.8) reduces to

\[
\log M(r) > n \left[ \log r - \exp \left[ q-2 \right] \left\{ \Phi \left( \frac{\log \left[ p-2 \right] n}{(t_\beta)^{1/\lambda}} \right) \right\} \right]
\]

\[
\cong n \left[ \varepsilon + \exp \left[ q-2 \right] \left\{ \frac{\Phi \left( \frac{\log \left[ p-2 \right] n}{t_\beta^{1/\lambda}} \right)}{t_\beta^{1/\lambda}} \right\} \exp \left[ q-2 \right] \left\{ \frac{\Phi \left( \frac{\log \left[ p-2 \right] n}{t_\beta^{1/\lambda}} \right)}{t_\beta^{1/\lambda}} \right\} \right]
\]

or,

(4.11) \( \log \left[ p-1 \right] M(r) > \left( \log \left[ q-1 \right] r \right)^{\lambda(r)} \left\{ t_\beta + o(1) \right\} \).

Thus, (4.9), (4.10) and (4.11) lead to
which is a contradiction. Thus, strict inequality in (4.7) cannot hold. This proves the result (2.1) and hence the theorem.

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APPLICATIONS OF SRIVASTAVA'S HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES IN HEAT CONDUCTION

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(Received: January 10, 1983; Revised: October 3, 1984)

1. INTRODUCTION

Singh [6] evaluated some integrals involving Kampe de Fériet's function and employed one of them to obtain a solution of a problem in heat conduction considered by Bhonsle [1]. Some expansion formulae involving above functions have also been obtained. Appell's functions and the functions related to them have many applications in mathematical physics. Here we evaluate an integral involving Srivastava's hypergeometric function \( F^{(3)}[x, y, z] \) of three variables ([7], P. 428) and discuss it's applications in solving a problem on heat conduction considered by Bhonsle [1] and in establishing an expansion formula involving Srivastava function.

2. AN INFINITE INTEGRAL

Multiplying both sides of the equation \([4, p, 74, E_{0} (4. 16. 1)]\) by \( e^{-z^2} H_{2n}(z) \) and using the orthogonality property of Hermite polyno-
mlals [5], we have

\[
\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z) dz = \frac{\sqrt{\pi} 2^{2(\nu-\rho)} \Gamma(2\rho+1)}{\Gamma(\nu-\rho+1)}, \quad \rho = 0, 1, 2, \ldots
\]

which, on application to Srivastava's triple series \( F^{(3)} [uz^2, vz^2, w] \), further gives

\[
(2.1) \int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z) F^{(3)} \left[ \begin{array}{c}
(a); (b); (b^*); (c); (c^*); (c^*);
(e); (g); (g^*); (h); (h^*); (h^*)
\end{array} \right] \left[ \begin{array}{c}
u, 1 + \rho - \nu
\end{array} \right] dz
\]

\[
= \frac{\Gamma(2\rho + 1) \sqrt{\pi}}{\Gamma(1 + \rho - \nu)} 2^{2(\nu - \rho)} F^{(3)} \left[ \begin{array}{c}
(a); \frac{2\rho + 1}{2}, \rho + 1 : (b); (b^*); (c);
(e); 1 + \rho - \nu : (g); (g^*); (g^*)
\end{array} \right]
\]

\[
(a); (c^*); (c^*)
\]

\[
(h); (h^*); (h^*)
\]

where \( \rho = 0, 1, 2, \ldots \) and \( A + B + B^* + C \leq E + G + G^* + H, \) etc.

An application of the above integral will be shown in our present investigation.

3. APPLICATION TO HEAT CONDUCTION

Bhonsle [1] employed Hermite polynomials in solving the partial differential equation

\[
(3.1) \quad \frac{\partial \Phi}{\partial t} - K \frac{\partial^2 \Phi}{\partial z^2} = K \Phi z^2,
\]

where \( \Phi (z,t) \) tends to zero for large value of it; and when \( |z| \rightarrow \infty \), this equation is related to the problem of heat conduction.
\(\frac{\partial \Phi}{\partial t} = K \frac{\partial^2 \Phi}{\partial z^2} - h_1 (\Phi - \Phi_0)\),

provided that

\[\Phi_0 = 0 \text{ and } h_1 = Kz^2.\]

The solution of equation (3.1) given by Bhonsle [1] is

\[
\Phi(z, t) = \sum_{r=0}^{\infty} \Phi_r e^{-(1+2r)Kt-z^2/2} H_r(z).
\]

We shall consider the problem of determining a function \(\Phi(z, t)\) such that

\[
\Phi(z, 0) = z^2 e^{-2z^2} F^{(3)} \left[ (a); (b); (b'); (c); (c'); (c''); (c'''); (e); (g); (g'); (h); (h'); (h''); \right. \\
\left. u z^2, v z^2, w z^2 \right].
\]

Now from (3.3) and (3.4), we have

\[
z^2 e^{-z^2} F^{(3)} \left[ (a); (b); (b'); (b''); (c); (c'); (c''); (e); (g); (g'); (h); (h'); (h''); \right. \\
\left. u z^2, v z^2, w z^2 \right]
\]

\[
= \sum_{r=0}^{\infty} Q_r e^{-z^2/2} H_r(z),
\]

and, by (2.1) and the orthogonality property of Hermite polynomials [2, p.289], we find that

\[
Q_{\mu} = \frac{\Gamma(2p+1)}{\Gamma(1+p-\mu/2)} \frac{2^{\mu-2}p^{1/2}}{\mu!} F^{(3)} \left[ (a), \frac{2p+1}{2}, p+1; (b); \right. \\
\left. (e), 1+p-\mu/2; (g); \right].
\]
Thus the solution of (3.3) of the problem reduces to

\[
(3.7) \quad \Phi(x, t) = \sum_{r=0}^{\infty} \frac{2^{r-2\rho-1/2} \Gamma(2\rho+1)}{\Gamma(1+\rho-r/2) \Gamma(r+1)} \frac{e^{-(1+2r)Kt-z^2/2}}{r!} \\
H_r(z) F^{(3)} \left[ (a), \frac{2\rho+1}{2}, \rho+1 : (b); (b'); (b''); (c); (c'); (c''); \right. \left. u, v, w \right] \left[ (e), 1-\rho-r/2 : (g); (g'); (g''); (h); (h'); (h''); \right. \left. u, v, w \right].
\]

where \( \rho = 0, 1, 2, \ldots \), and \( A+B+B'+C \leq E+G+G'+H \), etc.

4. EXPANSION FORMULA

By (3.5) and (3.6) we establish the following expansion formula

\[
(4.1) \quad z^2 e^{-z^2/2} \ F^{(3)} \left[ (a) : : (b); (b'); (b''); (c); (c'); (c''); \right. \left. u, v, w \right] \left[ (e) : : (g); (g'); (g''); (h); (h'); (h''); \right. \left. u, v, w \right] \] 

\[
= \sum_{r=0}^{\infty} \frac{\Gamma(2\rho+1)}{\Gamma(1+\rho-r/2) \Gamma(r+1)} \frac{2^{r-2\rho-1/2}}{r!} H_r(z) \ F^{(3)} \left[ (a), \frac{2\rho+1}{2}, \rho+1 : (h); (b'); (b''); (c); (c'); (c''); \right. \left. u, v, w \right] \left[ (e), 1-\rho-r/2 : (g); (g'); (g''); (h); (h'); (h''); \right. \left. u, v, w \right].
\]

ACKNOWLEDGEMENTS

Our sincere thanks are due to Prof. H. M. Srivastava for his suggestions. The first author is also thankful to the University Gran's
Commission, New Delhi, for financial assistance provided to him.

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INTEGRALS INVOLVING THE $H$-FUNCTION

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(Received : July 15, 1984; Revised : August 17, 1984)

ABSTRACT

The aim of this paper is to evaluate two new integrals involving Fox's $H$-function. Besides the $H$-function, our integrands also involve Bessel functions and Legendre functions. Since the $H$-function is sufficiently general in nature, a number of other integrals involving various special functions can also be obtained from these integrals simply on specializing the parameters of the $H$-functions involved in the main integrals. For the sake of illustration we also evaluate here two such integrals; these latter integrals are of interest in themselves and are also believed to be new.

1. INTRODUCTION

The following formulae will be required for evaluating the main integrals of this paper [1, p. 91, Eq. (3.1); 4, p. 3, Eq. (5)]

\[
(i) \int_{0}^{1} x^{\sigma-\lambda-\delta} (1-x^{2})^{-\frac{1}{2}\mu} p_{\nu}^{\mu} (x) J_{\delta}(xt) J_{\lambda}(xt) \, dx
\]

\[
= \frac{2^{\mu-\lambda-\delta-1} \Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\sigma\right) \Gamma(1+\frac{1}{2}\sigma) \Gamma(\lambda+\delta) \Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\sigma+1) \Gamma(\lambda+1) \Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)}
\]
\[
\text{where } \Re(\mu) < 1, \Re(\sigma) > -1; \]

\[
(ii) \int_0^{\infty} x^{\nu+1} (a^2 + x^2)^{-\lambda/2 - 5/4} \exp \left( -\frac{p^2 x}{a^2 + x^2} \right) J_\nu \left( \frac{p^2 x}{a^2 + x^2} \right) dx
\]

\[
= 2^{-\lambda-\nu-\frac{1}{2}} p^{2\nu} \sqrt{\pi} a^{-\lambda-\frac{1}{2}} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2} + 3/4)} \frac{\Gamma(\nu + \frac{1}{2} + 5/4)}{\Gamma(\lambda + 3/4)} \]

\[
\cdot \frac{1}{\Gamma(\lambda + \frac{1}{2} + \nu + 3/4)} \cdot \frac{1}{\Gamma(\nu + 3/4)} \cdot \frac{1}{\Gamma(\lambda + 3/4)}
\]

\[
(i) \int_0^1 x^{\sigma-\lambda-\delta} (1-x)^{-\frac{1}{2} + \mu} \left[ \frac{P^\mu_{\nu} (x)}{P^\mu_{\nu} (1)} \right] J_\delta(x) J_\lambda(x)
\]

\[
= 2^{2\nu - \lambda - \delta - \sigma - 1} \frac{\Gamma(\delta + 1) \Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + \nu) \Gamma(\lambda + 1 + \delta)}
\]

\[
\sum_{r=0}^{\infty} \binom{\lambda + \delta + 1}{r} \binom{\lambda + \delta + 2}{r} \frac{(-t^2)^r}{(\lambda + 1)_r (\delta + 1)_r (\lambda + \delta + 1)_r} r!
\]
\[ H_{p+1, q+2} \left[ 2^{-\rho} z \begin{bmatrix} A \\ B \end{bmatrix} \right], \]

where \( A = (-\sigma - 2r, \rho), (a_j, a_j)_{1,v}, \)
\[ B = (b_j, \beta_j)_{1,v} \left( -\frac{1}{2} \sigma + \frac{1}{2} \nu + \frac{1}{2} \mu - r, \frac{1}{2} \rho \right), \]
\[ \left( -\frac{1}{2} - \frac{1}{2} \sigma - \frac{1}{2} \nu + \frac{1}{2} \mu - r, \frac{1}{2} \rho \right), \]
and \( \Re(\mu) < 1, \Re(\sigma) > -1 \)
\[ \Re(\sigma + 1) + \rho \min_{1 \leq j \leq n} \Re \left( \frac{b_j}{\beta_j} \right) > 0 \]

(ii) \[ \int_0^\infty x^{\nu + 1} (a^2 + x^2)^{-\lambda/2 - 5/4} \left( -\frac{p^2 a}{a^2 + x^2} \right) J_\nu \left( \frac{p^2 x}{a^2 + x^2} \right) \]
\[ \cdot H_{m, n} \left[ z(a^2 + x^2)^{\alpha} \left( a_j, a_j \right)_{1,v} (b_j, \beta_j)_{1,v} \right] dx \]
\[ = 2^{-\lambda - \nu - \frac{1}{2}} p 2^{2\nu} \pi^{1/2} a^{\lambda - \frac{1}{2}} \]
\[ \sum_{r=0}^\infty \frac{\left( -\frac{p^2}{2a} \right)^r}{r!} H_{m+1, n} \left[ (2\alpha)^{2\mu} z \begin{bmatrix} E \\ F \end{bmatrix} \right], \]

where \( E = (a_j, a_j)_{1,v}, (\lambda/2 + 3/4, \mu), (\nu + \lambda/2 + 5/4 + r, \mu), \)
\[ F = (\lambda + \frac{1}{2} + r, 2\mu), (b_j, \beta_j)_{1,v} \]
and
\[ \Re(\alpha) > 0, p > 0, \Re(\lambda + 1/2) > 0 \]
\[ \Re(\nu + 1) > 0, 2\mu \max_{1 \leq j \leq n} \Re \left( \frac{a_j - 1}{a_j} \right) < \Re(\lambda + \frac{1}{2}) \]

The \( H \)-functions occurring in (2.1) and (2.2) stand for the well -
\[ m, n + 1 \quad \frac{H^{m,n+1}}{p+1, q+2} \begin{bmatrix} 2^{-\sigma} z & A \end{bmatrix} B, \]

where \( A = (-\sigma - 2r, \rho), (a_i, a_i)_{1,2}, \)

\[ B = (b_i, \beta_i)_{1,2}, \left( -\frac{1}{2} \sigma + \frac{1}{2} \nu + \frac{1}{2} \mu - r, \frac{1}{2} \rho \right), \]

\[ \left( -\frac{1}{2} - \frac{1}{2} \sigma - \frac{1}{2} \nu + \frac{1}{2} \mu - r, \frac{1}{2} \rho \right), \]

and \( \text{Re} (\mu) < 1, \text{Re} (\sigma) > -1 \)

\[
\text{Re} (\sigma + 1) + \rho \quad \min_{1 \leq j \leq n} \quad \text{Re} \left( \frac{b_i}{\beta_i} \right) > 0
\]

(ii) \[
\int_{0}^{\infty} x^{\nu+1} (\alpha^2 + x^2)^{-\lambda/2-5/4} \exp \left( -\frac{p^2 \alpha}{\alpha^2 + x^2} \right) J_{\nu} \left( \frac{p^2 x}{\alpha^2 + x^2} \right) d\alpha, \]

\[
H_{\nu}^{m,n} \left[ (a_i, a_i)_{1,2} \right] (b_i, \beta_i)_{1,2}
\]

\[= 2^{-\lambda-\nu-\frac{1}{2}} \left( \frac{p^2}{2\alpha} \right)^{r} \frac{(2\alpha)^{2\mu}}{\pi^{\frac{1}{2}}} a^{-\lambda-\frac{1}{2}}
\]

\[
\sum_{r=0}^{\infty} \left( \frac{p^2}{2\alpha} \right)^{r} \frac{(2\alpha)^{2\mu}}{r!} H_{\nu}^{m+1, n} \left[ (2\alpha)^{2\mu} z \right] F.
\]

where \( E = (a_i, a_i)_{1,2}, (\lambda/2 + 3/4, \mu), (\nu + \lambda/2 + 5/4 + r, \mu) \),

\[ F = (\lambda + \frac{1}{2} + r, 2\mu), (b_i, \beta_i)_{1,4} \]

and

\[ \text{Re} (\alpha) > 0, p > 0, \text{Re} (\lambda+1/2) > 0 \]

\[ \text{Re} (\nu+1) > 0, 2\mu \quad \max_{1 \leq j \leq n} \quad \text{Re} \left( \frac{a_j-1}{a_j} \right) < \text{Re} (\lambda+\frac{1}{2}) \]

The \( H \)-functions occurring in (2.1) and (2.2) stand for the well -
known (Fox's) $H$-function defined and represented as follows:

$$
H_{p, q}^{m, n} \left[ \left( a_1, a_2 \right) \ldots \left( a_p, a_q \right) \right] \\
\left( b_1, b_2 \right) \ldots \left( b_p, b_q \right)
$$

$$= \frac{1}{2\pi i} \int_{L} \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)} x^s \, ds, \quad (2.3)
$$

where the nature of contour $L$ in (2.3), the conditions of its convergence and some of the properties of the $H$-function can be found in Chapter 2 of a recent book on the subject [5]. We have restricted ourselves in the present study to the case where the parameters of the $H$-function occurring throughout the present paper satisfy conditions corresponding appropriately to the following conditions for (2.3)

(i) \[ A = \sum_{j=1}^{n} a_j - \sum_{n+1}^{p} a_j + \sum_{1}^{m} \beta_j - \sum_{m+1}^{q} \beta_j > 0 \]

(ii) \[ | \arg x | < \frac{1}{2} A \pi \]

**Proof.** To derive (2.1), we first express the $H$-function on the left-hand side of it with the help of (2.3), and obtain

$$
\int_{0}^{1} x^{\sigma-\lambda-\delta} (1 - x^\xi)^{-\frac{1}{2} \mu} P_{\mu}^{\lambda} J_{\delta}(xt) J_{\lambda}(\xi t) \, dx.
$$

$$
= \frac{1}{2\pi i} \int_{L} \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \xi)} z^\xi x^\xi \, d\xi. \quad (A)
$$
On changing the order of integration in (A) [which is easily permissible under the conditions stated in (2.1)], putting the value of $x$ integral in (A) from (1.1), we get

\[
\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)} z^\xi d\xi
\]

\[
= \frac{2^{\mu - \lambda - \delta - 1} \prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)} \frac{\Gamma(\delta + 1) \Gamma(\lambda + 1)}{\Gamma(\delta + 1 + \lambda + 1) \Gamma(1 + \frac{1}{2} (\sigma + \rho \xi) - \frac{1}{2} \mu - \frac{1}{2} v - \frac{1}{2} \mu)} \Gamma(3/2 + \frac{1}{2} (\sigma + \rho \xi) + \frac{1}{2} v - \frac{1}{2} \mu)
\]

\[
\cdot \frac{2^{\mu - \lambda - \delta - 1} \prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)} \frac{\Gamma(\delta + 1) \Gamma(\lambda + 1)}{\Gamma(\delta + 1 + \lambda + 1) \Gamma(1 + \frac{1}{2} (\sigma + \rho \xi) - \frac{1}{2} \mu - \frac{1}{2} v - \frac{1}{2} \mu, \delta + 1, \lambda + 1, \lambda + \delta + 1)} \frac{\Gamma(3/2 + \frac{1}{2} (\sigma + \rho \xi) + \frac{1}{2} v - \frac{1}{2} \mu)}{\Gamma(3/2 + \frac{1}{2} (\sigma + \rho \xi) + \frac{1}{2} v - \frac{1}{2} \mu, \delta + 1, \lambda + 1, \lambda + \delta + 1)}
\]

Now representing the function $4F_5$ involved in the above expression in a series form, and using the duplication formula for the gamma function, we easily get after a little simplification:

\[
\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)} z^\xi d\xi
\]

\[
= \frac{2^{\mu - \lambda - \delta - 1}}{\Gamma(\delta + 1) \Gamma(\lambda + 1)} \frac{\Gamma(\delta + 1) \Gamma(\lambda + 1)}{\Gamma(\delta + 1 + \lambda + 1) \Gamma(1 + \frac{1}{2} (\sigma + \rho \xi) - \frac{1}{2} \mu - \frac{1}{2} v - \frac{1}{2} \mu)} \Gamma(3/2 + \frac{1}{2} (\sigma + \rho \xi) + \frac{1}{2} v - \frac{1}{2} \mu)
\]

\[
\cdot \sum_{r=0}^\infty 2^{\lambda + \delta + 1} \frac{\Gamma(1 + \sigma + 2r + \rho \xi)}{(\delta + 1)_r (\lambda + 1)_r (\lambda + \delta + 1)_r} \frac{\Gamma(\lambda + \delta + 1)}{2^r} \frac{\Gamma(\lambda + \delta + 2)}{2^r}
\]

\[
\cdot \frac{1}{\Gamma(1 + \frac{1}{2} \sigma - \frac{1}{2} v - \frac{1}{2} \mu + r + \frac{1}{2} \rho \xi)} \frac{\Gamma(3/2 + \frac{1}{2} \sigma + \frac{1}{2} v - \frac{1}{2} \mu + r + \frac{1}{2} \rho \xi)}{r!}
\]
Changing the order of integration and summation in (C) and interpreting the result thus obtained with the help of (2.3), we get the required result (2.1). Proof of (2.2) follows on lines similar to that of (2.1) on using the formula (1.2).

3. SPECIAL CASES

(i) On specializing the parameters of the $H$–functions involved in (2.1) suitably, we get by means of a known formula [2, p. 600] the following interesting integral after a little simplification:

\[
\int_{0}^{1} x^{\sigma-\lambda-\delta} \left(1-x^{\mu}\right)^{-\frac{1}{2} \mu} P_{\nu}^{\mu}(x) J_{\delta}(xt) J_{\lambda}(xt)
\]

where \(\begin{array}{lcl}
\nu & = & \frac{3}{2} + \frac{1}{2} \sigma - \lambda + \frac{1}{2} \mu + r, \quad \frac{1}{2} + \frac{1}{2} \sigma - \lambda + \frac{1}{2} \mu + r
\end{array}\)

and \(\Re (\mu) < 1, \Re (\sigma) > - 1\) and \(\arg (1-z) < \pi\).

(ii) On specializing the parameter of the $H$–function involved in (2.2) suitably we get by means of a known formula [3, p, 11 (1. 7. 5)]:

\[
\int_{0}^{\infty} x^{\nu} \left(\frac{1}{a^2 + x^2}\right)^{-\lambda/2-5/4} \exp \left(-\frac{p^2 a}{a^2 + x^2}\right) J_{\nu} \left(\frac{p^2 x}{a^2 + x^2}\right)
\]

where \(K_{\nu} \left(2z^{\frac{1}{2}} (a^2 + x^2)^{\frac{\mu}{2}}\right) dx\)
\[ 2^{-\lambda-\nu+a-3/2} p^{2\nu} \pi^{\frac{1}{4}} a^{-\lambda-\frac{1}{2}} \]

\[ \sum_{r=0}^{\infty} \left( \frac{-p^2}{2a} \right)^r H_{3,0}^{2,3} \left( \frac{(2a)^2}{z^2} I \right) \]

where \( I = (\lambda/2+3/4, \mu), (\nu+\lambda/2+5/4+r, \mu), \)

\[ J = \left( \frac{a-\sigma}{2}, 1 \right), \left( \frac{a+\sigma}{2}, 1 \right), (\lambda+\frac{1}{2}+r, 2\mu), \]

and

\( \text{Re}(a) > 0, p > 0, R(\lambda+\frac{1}{2}) > 0, R(\nu+1) > 0. \)

**ACKNOWLEDGEMENTS**

The author is highly grateful to Dr. K. C. Gupta for his help and guidance in the preparation of the above paper. Thanks are also due to Prof. H. M. Srivastava for his encouragement and keen interest.

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ON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS 
IN COMPACT METRIC SPACES

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(Received: October 30, 1984)

ABSTRACT

In this paper, our main fixed point theorem, given in a compact metric space, improves a recent result of the second author using a condition of weak commutativity indebted to the first author. Further, we prove that other related results, established for commuting mappings, are not extendable to the case of weakly commuting mappings.

1. Main theorem

Let \((X, d)\) be a metric space and let \(S, I\) be two selfmaps of \(X\). Sessa [6], generalizing a result of Das and Naik [1], defined the pair \((S, I)\) to be weakly commuting if

\[
d(SIx, ISx) \leq d(Ix, Sx)
\]  

(1)

for all \(x\) in \(X\).

Clearly two commuting mappings weakly commute but the converse is not generally true as is shown in example 1 below.
In the following we study common fixed points of mappings defined on compact metric spaces. In this direction, we recall the following theorem of Fisher [2].

**Theorem 1.** Let $S$, $T$, $I$ and $J$ be continuous selfmaps of a compact metric space $(X, d)$ satisfying the inequality

$$
d(Sx, Ty) < \max \{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy),
\frac{1}{2} d(Ix, Sx), \frac{1}{2} d(Jy, Ty) \} \tag{2}
$$

for all $x, y$ in $X$ for which the right-hand side of the inequality (2) is positive. If

$$
S(Y) \subseteq I(X), \quad T(X) \subseteq J(X) \tag{3}
$$

and $S$ and $T$ commute with $I$ and $J$ respectively, then $S$, $T$, $I$ and $J$ have a unique common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$.

Slightly modifying the proofs of [2], we can show that

**Theorem 2.** Theorem 1 also holds if $S$ and $T$ weakly commute with $I$ and $J$ respectively.

**Proof.** We give it for the convenience of the reader. First of all, we observe that (2) implies that

$$
d(Sx, Ty) < \max \{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy),
\frac{1}{2} [ d(Ix, Jy) + d(Sx, Jy) ], \frac{1}{2} [ d(Ix, Jy) + d(Ix, Ty) ] \}
$$

and therefore

$$
d(Sx, Ty) < \max \{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy) \} \tag{4}
$$
for all $x, y$ in $X$ for which the right-hand side of inequality (4) is positive.

We distinguish two cases.

(i) If the right-hand side of (4) is positive for all $x, y$ in $X$, then the function

$$g(x, y) = \frac{d(Sx, Ty)}{\max \{d(Ix, Jy), d(Ix, Ty), d(Sx, Jy)\}}$$

is continuous on the compact metric space $X \times X$ and so achieves its maximum value $c$. In virtue of (4), $c < 1$. This means that

$$d(Sx, Ty) < c \cdot \max \{d(Ix, Jy), d(Ix, Ty), d(Sx, Jy)\}$$

(5)

for all $x, y$ in $X$. Let $x_0$ (resp. $y_0$) be an arbitrary point in $X$ and define a sequence $\{x_n\}$ (resp. $\{y_n\}$) inductively by choosing a point $x_n$ (resp. $y_n$) such that

$$Sx_{n-1} = Ix_n \text{ (resp. } Ty_{n-1} = Jy_n)$$

for $n = 1, 2, \ldots$. This can be done because of (3). It follows similarly to the proof of Theorem 1 of [2] that the sequences $\{Sx_n\} = \{Ix_{n+1}\}$ and $\{Ty_n\} = \{Jy_{n+1}\}$ have the same limit $z$ in $X$. Since $S$ and $I$ are continuous, the sequences $\{ISx_n\}$ and $\{ISx_n\}$ converge to $Iz$ and $Sz$ respectively. Further using (1) we have

$$d(ISx_n, Iz) \leq d(ISx_n, ISx_n) + d(ISx_n, Iz)$$

$$\leq d(Ix_n, Sx_n) + d(ISx_n, Iz).$$

Letting $n$ tend to infinity, we deduce that $\{ISx_n\}$ converges to $Iz$ so $Sz = Iz$. Using (5), it follows that

$$d(Sz, Ty_n) \leq c \cdot \max \{d(Iz, Jy_n), d(Iz, Ty_n), d(Sz, Jy_n)\}.$$
and on letting \( n \) tends to infinity we have

\[
d(S_z, z) \leq c. \max \{ d(I_z, z), d(Iz, z), d(Sz, z) \}
\]

\[
c. d(Sz, z).
\]

Since \( c < 1 \), we see that \( z \) is a fixed point of \( S \) and therefore also a fixed point of \( I \). Similarly, we can prove that \( z \) is a fixed point of \( T \) and \( J \).

(ii) Suppose that the right-hand side of (4) is zero for some \( x, y \) in \( X \). Then, as in the proof of Theorem 5 of [2], we have

\[
Ix = Jy = Sx = Ty
\]

Since \( S \) weakly commutes with \( I \), we have

\[
d(ISx, ISx) \leq d(Ix, Sx) = 0
\]

and so

\[
ISx = SISx = S^2x = STy.
\]

If \( S^2x \neq Ty \), then using (4) we have

\[
d(S^2x, Ty) < \max \{ d(ISx, Jy), d(ISx, Ty), d(S^2x, Jy) \}
\]

\[
= d(S^2x, Ty),
\]

a contradiction. Hence \( S^2x = Ty = STy \) and so \( Ty = z \) is a fixed point of \( S \). We now have

\[
Iz = ITy = ISx = SISx = STy = Sz = z
\]

and so \( z \) is also a fixed point of \( I \).

Similarly, using the weak commutativity of \( T \) and \( J \), we can prove
that $T$ and $J$ have a common fixed point $w$. We now claim that $z = w$.

Otherwise (4) implies

$$d(z, w) = d(Sz, Tw)$$

$$< \max \{ d(Iz, Jw), d(Iz, Tw), d(Sz, Jw) \}$$

$$= d(z, w),$$

a contradiction.

Thus in both cases we have shown that $S$, $T$, $I$ and $J$ have a common fixed point $z$. It follows easily that $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$. This completes the proof of the theorem.

The corollary follows easily.

**COROLLARY.** Let $S$ and $I$ be two continuous selfmaps of a compact metric space $(X, d)$ satisfying the inequality

$$d(Sx, Sy) < d(Ix, Iy)$$

for all distinct $x, y$ in $X$. If the range of $I$ contains the range of $S$ and if $S$ weakly commutes with $I$, then $S$ and $I$ have a unique common fixed point.

**EXAMPLE 1.** Let $x = [0, 1]$ with the euclidean metric $d$ and let $S$ and $I$ be two selfmaps of $X$ defined by

$$Sx = \frac{1}{4}x - \frac{1}{8}x^2, \quad Ix = \frac{1}{4}x$$

for all $x$ in $X$. Note that $S(x) = [0, 1/8] \subset [0, 1/4] = I(X)$ and

$$d(SIx, ISx) = \frac{x}{16} - \frac{x^2}{128} - \frac{x}{16} + \frac{x^2}{32} = 3\frac{x^2}{128} \leq \frac{x^2}{8} = d(Ix, Sx)$$
for all $x$ in $X$. Thus $S$ and $I$ weakly commute. Further

$$d(Sx, Sy) = \frac{1}{3} |x - y| [1 - \frac{1}{3} (x + y)]$$

$$< \frac{1}{3} |x - y| = d(Ix, Iy)$$

for all distinct $x, y$ in $X$. All the assumptions of the corollary are therefore satisfied but Theorem 1 is not applicable since

$$SIx = \frac{1}{16} x - \frac{1}{128} x^2 \neq \frac{1}{16} x - \frac{1}{32} x^2 = ISx$$

for any $x \neq 0$ and so $S$ and $I$ do not commute.

Results related to Theorem 2 can be found in [4] and [7].

2. Another fixed point theorem.

In this section we present a second generalization of Theorem 1 showing that condition (3) and the continuity of the mappings $I$ and $J$ are unnecessary.

We first recall the following result of Leader [5].

**Lemma.** For a continuous selfmap $S$ on a compact metric space $(X, d)$, the core

$$Y = \bigcap_{n=1}^{\infty} S^n(X)$$

is compact and $SY = Y$.

**Proof.** See proposition 2 of [5].

**Theorem 3.** Let $S$ and $I$ be commuting selfmaps and let $T$ and
$J$ be commuting selfmaps of a compact metric space $(X, d)$ satisfying inequality (2) for all $x, y$ in $X$ for which the right-hand side of inequality (2) is positive. If $S$ and $T$ are continuous, then $S$, $T$, $I$ and $J$ have a unique common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$.

**PROOF.** By the lemma, the core

$$Y = \bigcap_{n=1}^{\infty} S^n(X) = SY$$

is a compact subset of $X$ of course, $Y$ is non-empty because the family \{${S^n(X) : n = 1,2,\ldots}$\} has the finite intersection property. Now let $x$ be an arbitrary point in $Y$. Then $x$ is in $S^n(X)$ for $n = 1, 2, \ldots$ and so $Ix$ is in $IS^n(X) = S^n I(X)$ for $n = 1, 2, \ldots$ since $I$ commutes with $S$. Thus

$$Ix \in \bigcap_{n=1}^{\infty} S^n I(X) \subseteq \bigcap_{n=1}^{\infty} S^n(X) = Y$$

and so $I$ maps $Y$ into $Y$.

Similarly, the set

$$W = \bigcap_{n=1}^{\infty} T^n(X) = TW$$

is a non-empty compact subset of $X$ and $J$ maps $W$ into $W$.

Since $d$ is a continuous mapping of the compact set $Y \times W$ into the reals, there exist points $z'$ in $Y$ and $w'$ in $W$ such that

$$d(z', w') = \sup \{ d(x, y) : x \in X, y \in W \} = L.$$ 

Since $SY = Y$, there exists a point $z$ in $Y$ such that $Sz = z'$. 
Similarly, there exists a point \( w \) in \( W \) such that \( T_w = w' \).

We then have \( d(Sz, T_w) = L \). Suppose that

\[
\max \{ d(Iz, Jw), d(Iz, T_w), d(Sz, Jw) \} > 0.
\]

Then we have from (2):

\[
L = d(Sz, T_w)
\]

\[
< \max \{ d(Iz, Jw), d(Iz, T_w), d(Sz, Jw), \\
\frac{1}{2} d(Iz, Sz), \frac{1}{2} d(Jw, T_w) \}
\]

\[
< \max \{ d(Iz, Jw), d(Iz, T_w), d(Sz, Jw), \\
\frac{1}{2} [d(Iz, Jw) + d(Jw, Sz)], \\
\frac{1}{2} [d(Iz, Jw) + d(Iz, T_w)] \}
\]

\[
< L
\]

since \( Sz \) and \( Iz \) are in \( T \) and \( Yw \) and \( Tw \) are in \( W \). The foregoing inequality gives a contradiction and therefore

\[
d(Iz, Jw) = d(Iz, T_w) = d(Sz, Jw) = 0
\]

or equivalently

\[
Iz = Jw = Sz = Tw.
\] (6)

It now follows that \( d(Sz, T_w) = 0 = L \). The set \( Y = SY \) therefore consists of the single point \( z = Sz \) and the set \( W = TW \) consists of the single point \( w = T_w \). Since \( I \) maps \( Y \) into \( Y \), \( Iz = z \) and since \( J \) maps \( W \) into \( W \), \( Jw = w \). Further (6) implies \( z = w \), showing that \( z \) is a common fixed point of \( S, T, I \) and \( J \). The uniqueness of \( z \) is easily proved.

This concludes the proof.
To see the condition that $X$ is compact in Theorem 3 is necessary, consider the following example:

**EXAMPLE 2.** Let $X = [1, \infty)$ with the euclidean metric $d$ and define $S, T, I$ and $J$ by

$$Sx = 3x, \quad Tx = 6x, \quad Ix = 4x, \quad Jx = 8x$$

for all $x$ in $X$. Since

$$d(Sx, Ty) = \begin{cases} 3 \cdot \lvert x - 2y \rvert < 4 \cdot \lvert x - 2y \rvert = d(Ix, Jy) & \text{if } x \neq 2y \\ 0 < x/2 = (4x - 3x)/2 = d(Ix, Sx)/2 & \text{if } x = 2y \end{cases}$$

for all $x, y$ in $X$, it is easily seen that all the conditions of Theorem 3 are satisfied except the compactness of $X$, but $S, T, I$ and $J$ have no fixed points.

The next example shows that the condition that $S$ and $T$ commute with $I$ and $J$ respectively is necessary in Theorem 3.

**EXAMPLE 3.** Let $X = [0, \frac{1}{3}] \cup \{1/3\}$ with the euclidean metric $d$ and define $S, T, I$ and $J$ by

$$Sx = 1/4, \quad Tx = x, \quad Ix = 1/3, \quad Jx = x/2$$

for all $x$ in $X$. Since

$$d(Sx, Ty) = \begin{cases} 1/4 - y < 1/3 - y/2 = d(Sx, Jy) & \text{if } y \neq 1/3 \\ 1/3 - \frac{1}{6} < 1/3 - 1/6 = d(Ix, Jy) & \text{if } y = 1/3 \end{cases}$$

for all $x, y$ in $X$, then all the conditions of Theorem 3 are satisfied except the commutativity of $S$ and $I$, but $S, T, I$ and $J$ have no common fixed points.

The following example shows that the continuity of $S$ and $T$ is a
necessary condition in Theorem 3.

**EXAMPLE 4.** Let $X = [0,1]$ with the euclidean metric $d$ and define

$$
Sx = \begin{cases} 
1/2 & \text{if } x = 0, \\
x/4 & \text{if } x = 0,
\end{cases}
Ix = \begin{cases} 
1 & \text{if } x = 0, \\
x/2 & \text{if } x \neq 0,
\end{cases}
Tx = 0, 
Jx = x
$$

for all $x$ in $X$. Since

$$
d(Sx, Ty) = \begin{cases} 
\frac{1}{2} < 1 = d(I0, Ty) & \text{if } x = 0, y \in Y \\
x/4 < x/2 = d(Ix, Ty) & \text{if } x \neq 0, y \in Y
\end{cases}
$$

it is easily seen that all the conditions of Theorem 3 are satisfied except the continuity of $S$, but $S$, $T$, $I$ and $J$ have no common fixed points.

The next example shows that theorem 3 is not extendable to the case for weakly commuting mappings.

**EXAMPLE 5** Let $X = \{x, y, z, w\}$ be a finite set with metric $d$ defined by

$$
d(x, x) = d(y, y) = d(z, z) = d(w, w) = 0, \\
d(x, z) = d(x, w) = d(y, z) = d(y, w) = 1, \\
d(x, y) = d(z, w) = 2
$$

and define $S = T$, $I$ and $J$ by

$$
Sx = Sy = S2 = y, Sw = z, \\
Ix = Iy = Iz = x, Iw = w, \\
Jw = Jy = Jz = Jw = y.
$$
Obviously $X$ is compact and $S$ is continuous on $X$.

With a routine calculation, it is easily proved that $S$ and $I$ weakly commute whereas $S$ commutes with $J$.

Further, since

$$d(Sx, Sy) = d(Sx, Sz) = d(Sy, Sz) = d(y, y) = 0,$$

$$d(Sx, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Ix, Jw),$$

$$d(Sy, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Iy, Jw),$$

$$d(Sz, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Iz, Jw),$$

we see that condition (2) is satisfied but $S$, $I$ and $J$ do not have a common fixed point.

We also note that since

$$S(X) = \{y, z\} \subset \{x, w\} = I(X),$$

$$T(X) = \{y, z\} \supset \{y\} = J(X),$$

this example shows that condition (3) cannot be dropped in Theorem 2.

We conclude this section pointing out that Theorems 2 and 3 are two distinct generalizations of Theorem 1. Of course, in virtue of example 5, Theorems 2 and 3 are not comparable.

3. A related result.

Using different assumptions on the mappings under discussion and satisfying a more general condition than (2), Fisher proved the following result in [3].
**THEOREM 4.** Let $S$, $T$, $I$ and $J$ be selfmaps of a compact metric space $(X, d)$ satisfying the inequality

$$d(Sx, Ty) < \max \{ d(Ix, Jy), d(Ix, Sx), d(Jy, Ty) \}$$

for all $x, y$ in $X$ for which the right-hand side of the inequality (7) is positive. If $S$ and $T$ commute, if $I$ and $J$ commute with $ST$ and if $ST$ is continuous, then $S$, $T$, $I$ and $J$ have a unique common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$.

Note that in this theorem the assumption (3) and the continuity of $S$, $T$, $I$ and $J$ are unnecessary whereas the commutativity of the pairs $(S, T)$, $(I, ST)$ is assumed.

We now show in the next example that Theorem 4 fails if only the weak commutativity of the pair $(I, ST)$ is assumed, even if $S=T$.

**EXAMPLE 6.** Let $(X, d)$, $S=T$ and $J$ be as in example 5. Define $I$ by $Ix = Iy = Iz = Iw = x$.

It is easily seen that $J$ commutes with $S^2$ and $S^2$ is continuous. Since

$$IS^2x = IS^2y = IS^2z = IS^2w = Ix = x,$$

$$S^2Ix = S^2Iy = S^2Iz = S^2Iw = S^2x = y,$$

$S^2$ weakly commutes (but does not commute) with $I$.

Further, as already seen in example 5, condition (7) holds but $S$, $I$ and $J$ do not have a common fixed point.
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REMARKS ON SOME FIXED POINT THEOREMS AND THEIR EXTENSIONS

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(Received : December 20, 1984)

INTRODUCTION

The main aim of the present paper is to show that some results of Ghosh [4], Khan [6] and Ray [9] are particular cases of well known results. We also present a theorem, which extends of Ghosh and Chatterjee [5]. Also we remark that the results of Chatterjee [1] hold for a more general class of mappings.

1. Preliminaries and Basic Definitions

Definition 1.1. Let $X$ be a metric space. A mapping $T: X \rightarrow X$ is said to be a Banach operator if there exists $k, 0 \leq k < 1$, such that $d(T^2x, Tx) \leq k d(Tx, x)$ for all $x \in X$. $T$ is called a contraction if there exists $k, 0 \leq k < 1$, such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. 
Remark 1.1 Clearly any contraction is a Banach operator, but not conversely. Let \( X = \mathbb{R} \). Define \( T: \mathbb{R} \to \mathbb{R} \) by \( T(x) = x^2 \). Then \( T \) is a contraction on any closed interval \([a, b] \subset (-\frac{1}{2}, +\frac{1}{2})\), but a Banach operator on \([a, b] \subset (-1, 1)\). A Banach operator may not be continuous and may have more than one fixed point. In fact, let \( X = [0, 1] \).

Define \( T: X \to X \) by \( T(x) = 0 \), if \( x < \frac{1}{2} \), \( T(x) = 1 \), if \( x \geq \frac{1}{2} \). Then \( T \) is a discontinuous Banach operator with fixed points 0 and 1.

Definition 1.2 Let \( B \) be a bounded set in a normed linear space \( X \) and let \( \delta(B) \) be its diameter. A point \( x \in B \) is said to be a diametral point of \( B \) if \( \sup_{y \in B} \| x - y \| = \delta(B) \).

A convex set \( S \) of \( X \) is said to have normal structure if every bounded convex subset \( S_1 \) of \( S \) which contains more than one point, has a point that is not a diametral point of \( S_1 \).

Lemma 1.1. [8, Lemma 1.5] A Banach operator \( T: X \to X \), where \( X \) is a complete metric space, has a fixed point.

Lemma 1.2. [10, Theorem 1.3] Let \( C \) a non-empty weakly compact convex subset of a normed linear space, possess normal structure. If \( T: C \to C \) satisfies \( \| Tx - Ty \| \leq \frac{1}{2} [ \| x - Tx \| + \| y - Ty \| ] \) for all \( x, y \in C \), then it has a fixed point.

2. Results.

Ray [9] proved the following theorem.

Theorem [Ray]. Let \( K \) be a nonempty, bounded closed and convex subset of a reflexive Banach space \( X \) and let \( K \) have normal structure.
Let $T$ be a mapping of $K$ into itself, such that

\[(A) \quad \| Tx - Ty \| \leq \alpha \| x - y \| + \beta \| x - Tx \| + \| y - Ty \| + \gamma [ \| x - Ty \| + \| y - Tx \| ] \]

for all $x, y \in K$ and for some $\alpha, \beta, \gamma \in \mathbb{R}^+$ (nonnegative reals) with $3\alpha + 2\beta + 4\gamma \leq 1$, then $T$ has a unique fixed point.

**Lemma 2.1** If $\beta > 3, \gamma > 0$ and $T$ satisfies $A$, then there exists a constant $k, 0 < k < 1$, such that $\| Tx - T^2x \| \leq k \| x - Tx \|$.

**Proof.** Setting $y = Tx$ in ($A$) we have

\[
\| Tx - T^2x \| \leq \alpha \| x - Tx \| + \beta [ \| x - Tx \| + \| Tx - T^2x \| ] + \gamma [ \| x - T^2x \| + \| Tx - Tx \| ]
\]

\[
\leq (\alpha + \beta + \gamma) \| x - Tx \| + (\beta + \gamma) \| Tx - T^2x \| .
\]

Thus $[1 - (\beta + \gamma)] \| Tx - T^2x \| \leq [\alpha + \beta + \gamma] \| x - Tx \|$

or $\| Tx - T^2x \| \leq \frac{[\alpha + \beta + \gamma]}{1 - (\beta + \gamma)} \| x - Tx \|$

Let $\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} = k$. If $k \geq 1$, then $\alpha + \beta + \gamma \geq 1 - \beta - \gamma$, or $\alpha + 2\beta + 2\gamma \geq 1$, and hence $3\alpha + 2\beta + 4\gamma \geq 1 + 2\alpha + 2\gamma > 1$, contradiction.

**Remark 2.1** If $\beta > 0, \gamma > 0$, it follows from Lemma 2.1 that $T$ is a Banach operator and has a fixed point.

In view of Lemma 2.1 it is enough to prove the theorem for the cases $\beta = 0$ and $\gamma = 0$ only.

**Case 1.** If $\beta = 0$ then ($A$) reduces to
\[ \| Tx - Ty \| \leq \alpha \| x - y \| + \gamma [ \| x - Ty \| + \| y - Tx \| ]. \]

**Lemma 2.2** If \( \gamma > 0 \), there exists a constant \( k, 0 < k < 1 \), such that \( \| Tx - T^2x \| \leq k \| x - Tx \| \).

**Proof.** As in Lemma 2.1 we have \( \| Tx - T^2x \| \leq \frac{\alpha + \gamma}{1 - \gamma} \| x - Tx \| \).

Let \( \frac{\alpha + \gamma}{1 - \gamma} = k, k < 1 \). Suppose not, \( k \geq 1 \) will imply \( \alpha + 2\gamma \geq 1 \), which in turn implies \( 3\alpha + 4\gamma \geq 1 + 2\alpha + 2\gamma > 1 \), a contradiction.

If \( \gamma > 0 \), the result follows from Lemma 1.1. If \( \gamma = 0, \alpha = 1/3 \), the result is a particular case of Banach's contraction principle.

**Case 2.** \( \gamma = 0 \). The case \( \beta = \gamma = 0 \) need not be considered, due to Banach's contraction principle, and \( \gamma = \alpha = 0 \) follows from Lemma 1.2. Thus we may assume \( \gamma = 0 \) and \( \alpha > 0, \beta > 0 \).

**Lemma 2.3** If \( \alpha > 0, \beta > 0 \), then there exists a \( k, 0 < k < 1 \), such that \( \| Tx - T^2x \| \leq k \| x - Tx \| \).

**Proof** \( \| Tx - T^2x \| \leq \alpha \| x - Tx \| + \beta [ \| x - Tx \| + \| Tx - T^2x \| ] \).

Thus \( \| Tx - T^2x \| \leq \frac{\alpha + \beta}{1 + \beta} \| x - Tx \| \). Let \( k = \frac{\alpha + \beta}{1 + \beta} \). Clearly \( k < 1 \). Suppose not, that is \( k \geq 1 \), then \( \alpha + 2\beta \geq 1 \) implies \( 3\alpha + 2\beta > 1 + 2\alpha > 1 \), a contradiction. Thus in this case \( T \) is a Banach operator and has a fixed point.

**Remark 2.2.** Result of Kirk is not a special case of [9] as remarked on page 904. In fact \( 3\alpha = 1 \) implies \( \alpha = 1/3 \), the result of Ray becomes a particular case of Banach's contraction principle for \( \beta = \gamma = 0 \).

The following is proved in [4].

**Theorem B** [4, Theorem 3]. Let \( E \) be a rotund Banach space,
\( M \) be a compact convex subset of \( E \) and let \( T \) be a self-mapping of \( M \).

If \( T \) is continuous and satisfies
\[
\| TT_\lambda x - TT_\lambda y \| \leq a_1 \| x - y \| + a_2 \| x - TT_\lambda x \| + a_3 \| y - TT_\lambda y \|
\]
\[
+ a_5 \| x - TT_\lambda y \| + a_6 \| y - TT_\lambda x \|
\]
for all \( x, y \in M \), where \( a_i (i = 1, 2, \ldots, 5) \) are non-negative real numbers such that
\[
\sum_{i=1}^{5} a_i = 5 \quad \text{and} \quad T_\lambda x = \lambda x + (1 - \lambda) Tx, \quad 0 < \lambda \leq 1,
\]
and if \( T \) has a fixed point, then \( T \) has a fixed point.

\textbf{Remark 2.3.} The condition (C) and rotundity of the space \( E \) is unnecessary in Theorem B. Since \( T \) is a continuous self-map of a compact convex subset of a Banach space, \( T \) has a fixed point by Schauder's fixed point theorem. In fact \( F(T) \) and \( F(T_\lambda) \) are the same, where \( F(T) = \{ x \in M : T(x) = x \} \). Indeed let \( y \in F(T) \), then \( Ty = y \).

Now \( T_\lambda (y) = \lambda y + (1 - \lambda) y = y \), thus \( y \in F(T_\lambda) \), hence \( F(T) \subseteq F(T_\lambda) \).

Conversely suppose \( z \in F(T_\lambda) \), then \( T_\lambda z = z \), which in turn implies \( z = \lambda z + (1 - \lambda) Tz \) which implies \( z = Tz \). Thus \( F(T_\lambda) \subseteq F(T) \).

\textbf{Remark 2.4.} Theorem 1 [1, p. 91] can be extended to the following type of mapping: There exists a constant \( k, 0 \leq k < 1 \), such that
\[
d(TT_\lambda x, TT_\lambda y) \leq k \max \{ d(x, y), d(x, TT_\lambda x), d(y, TT_\lambda y)\}
\]
\[
d(x, TT_\lambda y), d(y, TT_\lambda x)\}
\]
for all \( x, y \in X \), where \( X \) and \( T_\lambda x, T_\lambda y \) are as in [1].

In fact the result is true for any contractive type mapping [11] for which the fixed point is unique.

\textbf{Remark 2.5.} The claim of lines \( B_1, B_2 \) [1, p. 89] is incorrect.
Let $X = \{a, b\}$ with the usual norm.

Define $T: X \to X$ by $Ta = b$ and $Tb = a$, $X$ is compact $T$ is continuous, but $T$ is without fixed point.

**Remark 2.6.** The statement line $T_1$ [1, p. 90] is incorrect. Indeed, let $X$ be any normed linear space. Let $B = \{x \in X: \|x\| < 1\}$. Clearly $B$ is convex. Define $T: B \to B$ by $T(x) = x/2 + a/2$, where $a \neq 0$ and $\|a\| = 1$. Clearly $T$ is nonexpansive and satisfies the conditions of $T_1$ [1, p. 90]. However, $T$ does not have a fixed point.

**Definition 2.1.** A metric space $X$ is said to be **metrically convex** if for any $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$ such that $d(x, z) + d(z, y) = d(x, y)$.

**Definition 2.2.** Let $K$ be a nonempty closed subset of a metric space $X$ and let $S, T$ be mappings of $K$ into $CB(X)$ (the set of nonempty closed bounded subsets of $X$). Then $(S, T)$ is said to be a **generalized contraction pair** of $K$ into $CB(X)$ if there exist nonnegative reals $\alpha, \beta, \gamma$ with $\alpha + 2\beta + 2\gamma < 1$ such that for any $x, y \in K$,

\[
H(Sx, Ty) \leq \alpha d(x, y) + \beta [D(x, Sx) + D(y, Ty)] + \\
\gamma [D(x, T)x) + D(y, Sx)].
\]

**Note**

For any nonempty subsets $A, B$ of $X$ we define

\[
D(A, B) = \inf \{d(a, b) : a \in A \text{ and } b \in B\}
\]

and

\[
H(A, B) = \max \{\sup \{D(a, B) : a \in A\}, \sup \{D(A, b) : b \in B\}\}.
\]

**Theorem D.** [6, Theorem 3, 1] Let $X$ be a complete and metri-
cally convex metric space, $K$ a nonempty closed subset of $X$. Let $(S, T)$ be a generalized contraction pair of $K$ into $CB(X)$.

If for any
\[
x \in \partial K, S(x) \subset K, T(x) \subset K \quad \text{and} \quad (a+\beta+\gamma)(1+\beta+\gamma)/(1-\beta-\gamma)^2 < 1,
\]
then there exists $z \in K$ such that $z \in S(z)$ and $z \in T(z)$.

**Theorem E** [12, Theorem 3.2]. Let $X$ be a complete, metrically convex metric space, $K$ be a closed subset of $X$, and $S, T$ be two mappings of $K$ into $CB(X)$ satisfying the boundary condition: $Sx \subset K, Tx \subset K, (V x \in \partial K)$. Suppose there exist nonnegative numbers $a_1, \ldots, a_5 (\Sigma a_i < 1)$ with
\[
a_i + a_j < \frac{1 - a_5}{3 + a_5}, \quad (i = 1, 2; j = 3, 4)
\]
and
\[
\ell_1(Sx, Ty) \leq a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Sx) + a_5 d(x, y)
\]
for all $x, y \in K$.

Then the fixed point set of each $S, T$ is nonempty and these two sets coincide.

**Lemma 2.4.** The condition
\[
\frac{(a + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1
\]
implies
\[
a_i + a_j < \frac{1 - a_5}{3 + a_5}, \quad (i = 1, 2; j = 3, 4).
\]

**Proof.** $\frac{(a + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1$ implies
\[
a + \beta + \gamma + a\beta + \beta^2 + \gamma \beta + a\gamma + \beta \gamma + \gamma^2 < 1 - 2\beta - 2\gamma + \beta^2 + \gamma^2 + 2\beta \gamma.
\]
Thus, \( \alpha + \beta + \gamma + 3\beta + 3\gamma < 1 \). Let \( \alpha = a_5, \beta = a_1 + a_2, \gamma = a_3 + a_4 \).

Then we have,

\[
\begin{align*}
\sum a_i & \leq \frac{1 - a_5}{3 + a_5}, \\
\text{or } a_5 + 3(a_1 + a_2) + a_5(a_1 + a_2) & < 1.
\end{align*}
\]

By the Lemma.

Using symmetric properties of a metric, it can be easily seen that Definition 2.2 is equivalent to the condition of Theorem E.

Remark 2.7. It follows from Lemma 2.4 that Theorem 3.1 [6] is a special case of Theorem 3.2 [12].

The following result extends Theorem 5, 6 and 7 of [5].

**Theorem 1.** Let \( S \) be a compact metric space and \( T \) be a continuous mapping of \( S \) into itself. Suppose there exists a family of functions \( F = \{ f_\alpha \}_{\alpha \in S} \) from \( [0, 1] \) into \( S \) such that for each \( \alpha \in S, f_\alpha(1) = \alpha \) and for a function \( \Phi: (0, 1) \to (0, 1), \quad d(f_{T_\alpha}(t), f_{T_\beta}(t)) \leq \| \Phi(t) \| \),
\[ \phi(t) \max \{ d(x, y), d(x, f_{T^n}(t)), d(y, f_{T^n}(t)), d(x, f_{T^n}(t)), d(y, f_{T^n}(t)) \} \]
for all \( x, y \in S \) and for all \( t \in [0,1] \). Further, assume that for \( t \to t_0 \) in \([0,1]\) and \( a \to a_0 \) in \( S \), \( f_\alpha(t) \to f_{a_0}(t_0) \) in \( S \), then \( T \) has a fixed point.

**Proof.** For each \( n = 1,2,3, \ldots \), let \( k_n = \frac{n}{n+1} \), and let
\[ T_n : S \to S \]
be defined by \( T_n(x) = f_{T^n}(k_n) \) for all \( x \in S \).

Since \( T(S) \subset S \) and \( 0 < k_n < 1 \), each \( T_n \) is well defined and maps \( S \) into \( S \). Now for each \( x, y \in S \) we have
\[ d(T_n x, T_n y) = d(f_{T^n}(k_n), f_{T^n}(k_n)) \leq \phi(k_n) \max \{ d(x, y), d(x, f_{T^n}(k_n)), d(y, f_{T^n}(k_n)), d(x, f_{T^n}(k_n)), d(y, f_{T^n}(k_n)) \} \]
\[ = \phi(k_n) \max \{ d(x, y), d(x, T_n(x)), d(y, T_n(y)), d(x, T_n(x)), d(y, T_n(x)) \}. \]

Thus each \( T_n \) satisfies Ciric's condition. Since \( S \) is compact (hence complete), it follows from Theorem 1 [2] each \( T_n \) has a unique fixed point \( x_n \in S \). Since \( S \) is compact, there is a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( x_{n_j} \to x \in S \). Now \( T_{n_j} x_{n_j} = x_{n_j} \to x \in S \).

Using the continuity of \( T \), we conclude that \( T x_{n_j} \to T x \).

Finally \( T_{n_j} x_{n_j} = f_{T^n_{n_j}}(k_{n_j}) \to f_{T^n}(1) = T x \). It follows that \( T x = x \).

**REFERENCES**


ON $|V, \lambda |_k$ SUMMABILITY OF ULTRASPHERICAL SERIES

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(Received: March 29, 1982; Revised [final]: November 15, 1984)

In the present paper, a new theorem on $|V, \lambda |_k$ factors of ultraspherical series has been proved. The work is motivated by the recent papers [1], [2], [5] and [6].

1. Let $\Sigma u_n$ be a given series with the sequence of partial sums $\{S_n\}$ and let $\lambda = \{\lambda_n\}$ be a monotonic non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_1 = 1$. The sequence to sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} S_v,$$

defines the generalized de la Vallée Poussin means of the sequence $\{S_n\}$ generated by $\lambda$.

The series $\Sigma u_n$ is said to be summable $|V, \lambda |_k$, if the sequence $\{V_n(\lambda)\}$ is of bounded variation, i.e.,

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda) - V_n(\lambda)| < \infty.$$

We say that the series $\Sigma u_n$ is summable $|V, \lambda |_k$, $k \geq 1$, if

$$\Sigma \lambda_n^{k-1} |V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty.$$
On taking $\lambda_n = n$, this summability reduces to $|C, 1|_k$ summability and for $k = 1$ this is the same as summability $|V, \lambda|$

Let $f(\theta, \phi)$ be a function defined for the range $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The ultraspherical series corresponding to $f(\theta, \phi)$ on the sphere $S$ is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \int_S \frac{f(\theta', \phi') P_n(\lambda)(\cos \omega) \sin \theta' d\theta' d\phi'}{[\sin^2 \theta' \sin^2 (\phi-\phi')]^{(1-2\lambda)/2}}$$

$$\equiv \sum_{n=0}^{\infty} u_n$$

where

$$\cos w = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi-\phi').$$

A generalized mean value of $f(\theta, \phi)$ on the sphere $S$ has been defined by Kogbetliantz [3] as follows:

$$f(w) = \frac{1}{2\pi (\sin w)^{2\lambda}} \int_{C_w} \frac{f(\theta', \phi') ds'}{[\sin^2 \theta' \sin^2 (\phi-\phi')]^{(1-2\lambda)/2}}$$

where the integral is taken around the small circle $C_w$ whose centre is $(\theta, \phi)$ on the sphere $S$ and whose curvilinear radius is $w$.

We write

$$\phi(w) = f(w)(\sin w)^{2\lambda-1}$$

$$\phi_p(v) = \frac{1}{\Gamma(p)} \int_0^v (x - t)^{(p-1)} \phi(t) dt, \quad p > 0$$

$$\Phi_{k, p}(x) = \frac{1}{\Gamma(kp-k+1)} \int_0^x (x - t)^{(kp-k-1)} \phi(t)^k dt, \quad p > 1 - (1/k)$$
Theorem 1. If $\phi(w)$ is of bounded variation in $(\gamma, \pi)$, where

$$\eta = \mu', \quad \frac{1 - \lambda}{\lambda} = \Delta, \quad 0 < \lambda \leq \frac{1}{2},$$

$\mu'$ is a large constant, and if

$$\int_0^t |\phi(u)|^\alpha \, du = 0 \{t^{1 + (a - 1)k} (\log^{a-1}/t)\}, \text{ as } t \to 0, \quad (1.2)$$

$$a = \frac{2\lambda + 1 - \Delta}{\Delta}, \quad \beta > 0, \quad k \text{ and } \mu \text{ are integers such that }$$

$$1 \leq k, \mu \leq \infty,$$

then the series

$$\sum \frac{\mu_n u_n(t)}{[\log^{\alpha} (n + 1)]^{\beta + \delta - \frac{1}{2}}}$$

at $t = x$ is summable $|V, \lambda |$, provided $\{\mu_n\}$ is a convex sequence such that

$$\sum n \frac{\mu_n (\log^{\alpha} n)^{1/2 - \delta}}{\lambda_n^{\delta - \frac{1}{2}}} < \infty \quad (|\delta| < \frac{1}{2}),$$

and

$$\sum n \frac{(\log^{\alpha} n)^{1/2 - \delta}}{\lambda_n} \Delta \mu_n < \infty.$$

Theorem 2. If $\phi(w)$ is of bounded variation in $(\gamma, \pi)$ where $\gamma = \mu/n^2$,

$\Delta$ is a positive real number less than unity satisfying
\[ \frac{1 - \lambda}{\lambda} > \Delta > \frac{1 + 2\lambda}{3 + \beta - \alpha}, \quad (0 \leq \alpha \leq \beta < 1) \]

\( \mu' \) is a large constant, and if

\[ \Phi_k, \alpha (t) = O \left( t^{1+\beta} (\log^\gamma 1/t) \right), \quad \text{as } t \to 0, \gamma \geq 0, \text{ integer}, \]

then the series

\[ \sum \frac{\mu_n u_n (t)}{[\log^\alpha (n+1)]^{\gamma+\beta-1/2}} \]

at \( t = x \) is summable \( |V, \lambda |_k \), provided \( \{\mu_n\} \) is a convex sequence such that (1.3) and (1.4) hold.

2. We require the following lemmas:

**Lemma 1.** The conditions (1.2) and (1.5) imply, respectively, that

\[ \int_0^t |\phi(u)| \, du = O \left( t^\alpha (\log^{\gamma+\beta} 1/t) \right) \] \hspace{1cm} (2.1)

and

\[ \Phi_k \lambda (t) = \Phi_k (t) = O \left( t^{1+\beta} (\log^\alpha 1/t) \right). \] \hspace{1cm} (2.2)

**Proof.** For \( k = 1 \) the lemma follows directly from (1.2) and (1.5). Using Hölder’s inequality we have, for \( 1 < k < \infty \),

\[ \int_0^t |\phi(u)| \, du \leq \left\{ \int_0^t |\phi(u)|^k \, du \right\}^{1/k} \left\{ \int_0^t \, du \right\}^{1-(1/k)} \]

\[ = O \left( t^\alpha (\log^\gamma 1/t) \right). \]

By a similar proof we obtain (2.2).

**Lemma 2.** If \( \{\mu_n\} \) is a convex sequence such that
\[ \sum \frac{n \mu_n \log n}{\lambda_n^2} < \infty, \]

then

\[ \sum_{n=1}^{m} \log^\alpha (n+1) \cdot \Delta \mu_n = O(1), \quad m \to \infty \quad (2.3) \]

\[ \sum_{n=1}^{m} n \log^\alpha (n+1) \cdot \Delta^2 \mu_n = O(1), \quad m \to \infty. \quad (2.4) \]

**Proof.** The convergence of \( \sum \frac{n \mu_n \log n}{\lambda_n^2} \) implies the convergence of \( \sum \frac{\mu_n \log n}{n} \), and the latter implies \( \mu_n \log n \to 0 \) and \( n \log n \Delta n \to 0 \).

Now using Abel's transformation, we have

\[ \sum_{n=1}^{m} \frac{\mu_n}{n \log^\alpha (n+1)} = \sum_{n=1}^{m} \left[ \sum_{r=1}^{n} \frac{1}{r} \right] (\Delta \mu_n) + \left[ \sum_{r=1}^{m} \frac{1}{r} \right] u_m \]

which implies

\[ \sum_{n=1}^{m} \log n \Delta \mu_n = O(1). \]

Again, by Abel's transformation, we have

\[ \sum_{n=1}^{m} \log n \Delta^2 \mu_n = \sum_{n=1}^{m} \left[ \sum_{r=1}^{n} \log r \right] \Delta^2 \mu_n + \left[ \sum_{r=1}^{m} \log r \right] \Delta \mu_m \]

which implies

\[ \sum_{n=1}^{m} n \log n \Delta^2 \mu_n = O(1). \]
3. Proof of Theorem 1. Let \( S_n \) be the \( n^{th} \) partial sum of the series (1.1). Then we have (Szegö [4], p. 84)

\[
S_n = \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \int_0^\pi f(w) \sum_{k=0}^n (k+\lambda)P_k^{(\lambda)}(\cos w) (\sin w)^{2\lambda} \, dw
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \int_0^\pi f(w) \frac{d}{dx}\left\{P_{n+1}^{(\lambda)}(x) + P_n^{(\lambda)}(x)\right\} (\sin w)^{2\lambda} \, dw
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \int_0^\pi \phi(w) \frac{d}{dw}\left\{P_{n+1}^{(\lambda)}(\cos w) + P_n^{(\lambda)}(\cos w)\right\} \, dw
\]

\[
= S_n^1 + S_n^2, \text{ say.}
\]

\[
S_n^1 = \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \left\{\int_0^\pi \int_0^\eta \phi(w) \, dw\right\} = J_1 + J_2, \text{ say.}
\]

\[
J_1 = O \left(n^{2\lambda+1}\right) \int_0^\eta \phi(w) \, dw = O \left(n^{2\lambda+1-\Delta (\lambda+1)} (\log n)^{\theta/\kappa}\right)
\]

\[
= O \left( (\log n)^{\theta/\kappa}\right).
\]

\[
J_2 = \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \int_0^\pi \phi(w) \frac{d}{dw}\left\{P_{n+1}^{(\lambda)}(\cos w)\right\} \, dw
\]

\[
= A \left\{\int_{\eta}^{\pi/2} \int_{\pi/2}^{\pi-\eta} \int_{\pi-\eta}^{\pi} \phi(w) \frac{d}{dw}\left\{P_{n+1}^{(\lambda)}(\cos w)\right\} \, dw
\]

\[
= J_{2,1} + J_{2,2} + J_{2,3}, \text{ say.}
\]

\[
J_{2,1} = A \left[\phi(w) P_{n+1}^{(\lambda)}(\cos w)\right]_{\pi/2}^{\pi/2} = A \int_\eta^{\pi/2} P_{n+1}^{(\lambda)}(\cos w) \, d\phi(w)
\]

\[
= O \left(n^{\lambda-1+\Delta\lambda}\right) = O \left(1\right).
\]
\[ J_{2, 2} = A \left[ \phi \left( w \right) P_{n+1}^{(\lambda)} \left( \cos w \right) \right]_{\pi/2}^{\pi-\eta} - A \int_{\pi/2}^{\pi-\eta} P_{n+1}^{(\lambda)} \left( \cos w \right) d\phi \left( w \right) \]

\[ = A \left[ \phi \left( \pi - w \right) \left( -1 \right)^{n+1} P_{n+1}^{(\lambda)} \left( \cos w \right) \right]_{\pi/2}^{\pi-\eta} + \]

\[ (-1)^{n} A \int_{\pi/2}^{\eta} P_{n+1}^{(\lambda)} \left( \cos w \right) d\phi \left( \pi - w \right) = O(n^{\lambda-1+\Delta\lambda}) = O(1). \]

\[ J_{2, 3} = A \left[ \phi \left( w \right) P_{n+1}^{(\lambda)} \left( \cos w \right) \right]_{\pi}^{\pi-\eta} - A \int_{\pi}^{\pi-\eta} P_{n+1}^{(\lambda)} \left( \cos w \right) d\phi \left( w \right) \]

\[ = O \left( n^{2\lambda-1} \right) = O(1). \]

Therefore

\[ J_2 = O(1) \]

and

\[ S_n^1 = O \left\{ \left( \log n \right)^{B/\alpha} \right\}. \]

Other parts of the sum are treated similarly, and consequently we have

\[ S_n = O \left\{ \left( \log n \right)^{B/\alpha} \right\} \]

and

\[ \sum_{v=0}^{n} \left| S_v \left( x \right) \right|^2 = O \left\{ n \left( \log n \right)^{B/\alpha} \right\}. \]

Now let

\[ T_n \left( x \right) = \frac{1}{n+1} \sum_{v=1}^{n} y u_v \left( x \right). \]
By Abel's transformation, we have
\[ T_n(x) = S_n(x) - \frac{1}{n+1} \sum_{\nu=0}^{n} S_{\nu}(x) = O\left( (\log n)^{\beta/\varepsilon}\right), \]
which implies
\[ \sum_{\nu=1}^{n} |T_{\nu}(x)|^k = O\left( n(\log n)^\beta \right). \]

Let \( C_n = V_{n+1}(\lambda; x) - V_n(\lambda; x) \), where \( V_n(\lambda; x) \) is the \( n^{th} \) de la Vallée Poussin mean of the series
\[ \sum \frac{\mu_n \, u_n(x)}{[\log^* (n+1)]^{\beta+\varepsilon-1/2}}. \]

By an easy computation, we have
\[ C_n = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{\nu=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(\nu - n - 1) + \lambda_n \} \]
\[ \frac{\mu_n \, u_n(x)}{[\log^* (n+1)]^{\beta+\varepsilon-3/3}}. \]

Therefore, in order to prove the theorem, it is sufficient to show that
\[ \sum_{n=1}^{\infty} \lambda_n^{k-1} |C_n|^k < \infty. \]

(i) Let \( \sum \) be the summation over all \( n \) satisfying \( \lambda_{n+1} = \lambda_n \) and \( \sum \)

(ii) be the summation over all \( n \), where \( \lambda_{n+1} > \lambda_n \).

When \( \lambda_{n+1} = \lambda_n \), Abel's transformation gives that
By Minkowski's inequality it is therefore sufficient to prove that

\[
\sum_{n \geq 1} \lambda_n^{k-1} \mid L_n \mid^k \quad < \infty \quad \text{for} \quad r = 1, 2, 3.
\]

Now

\[
\sum_{n \geq 1} \lambda_n^{k-1} \mid L_n \mid^k = O(1) \sum_{n \geq 1} \lambda_n \left[ \sum_{v = n - \lambda_n + 2}^{n} v \mid T_v(x) \mid \right]^k \Delta \left\{ \frac{\mu_v}{v[\log^\rho(v + 1)]^{\beta + \varepsilon - 1/2}} \right\}.
\]

Using Abel's transformation again, by (3.1), we easily have
(3.2)

by (2.3) and (2.4); and hypothesis (1.3).

Further, applying Abel’s transformation and (3.1), it is easy to see that

\[
(3.3)
\]
by hypothesis (1.3) and (1.4).

Now, in order to estimate \( \sum_{n}^{(ii)} \) we have, with the aid of Abel's transformation,

\[
|C_n| \leq \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{v=n-\lambda_n+2}^{n} v \ | T_v(x) | \right|
\]

\[
\left| \Delta \left\{ (\lambda_n + v - n - 1) \frac{\mu_v}{v \log^\theta (v + 1) \beta + \delta - 1/2} \right\} \right|
\]

\[
+ (n-\lambda_n+1) |T_{n-\lambda_n+1}(x)| \frac{\mu_n-\lambda_n+2}{(n-\lambda_n+2)[\log^\theta(n-\lambda_n+3)]^{\beta+\delta-1/2}}
\]

\[
+ (n+1) |T_{n+1}(x)| \frac{\lambda_n \mu_{n+1}}{(n+1)[\log^\theta(n+2)]^{\beta+\delta-1/2}}
\]

\[= M_1^n + M_2^n + M_3^n, \text{ say.} \]

By Minkowski's inequality, it is therefore sufficient to prove that

\[
\sum_{n}^{(ii)} \lambda_n^{k-1} |M_n^r|^k < \infty \text{ for } r = 1, 2, 3.
\]

Now

\[
\sum_{n}^{(ii)} \lambda_n^{k-1} |M_n^r|^k \leq \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \left[ \sum_{v=n-\lambda_n+1}^{n} v \ | T_v(x) | \right]
\]

\[
\left\{ \lambda_v \Delta \left( \frac{\mu_v}{v \log^\theta (v + 1) \beta + \delta - 1/2} \right) + \frac{\mu_v}{v \log^\theta (v + 1) \beta + \delta - 1/2} \right\}^{-k}
\]

\[
\leq \left( \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \left[ \sum_{v=n-\lambda_n+2}^{n} v \ | T_v(x) | \lambda_v \Delta \right] \right)^k
\]
\[
\left( \frac{\mu_v}{v[\log^*(v+1)]^{\beta+\delta-1/2}} \right)^k 1/k
\]

\[
+ \left[ \sum \frac{1}{n} \frac{\lambda_{n+1}^{k+1}}{\lambda_n^k} \left\{ \sum_{v=n-\lambda_n+2}^{n} \lambda_v \mid T_v(x) \right\} \right]^{1/k} \left[ \lambda_v [\log^*(v+1)]^{\beta+\delta-1/2} \right]^{k^{1/k}} = \left( N_1^{1/k} + N_2^{1/k} \right)^k, \text{ say.}
\]

We observe that

\[
N_1 = O(1) \sum_{v=1}^{\infty} v \mid T_v(x) \mid^k \lambda_v^k \triangleq \left\{ \frac{\mu_v}{v[\log^*(v+1)]^{\beta+\delta-1/2}} \right\}
\]

\[
= O(1) \sum_{v=1}^{\infty} v \mid T_v(x) \mid^k \lambda_v^k \triangleq \left\{ \frac{\mu_v}{v[\log^*(v+1)]^{\beta+\delta-1/2}} \right\}.
\]

And similarly

\[
N_2 = O(1) \sum_{n} \frac{1}{\lambda_n^k+1} \left\{ \sum_{v=n-\lambda_n+2}^{n} \lambda_v \mid T_v(x) \mid^k \lambda_v^{k-1} \mu_v \right\} \frac{\log^*(v+1)^{\beta+\delta-1/2}}{\lambda_v^{k+1}}.
\]
\[
= O(1) \sum_{v=1}^{\infty} \frac{\left| T_{v}(x) \right|^k \mu_v}{\lambda_v \left[ \log^\mu (v+1) \right]^{\alpha+\delta-1/2}} = O(1), \text{ by (3.3).}
\]

Therefore

\[(ii) \quad \sum_{n} \lambda_n^k - 1 \left| M_n^1 \right|^k = O(1).\]

Finally,

\[(ii) \quad \sum_{n} \lambda_n^k - 1 \left| M_n^2 \right|^k + (ii) \sum_{n} \lambda_n^k - 1 \left| M_n^3 \right|^k\]

\[= O \sum_{n=1}^{\infty} \frac{\left| T_n(x) \right|^k \mu_n}{\lambda_n \left[ \log^\mu (n+1) \right]^{\alpha+\delta-1/2}} = O(1), \text{ by (3.3).} \]

4. **Proof of Theorem 2.** To prove this theorem it is sufficient to calculate the order of the integral

\[I = \int_{0}^{\eta} f(w) \left\{ \frac{d}{dw} P_{n+1}^{(\lambda)} (\cos w) \right\} (\sin w)^{2\lambda-1} dw\]

under the hypothesis of the theorem.

We have

\[I = \int_{0}^{\eta} \phi(w) \left\{ \frac{d}{dw} P_{n+1}^{(\lambda)} (\cos w) \right\} dw\]

\[= \left[ 2\lambda \sin w \cdot P_{n+1}^{(\lambda+1)} (\cos w) \Phi_1(w) \right]_0^{\eta}\]

\[- 2\lambda \int_{0}^{\eta} \Phi_1(w) \frac{d}{dw} \left\{ \sin w P_{n}^{(\lambda+1)} (\cos w) \right\} dw\]

\[= I_1 - I_2, \text{ say.}\]
But,

\[ I_1 = O \left\{ n^{2\lambda+1} \eta \eta^{2+\beta-\alpha} \left( \log^\alpha 1/\eta \right)^{\gamma/k} \right\} = O \left\{ (\log^\alpha n)^{\gamma/k} \right\} . \]

\[ I_2 = 2\lambda \int_0^\eta \frac{d}{dw} \left\{ \sin w P_n^{(\lambda+1)}(\cos w) \right\} \]

\[ = \frac{2\lambda}{\Gamma(1-\beta)} \int_0^\eta \Phi_\alpha(u) \left( \int_0^{u} \frac{d}{dw} \left\{ \sin w P_n^{(\lambda+1)}(\cos w) \right\} \right) (w-u)^{-\alpha} \sin w \] \[ dw \]

\[ = I_2, \eta \leq I_2, 2, \text{ say}. \]

We approximate \( F(\eta, u) \) for the first integral \( I_2, \eta \leq I_2, 2 \), when \( u \) is ranging from 0 to 1/n

\[ F(\eta, u) = \int_0^\eta (w-u)^{-\alpha} \frac{d}{dw} \left\{ \sin w P_n^{(\lambda+1)}(\cos w) \right\} dw \]

\[ = \left\{ \int_0^{2u} + \int_{2u}^\eta \right\} (w-u)^{-\alpha} \frac{d}{dw} \left\{ \sin w P_n^{(\lambda+1)}(\cos w) \right\} dw \]

\[ = \int_0^{2u} \left( w-u \right)^{-\alpha} \cos w P_n^{(\lambda+1)}(\cos w) \]

\[ + \sin w \frac{d}{dw} \left[ P_n^{(\lambda+1)}(\cos w) \right] dw + \]

\[ + O u^{-\alpha} \int_{2u}^{\xi} \frac{d}{dw} \left[ \sin w P_n^{(\lambda+1)}(\cos w) \right] dw \quad 2u \leq \xi \leq \eta \]
\[= O \left(n^{2\lambda+1}\right) u^{1-\alpha} + O \left(n^{2\lambda+3}\right) u^{3-\alpha} + O \left(n\right) \left[ \sin w \, P_{n}^{(\lambda+1)}(\cos w) \right] 2u \]

\[= O \left(n^{2\lambda+1}\right) u^{1-\alpha} + O \left(n^{2\lambda+3}\right) u^{3-\alpha} + O \left(n^{2\lambda+1-\delta}\right) u^{-\alpha}.\]

Again we have the following order estimate for \(F(\eta, u)\) when \(u\) ranges from \(1/n\) to \(\eta\).

\[F(\eta, u) = \left\{ \int_{u}^{\eta} (w-u)^{-\alpha} \right\}_{u+(1/n)} \]

\[\frac{d}{dw} \left\{ \sin w \, P_{n}^{(\lambda+1)}(\cos w) \right\} dw = O(n^{\alpha-1}) u^{-\lambda-1} + u^{2-\lambda-3} n^{\lambda+1} O(n^{\alpha-1}) + \]

\[+ O(n^{\alpha}) \max \left[ \sin w \, P_{n}^{(\lambda+1)}(\cos w) \right]_{u+(1/n)} \]

\[= O \left(n^{\lambda+\alpha-1}\right) u^{-\lambda-1} + O \left(n^{\lambda+\alpha}\right) u^{-\lambda} + O(n^{\alpha+\lambda}) u^{-\lambda-1}.\]

If \(u + (1/n)\) is not less than \(\eta\) we do not need to break the integral into two parts.

Now

\[I_{2, 1} = \int_{0}^{1/n} O \left(u^{2+\beta-\alpha}\right) \left(\log^a 1/u\right)^{\gamma/\beta} n^{2\lambda+1} \, du + \]

\[+ \int_{0}^{1/n} O \left(u^{4+\beta-\alpha}\right) \left(\log^a 1/u\right)^{\gamma/\beta} n^{2\lambda+3} \, du \]

\[+ \int_{0}^{1/n} O \left(u^{1+\beta-\alpha}\right) \left(\log^a 1/u\right)^{\gamma/\beta} n^{2\lambda+1-\delta} \, du \]
\[
\begin{align*}
&= O\left\{ n^{-3+2\lambda(-\alpha)} + 2\lambda + 1 \left( \log^\mu n \right) \gamma/k \right\} + O\left\{ n^{-(5\lambda+2\alpha)^2 + 3 + 2\lambda} \left( \log^\mu n \right) \gamma/k \right\} \\
&\quad + O\left\{ n^{-(2+2\alpha)} + 2\lambda + 1 - \Delta \left( \log^\mu n \right) \lambda/k \right\} \\
&= O\left\{ n^{-3+2\lambda(-\alpha)} + 2\lambda + 1 \left( \log^\mu n \right) \gamma/k \right\} + \\
&\quad + O\left\{ n^{2\lambda+1-\Delta(3+\beta-\alpha) + \Delta(2+\beta-\alpha) - (2+\beta-\alpha)} \left( \log^\mu n \right) \gamma/k \right\} \\
&= O\left\{ (\log^\mu n) \gamma/k \right\},
\end{align*}
\]

and

\[
I_{2,2} = O\left\{ n^{\lambda+\alpha-1} \int_{1/n}^{\eta} u^{\beta-\lambda} \left( \log^\mu 1/u \right) \lambda/k \ du \right\} + \\
\quad + O\left\{ n^{\lambda+\alpha} \int_{1/n}^{\eta} u^{1+\beta-\lambda} \left( \log^\mu 1/u \right) \gamma/k \ du \right\} + \\
\quad + O\left\{ n^{\alpha+\lambda-\Delta} \int_{1/n}^{\eta} u^{1+\beta-\lambda-1} \left( \log^\mu 1/u \right) \gamma/k \ du \right\} \\
= O\left\{ n^{\lambda+\alpha-1-\Delta(1+\beta-\lambda)} \left( \log^\mu n \right) \gamma/k \right\} + \\
\quad + O\left\{ n^{2\lambda+1-\Delta(3+\beta-\alpha) - (1+\lambda-\alpha) + \Delta(1+\lambda-\alpha)} \left( \log^\mu n \right) \gamma/k \right\} + \\
\quad + O\left\{ n^{2\lambda+1-\Delta(3+\beta-\alpha) - 1-\lambda+\alpha-\Delta(-1+\omega+\lambda)} \left( \log^\mu n \right) \gamma/k \right\} \\
= O\left\{ (\log^\mu n) \gamma/k \right\},
\]

and therefore

\[
I = O\left\{ (\log^\mu n) \gamma/k \right\}.
\]

Now, using steps parallel to those used in the proof of Theorem 1, we see that the proof is complete.
REFERENCES


ON STRONG APPROXIMATION TO A FUNCTION BY ITS JACOBI SERIES

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(Received: December 21, 1982; Revised [final]: November 15, 1984)

1. DEFINITIONS

An infinite series $\sum u_n$ is said to be strongly summable with index $k > 0$ to the sum $S$, if

$$\sum_{r=0}^{n} | S_r - S|^k = O(n), \ n \to \infty$$

where $S_r$ denotes the $r^th$ partial sum of the series $\sum u_n$.

Let $f(x)$ be a function defined on the interval $-1 \leq x \leq 1$, such that the integral

$$\int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} f(x) \, dx$$

exists in the sense of Lebesgue. The Fourier–Jacobi expansion, generally known as Jacobi series corresponding to the function $f(x)$ given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \quad (1.1)$$
where \( P_n^{(a,\beta)}(x) \) denotes the Jacobi polynomial of order \((a,\beta)\) and degree \(n\) in \(x\),

\[
a_n = \frac{1}{g_n} \int_{-1}^{1} (1-x)^a (1+x)^\beta f(x) P_n^{(a,\beta)}(x)
\]

and

\[
g_n = \frac{2^{2a+\beta+1}}{2^{2n+\alpha+\beta+1}} \frac{\Gamma(n+a+1) \Gamma(n+\beta+1)}{\Gamma(n+1) \Gamma(n+\alpha+\beta+1)}.
\]

Ultraspherical and Legendre series are particular cases of the series (1.1) when \(\alpha = \beta = \lambda - \frac{1}{2}\) and \(\alpha = \beta = 0\), respectively.

We write

\[
\psi(w) = f(\cos w) - f(1),
\]

and establish the following result:

**Theorem.** If \(\alpha > -1\), \(\beta > -1\), \(\alpha + \beta > 0\), and if \(\Psi(t)\) is a positive increasing function such that

\[
\int_0^t |d\phi(u)| \leq A \Psi(t), \quad A \text{ being a constant,}
\]

\[
\int_t^\infty w^{\alpha+1/2} \Psi(w) \, dw = O \{t^{\alpha+1/2} \Psi(t)\}, \quad (1.2)
\]

and

\[
\Phi(t) = \int_t^\infty w^{\beta+1/2} \phi(\pi - w) \, dw = O \{t^{\beta+1/2} \Psi(t)\}, \quad (1.3)
\]

then
\[
\frac{1}{n+1} \sum_{r=0}^{n} |S_r(1) - f(1)| = O(\Psi(1/n)).
\]

2. LEMMAS

We require the following lemmas for the proof of the theorem.

**Lemma 1 (Szegö [2]).** Let \( \alpha, \beta \) be arbitrary real numbers and \( C \) a fixed positive constant. Then, for \( n \to \infty \),

\[
P_n^{(\alpha, \beta)}(\cos \theta) = \begin{cases} 
\theta^{\alpha-1/2} O(n^{-1/2}), & C/n \leq \theta \leq \pi/2 \\
O(n^\alpha), & 0 \leq \theta \leq C/n
\end{cases}
\]

**Lemma 2 (Szegö [2]).** If \( \alpha > -1, \beta > -1 \), then

\[
P_n^{(\alpha, \beta)}(\cos \theta) = n^{-1/2} K(\theta) (\cos (N\theta + \gamma)) + (n \sin \theta)^{-1}. O(1)
\]

where

\[
K(\theta) = \pi^{-1/2} (\sin \theta/2)^{-\alpha-1/2} (\cos \theta/2)^{-\beta-1/2}
\]

\[
N = n + \frac{1}{2} (\alpha + \beta + 1), \quad \gamma = - (\alpha + \frac{1}{2}) \pi/2.
\]

**Lemma 3.** The condition (1.3) implies

\[
\int_0^\pi w^{2\beta} |\phi(\pi - w)| \, dw = O(t^{x+\alpha}) = O(t^{-1/2}).
\]  \hspace{1cm} (2.1)

Proof of Lemma 3 is fairly straightforward, and we omit the details.

3. PROOF OF THE THEOREM

Let \( S_n \) be the \( n^{th} \) partial sum of the series (1.1). Hence we have

\[
S_n(1) = \int_0^\pi (1 - \cos w)^{\alpha+1/2} (1 + \cos w)^{\beta+1/2} f(\cos w).
\]
\[
\sum_{m=0}^{n} \frac{1}{g_m} P_m^{(\alpha, \beta)}(\cos w) P_m^{(\alpha, \beta)}(1) \, dw
\]

\[
S_n(1) - f(1) = \int_{0}^{\pi} (\sin \frac{w}{2})^{2\alpha+1}(\cos \frac{w}{2})^{2\beta+1} \phi(w),
\]

\[
\frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} P_n^{(\alpha+1,\beta)}(\cos w) \, dw
\]

\[
= \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)}.
\]

\[
\cdot \left\{ \int \frac{1}{n} \phi(w) \right\} \delta + \int_{1/n}^{\pi} \left\{ \frac{\pi}{n-1/n} \phi(w) \right\} \delta
\]

\[
= J_1 + J_2 + J_3 + J_4, \text{ say.}
\]

\[
J_1 = O (n^{2\alpha+2}) \int_{0}^{1/n} (\sin \frac{w}{2})^{2\alpha+1}(\cos \frac{w}{2})^{2\beta+1} \phi(w) \, dw
\]

As \( \phi(0) = 0 \), we see that

\[
| \phi(w) | \leq \int_{0}^{w} d | \phi(u) | \leq A \Psi(w).
\]

Therefore

\[
J_1 = O (n^{2\alpha+2}) \int_{0}^{1/n} (\sin \frac{w}{2})^{2\alpha+1}(\cos \frac{w}{2})^{2\beta+1} \Psi(w) \, dw = O \{ \Psi(1/n) \}.
\]

\[
J_2 = O (n^{\alpha+1}) \int_{1/n}^{\delta} (\sin \frac{w}{2})^{2\alpha+1}(\cos \frac{w}{2})^{2\beta+1} \phi(w) \, [O(n^{-1/2})].
\]
\[
\sum_{m=0}^{n} \frac{1}{g_m} P^{(\alpha, \beta)}_m (\cos w) \, P^{(\alpha, \beta)}_m (1) \, dw
\]

\[
S_n(1) - f(1) = \int_0^\pi (\sin w/2)^{2\alpha+1}(\cos w/2)^{2\beta+1} \phi(w),
\]

\[
\frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} \quad p_n^{(\alpha+1, \beta)} (\cos w) \, dw
\]

\[
= \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} \cdot
\]

\[
\left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\pi-1/n}^{\pi} \right\}
\]

\[
= J_1 + J_2 + J_3 + J_4, \text{ say.}
\]

\[
J_1 = O (n^{2\alpha+2}) \int_0^{1/n} (\sin w/2)^{2\alpha+1}(\cos w/2)^{2\beta+1} \phi(w) \, dw
\]

As \( \phi(0) = 0 \), we see that

\[
| \phi(w) | \leq \int_0^w d | \phi(u) | \leq A \Psi(w).
\]

Therefore

\[
J_1 = O (n^{2\alpha+2}) \int_0^{1/n} (\sin w/2)^{2\alpha+1}(\cos w/2)^{2\beta+1} \Psi(w) \, dw = O (\Psi(1/n)).
\]

\[
J_2 = O (n^{\alpha+1}) \int_{1/n}^{\delta} (\sin w/2)^{2\alpha+1}(\cos w/2)^{2\beta+1} \phi(w) \, [O(n^{-1/2})].
\]
\[
\begin{align*}
\cdot \{\sin \frac{w}{2}\}^{-\alpha - 3/2} \cdot (\cos \frac{w}{2})^{-\beta - 1/2} & (O(1) + n(\sin \frac{w}{2})^{-1})
\end{align*}
\]

\[
\begin{align*}
&= O \left( n^{\alpha + 1/2} \right) \int_{\delta}^{\pi} \left| w^{\alpha - 1/2} \phi(w) \right| dw + O \left( n^{\alpha - 1/2} \right) \int_{\delta}^{\pi} w^{\alpha - 3/2} \left| \phi(w) \right| dw \\
&= O \left( n^{\alpha + 1/2} \right) \int_{\delta}^{\pi} w^{\alpha - 1/2} \Psi(w) dw \\
&= O \{\Psi(1/n)\} \ \text{by (1.2)}
\end{align*}
\]

\[
J_3 = O \left( n^{\alpha + 1/2} \right) \int_{\delta}^{\pi} (\sin \frac{w}{2})^{\alpha - 1/2}(\cos \frac{w}{2})^{\beta + 1/2} \left| \phi(w) \right| dw
\]

\[
\begin{align*}
&= O \left( n^{\alpha + 1/2} \right) \int_{\delta}^{\pi} (\cos \frac{w}{2})^{\alpha - 1/2}(\sin \frac{w}{2})^{\beta + 1/2} \left| \phi(\pi - w) \right| dw \\
&= O \left( n^{\alpha + 1/2} \right), O \left( \frac{1}{n} \right)^{\alpha + 1/2} \Psi \left( \frac{1}{n} \right), \ \text{by (1.3)}
\end{align*}
\]

\[
J_4 = \lambda_n \int_{\delta}^{\pi} (\sin \frac{w}{2})^{2\alpha + 1}(\cos \frac{w}{2})^{2\beta + 1} \left| \phi(w) \right| P_n^{(\alpha + 1, \beta)}(\cos w) dw
\]

where

\[
\lambda_n = \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)}
\]

Therefore

\[
J_6 = (-1)^n \lambda_n \int_{\delta}^{\pi} (\cos \frac{w}{2})^{2\alpha + 1}(\sin \frac{w}{2})^{2\beta + 1} \phi(\pi - w).
\]
\[ P_n^{(\beta, \alpha+1)} \cos \omega \, dw. \]

Since \( P_n^{(\alpha, \beta)} (-x) = (-1)^n P_n^{(\beta, \alpha)} (x) \), it follows that

\[
J_4 = O \left( n^{\alpha+\beta+1} \right) \int_0^{1/n} w^{2\beta+1} | \phi (\pi - w) | \, dw
\]

\[
= O \left( n^{\alpha+\beta} \right) O \left( (1/n)^{\alpha+\beta} \right) \Psi(1/n), \quad \text{by (2.1)}
\]

\[
= O \{ \Psi(1/n) \}. \tag{3.4}
\]

In view of (3.1), (3.2), (3.3) and (3.4), we get

\[
| S_n(1) - f(1) | = O \{ \Psi(1/n) \}.
\]

Therefore

\[
\frac{1}{n+1} \sum_{r=0}^{n} | S_r(1) - f(1) | = O(\Psi(1/n))
\]

This completes the proof of the theorem.

For several analogous results, see the works of Singh [1] and Yadav [3].

REFERENCES


A NOTE ON A CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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ABSTRACT

It is the purpose of the present paper to prove some distortion inequalities for the derivative of functions belonging to a certain class of analytic functions considered earlier by K. S. Padmanabhan [3], and by S. Chandra and P. Singh [1].

1. INTRODUCTION

Let $A(\alpha)$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and satisfy the following condition

$$(2) \quad \left| \frac{f(z)}{z} - 1 \right| < \alpha \quad (z \in U)$$

* This research was completed at the University of Victoria (Victoria, British Columbia, Canada) while the author was on study leave from Kinki University.
for some $a$ ($0 < a \leq 1$).

This class $A (a)$ was studied by Padmanabhan [3], and by Chandra and Singh [1].

Now, we need the following lemma which is immediately obtained from Nehari's lemma [2].

**Lemma.** Let $\phi (z)$ be analytic in the unit disk $U$ and satisfy

$$| \phi (z) | \leq \alpha \text{ for } z \in U.$$

Then

$$| \phi' (z) | \leq \frac{1 - | \phi (z) |^2 / \alpha^2}{1 - | z |^2} \quad \text{for } z \in U,$$

where $0 < \alpha \leq 1$.

## 2. MAIN RESULTS

We now state and prove our results, which provide upper and lower bounds for the derivative of functions belonging to the class $A (\alpha)$.

**Theorem 1.** Let the function $f(z)$ defined by (1) belong to the class $A (\alpha)$. Then

$$| f' (z) | \leq 1 + 2\alpha | z | + \frac{\alpha | z |^2}{1 - | z |^2} \quad \text{and}$$

$$| f' (z) | \geq 1 - 2\alpha | z | - \frac{\alpha | z |^2}{1 - | z |^2}$$

for $0 < \alpha \leq 1$ and $z \in U$.

**Proof.** Let the function $g(z)$ be defined by
\[(6) \quad g(z) = \frac{f(z)}{z} - 1.\]

Then \(g(z)\) is analytic in the unit disk \(U\) and has simple zero at the origin. Hence we can write that

\[(7) \quad g(z) = \frac{f(z)}{z} - 1 = z\phi(z),\]

where \(\phi(z)\) is analytic in the unit disk \(U\) and satisfies \(|\phi(z)| < \alpha\) for \(z \in U\). Thus we have

\[(8) \quad f(z) = z + z^2\phi(z) ;\]

hence further,

\[(9) \quad f(z) = 1 + 2z\phi(z) + z^2\phi(z).\]

Consequently, by using the lemma, we obtain two inequalities (4) and (5) for \(0 < \alpha \leq 1\) and \(z \in U\).

Next we shall consider functions with initial zero coefficients.

**THEOREM 2.** Let the function defined by

\[(10) \quad f(z) = z + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N = \{1, 2, 3, \ldots\})\]

belong to the class \(A(\alpha)\). Then

\[(11) \quad |f(z)| \leq 1 + (p + 1)\alpha |z|^p + \frac{\alpha |z|^{p+1}}{1 - |z|^2}\]

and

\[(12) \quad |f'(z)| \geq 1 -(p+1) \alpha |z|^p - \frac{\alpha |z|^{p+1}}{1 - |z|^2}\]

for \(0 < \alpha \leq 1\) and \(z \in U\).
PROOF. Let the function \( h(z) \) be defined by

\[
(13) \quad h(z) = \frac{f(z)}{z} - 1.
\]

Then \( h(z) \) is analytic in the unit disk \( U \) and has \( p \) zeros at the origin. Therefore, we can write

\[
(14) \quad h(z) = \frac{f(z)}{z} - 1 = z^p \phi(z),
\]

where \( \phi(z) \) is analytic in the unit disk \( U \) and satisfies \( |\phi(z)| < \alpha \) for \( z \in U \). This gives two inequalities of the theorem with the aid of Lemma.

3. REMARK

We have not been able to obtain sharp estimates for \( |f'(z)| \) in our theorems.

REFERENCES


A NOTE ON THE RADII OF STARLIKENESS AND CONVEXITY

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1. INTRODUCTION AND DEFINITIONS

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$. We denote by $S$ the subclass of univalent functions $f(z)$ of $A$ and by $S^*$ and $K$ the subclasses of $S$ whose members are starlike with respect to the origin and convex in the unit disk $U$, respectively. A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ ($0 \leq \alpha < 1$) in the unit disk $U$ if and only if

$$Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for $z \in U$. Further, a function $f(z)$ in $A$ is said to be convex of order $\alpha$ ($0 \leq \alpha < 1$) in the unit disk $U$ if and only if

*This research was completed at the University of Victoria (Victoria, British Columbia, Canada) while the author was on study leave from Kinki University.
(1.3) \[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \]

for \( z \in U \). We denote by \( S^*(\alpha) \) and \( K(\alpha) \) the subclasses of \( A \) whose members satisfy (1.2) and (1.3), respectively. Then it is well known that \( S^*(\alpha) \subset S^*, K(\alpha) \subset K \) for \( 0 < \alpha < 1 \) and that \( S^*(0) \equiv S^*, K(0) \equiv K \) for \( \alpha = 1 \).

The classes \( S^*(\alpha) \) and \( K(\alpha) \) were first introduced by Robertson [9], and were latter studied by Merkes, Robertson and Scott [5], Schild [12], Bogowski, Jablonski and Stankiewicz [1] and Jack [3]. In particular, the class \( S^*(1/2) \) was studied by Schild [13] and MacGregor [4].

Now, several essentially equivalent definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals, have been given in the literature (cf., e.g., [2, Chapter 13], [6], [7], [10] [11], and [14, p. 28 et seq.]). We find it convenient to restrict ourselves to the following definitions used recently by Owa [8].

**DEFINITION 1.** The fractional integral of order \( \lambda \) is defined by

\[
D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\lambda}} \, d\zeta,
\]

where \( \lambda > 0 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \( (z - \zeta)^{-1} \) is removed by requiring \( \log (z - \zeta) \) to be real when \( (z - \zeta) > 0 \).

**DEFINITION 2.** The fractional derivative of order \( \lambda \) is defined by

\[
D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\lambda}} \, d\zeta,
\]
where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z - \zeta)^{-\lambda}$ is removed by requiring $\log (z - \zeta)$ to be real when $(z - \zeta) > 0$.

**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order $(n + \lambda)$ is defined by

\[
D^\lambda_{z^n} f(z) = \frac{d^n}{dz^n} D_z \lambda f(z),
\]

where $0 \leq \lambda < 1$, $n \in N \cup \{0\}$ and $N = \{1, 2, 3, \ldots\}$.

Recently, by using the fractional derivative $D_z \lambda f(z)$ of $f(z)$ of order $\lambda$, Srivastava and Owa [15] introduced the following class.

**Definition 4.** We say that $f(z)$ is in the class $P_{\lambda}^\star(\alpha, \beta)$ if $f(z)$ of $S$ defined by

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)
\]

satisfies the following condition

\[
\left| \frac{\Gamma(2 - \lambda) z^{\lambda - 1} D_z \lambda f(z) - 1}{\Gamma(2 - \lambda) z^{\lambda - 1} D_z \lambda f(z) + (1 - 2\alpha)} \right| < \beta \quad (z \in \mathbb{U})
\]

for $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

2. Radii of starlikeness and convexity

In this section, we determine the radii of starlikeness and convexity of functions $f(z)$ belonging to the class $P_{\lambda}^\star(\alpha, \beta)$. We first need the following lemma given by Srivastava and Owa [15].
Lemma. A function \( f(z) \) defined by (1.7) is in the class \( \mathcal{P}_{\lambda}^*(\alpha, \beta) \) if and only if
\[
(2.1) \quad \sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (1+\beta) a_n \leq 2\beta (1-\alpha).
\]

The result (2.1) is sharp.

Now we state and prove

Theorem 1. Let the function \( f(z) \) defined by (1.7) be in the class \( \mathcal{P}_{\lambda}^*(\alpha, \beta) \) for \( 0 \leq \lambda < 1, 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \). Then \( f(z) \) is starlike of order \( \gamma \) \((0 \leq \gamma < 1)\) in the disk \( |z| < r_0 \), where
\[
(2.2) \quad r_0 = \inf_{n \in \mathbb{N}} \left\{ \frac{(1+\beta)(1-\gamma)\Gamma(n+1)\Gamma(2-\lambda)}{2\beta(n-\gamma)(1-\alpha)\Gamma(n+1+\lambda)} \right\}^{1/(n-1)}
\]

with equality for the function \( f(z) \) given by
\[
(2.3) \quad f(z) = z - \frac{2\beta (1-\alpha) \Gamma(n+1-\lambda)}{(1+\beta) \Gamma(n+1) \Gamma(2-\lambda)} \quad z^n.
\]

Proof. It suffices to show that
\[
(2.4) \quad \left| \frac{zf''(z)}{f'(z)} - 1 \right| < 1 - \gamma
\]
for \( |z| < r_0 \). Now, we can observe that
\[
(2.5) \quad \left| \frac{zf(z)}{f'(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{(n-1) a_n |z| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}
\]
Consequently, in view of the lemma, we need only find values of $|z|$ for which

$$
(n - \gamma) \frac{a_n}{|z|^{n-1}} \leq \left( \frac{1 + \beta}{2n(1 - \alpha)} \right) \frac{\Gamma(n + 1) \Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} \quad (n \geq 2)
$$

which will be true when $|z| \leq r_0$. This completes the proof of the theorem.

**Theorem 2.** Let the function $f(z)$ defined by (1.7) be in the class $P_{\lambda \ast}(a, \beta)$ for $0 \leq \lambda < 1$, $0 \leq a < 1$ and $0 < \beta \leq 1$. Then $f(z)$ is convex of order $\gamma$ ($0 \leq \gamma < 1$) in the disk $|z| < r_1$, where

$$
r_1 = \inf_{n \in \mathbb{N} - \{1\}} \left\{ \frac{\left(1 + \beta\right) \Gamma(n + 1) \Gamma(2 - \lambda)}{2n(1 - \alpha) \Gamma(n + 1 - \lambda)} \right\}^{1/(n-1)}
$$

with equality for the function $f(z)$ given by

$$
f(z) = z - \frac{2\beta(1 - \alpha) \Gamma(n + 1 - \lambda)}{(1 + \beta) n \Gamma(n + 1) \Gamma(2 - \lambda)} |z|^n.
$$

**Proof.** Since $f(z) \in K(\gamma)$ if and only if $zf'(z) \in S(\gamma)$, we can see that the theorem follows that of Theorem 1. with $a_n$ replaced by $na_n$.

**Corollary.** Let the function $f(z)$ defined by (1.7) be in the class $P_{\lambda \ast}(a, \beta)$ for $0 \leq \lambda < 1$, $0 \leq a < 1$ and $0 < \beta \leq 1$. Then $f(z)$ in univalent and starlike for $|z| < r_2$, where

$$
r_2 = \inf_{n \in \mathbb{N} - \{1\}} \left\{ \frac{(1 + \beta) \Gamma(n) \Gamma(2 - \lambda)}{2\beta(1 - \alpha) \Gamma(n + 1 - \lambda)} \right\}^{1/(n-1)}
$$
Proof. By taking $\gamma = 0$ in Theorem 1, we have the above corollary.

Acknowledgements

The author wishes to express his thanks to Professor H. M. Srivastava, for his kind encouragement and helpful guidance.

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SOME REMARKS ON COMMON FIXED POINTS OF FOUR MAPPINGS

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ABSTRACT

In this paper, inspired by a recent result of Fisher [4], we point out a common fixed point theorem for four self mappings of a complete metric space, using a well known contractive condition of Meade and Singh [13] and the concept of weak commutativity of the second author [16]. Our theorem generalizes results of Chang [1], Imdad and Khan [8], Sessa and Fisher [17] and Singh and Singh [19].

1. INTRODUCTION

Let $R^+$ be the set of nonnegative reals and let $(X, d)$ be a complete metric space. Meade and Singh [13], improving a result of Husain and Sehgal [6], proved a common fixed point theorem for two self mappings of $(X, d)$ considering a real function $f : (R^+)^5 \to R^+$ satisfying the following properties:

(i) $f$ is upper semi-continuous

AMS (MOS) Subject Classification: Primary 54H25, Secondary 47H10.
(ii) \( f \) is non-decreasing in each coordinate variable.

(iii) \( f(t, t, at, bt, t) < t \) for any \( t > 0 \) where \( a \geq 0, b \geq 0 \) and \( a + b = 3 \).

Let \( F \) be the family of such functions \( f \). Many authors studied contractive conditions with functions \( f \in F \) or using functions with similar properties: for instance, see C. Chang [1], S. Chang [2], Danes [3], Guay, Singh Whitfield [5], Husain and Sehgal [7], Imdad and Khan [8], Imdad, Khan and Sessa [9], Kasahara and Singh [10], Matkowski [12], Park and Rhoades [14], Rhoades [15], Sessa and Fisher [17], Sharma [18], and Yeh [20, 21].

Inspired by a recent paper of Fisher [4], we prove a common fixed point theorem for four self mappings of \( (X, d) \) extending the results of [1], [8], [13], [17], and [18].

We also use the following notion of weakly commuting selfmappings of \( (X, d) \) given in [16].

**Definition.** Two selfmappings \( S \) and \( I \) of \( (X, d) \) weakly commute if

\[
d(SIx, ISx) \leq d(IX, SX)
\]

for any \( x \in X \).

Obviously, if \( S \) commutes with \( I \), then \( S \) also weakly commutes with \( I \) but when \( S \) weakly commutes with \( I \), then \( S \) does not necessarily commute with \( I \) as is shown in Example 1 below.

**2. A fixed point theorem.**

As in [1], [5] we put
\( \gamma(t) = \max \{ f(t, t, t, t), f(t, t, 2t, 0, t), f(t, t, 0, 2t, t) \} \)
for any \( t > 0 \) and further, we assume a slight modified version of the property (iii), i.e.

(iii') \( \gamma(t) < t \) for any \( t > 0 \).

Now let \( S, T, I \) and \( J \) be four self mappings of \((X, d)\) such that

(1) \( T(X) \subseteq I(X) \) and \( S(X) \subseteq J(X) \)

(2) \( d(Sx, Ty) \leq f( d(Ix, Jy), d(Ix, Sx), d(Ix, Ty), \\
d(Jy, Sx), d(Jy, Ty)) \)

for all \( x, y \) in \( X \), where \( f \) satisfies (i), (ii) (iii').

Let \( x_0 \) be an arbitrary point of \( X \) and \( x_1, x_2 \) in \( X \) such that \( Sx_0 = Jx_1, Tx_1 = Ix_2 \). This can be done since (1) holds. In according to Fisher [4], we can inductively define a sequence

(3) \( Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, Sx_{2n+2}, \ldots \)

such that \( Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2} \) for each integer \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \). Employing the method of proof of [13], it is proved that

\textbf{Lemma.} The sequence (3) is a Cauchy sequence.

See also an analogous result in [17].

Meade and Singh [13] established the following result:

\textbf{Theorem 1.} Let \( S \) and \( T \) two self mappings of \((X, d)\) satisfying

\[
d(Sx, Ty) \leq f(d(x, y), d(x, Sx), d(x, Ty), \\
d(y, Sx), d(y, Ty))
\]
for all \( x, y \) in \( X \), where \( f \in F \). Then \( S \) and \( T \) have a unique common fixed point.

Bearing in mind the proofs of the results of [1], [5], [18], it is not hard to verify that Theorem 1 holds also under the assumption \((iii')\) instead of \((iii)\). Analogous consideration holds for the main Theorem of [8].

Drawing inspiration from Fisher [4], we generalize Theorem 1 with the following

**Theorem 2.** Let \( S, T, I \) and \( J \) are four self mappings of \((X,d)\) satisfying conditions (1) and (2), where \( f \) satisfies properties (i), (ii), \((iii')\). If one of \( S, T, I \) and \( J \) is continuous and if \( S \) and \( T \) weakly commute with \( I \) and \( J \) respectively, then \( S, T, I \) and \( J \) have a unique common fixed point \( z \). Further, \( z \) is the unique common fixed point of \( S \) and \( I \) and of \( T \) and \( J \).

**Proof.** It is similar to that of Fisher and Sessa [17]. However we outline the essential steps in order to show where the weak commutativity plays the key role.

By lemma, the sequence \((3)\) converges to a point \( z \).

Suppose that \( I \) is continuous. Since the sequences

\[
\{Sx_{2n}\} = \{Ix_{2n+1}\} \quad \text{and} \quad \{Tx_{2n-1}\} = \{Ix_{2n}\}
\]

converge also to \( z \), we have that the sequence \(\{ISx_{2n}\}\) converges to \( Iz \). \( S \) being weakly commuting with \( I \), we deduce

\[
d(ISx_{2n}, Iz) \leq d(ISx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz)
\]

\[
\leq d(Ix_{2n}, Sx_{2n}) + d(ISx_{2n}, Iz),
\]

which implies, as \( n \to \infty \), that \(\{ISx_{2n}\}\) converges to \( Iz \). As in [17], using
twice (2) and properties (i), (ii), (iii') and the fact that \( \{I^2x_{2n}\} \) converges also to \( Ix \), we ascertain \( Ix = Sz = z \). Since the range of \( J \) contains the range of \( S \) let \( z \) be a point in \( X \) such that \( Jz' = z \). Then using (2) we have

\[
d(z, Tz') = d(Sz, Tz') \leq f(0, 0, d(z, Tz'), 0, d(z, Tz')) \leq \gamma (d(z, Tz'))
\]

which implies \( z = Tz' \) by property (iii'). Since \( T \) is weakly commuting with \( J \), we have

\[
d(TJz', JTz') \leq d(Jz', Tz') = d(z, z) = 0
\]

and then

\[
Ja = JTz' = TJz' = Tz.
\]

Using again (2) and (iii'), one deduces \( Tz = Jz = z \). Therefore \( z \) is a common fixed point of \( S, T, I \) and \( J \).

Analogous proof can be given if one supposes the continuity of \( J \) instead of \( I \).

Now we suppose the continuity of \( S \). Then the sequence \( \{SIx_{2n}\} \) converges to \( Sz \). Since \( S \) weakly commutes with \( I \), we have

\[
d(ISx_{2n}, Sz) \leq d(ISx_{2n}, SIx_{2n}) + d(SIx_{2n}, Sz) \leq d(Sx_{2n}, Ix_{2n}) + d(SIx_{2n}, Sz)
\]

which implies, as \( n \to \infty \), that the sequence \( \{ISx_n\} \) converges to \( Sz \). By (2) and properties (i), (ii), (iii), and observing that \( \{S^2x_{2n}\} \) converges also to \( Sz \), one proves that \( Sz = z \). As above, one shows that \( Jz = Tz = z \). Since the range of \( I \) contains the range of \( T \), let \( z' \) be a
point in $X$ such that $Iz'' = z$. Using again (2), we have

$$d(Sz'', z) = d(Sz'', Tz) \leq f(d(Iz'', Jz), d(Iz'', Sz''), d(Iz'', Tz), d(Jz, Sz''), d(Jz, Tz)) \leq$$

$$f(0, d(z, Sz''), 0, d(z, Sz'', 0) \leq \gamma (d(z, Sz'')) ,$$

which implies $Sz'' = z$ by property (iii'). Since $S$ weakly commutes with $I$, we have

$$d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0$$

and therefore

$$Iz = ISz'' = SIz'' = Sz = z.$$

Thus $z$ is a common fixed point of $S$, $T$, $I$ and $J$ and making a similar proof, the same conclusion is achieved supposing the continuity of $T$ instead of $S$.

The uniqueness of $z$ is easily proved.

3. Some remarks.

Remark 1. Assuming $I = J = \text{identity of } X$, Theorem 2 becomes Theorem 1.

Remark 2. Recently S. L. Singh and S. P. Singh [19] proved a common fixed point theorem for three selfmappings $S$, $T$, $I$ of $(X, d)$ satisfying the following condition:

$$(4) \quad d(Sx, Ty) \leq h \max \{ d(Ix, Iy), d(Ix, Sx), \frac{1}{2} [d(Ix, Ty) + d(Iy, Sx)], d(Iy, Ty) \}$$

for all $x, y$ in $X$, where $0 \leq h < 1$. 


C. C. Chang [1] studied the following condition

\[(5) \quad d(Sx, Ty) \leq f( d(Ix, Iy), d(Ix, Sx), d(Ix, Ty), d(Iy, Sx), d(Iy, Ty) ) \]

for all \(x, y\) in \(X\), where \(f\) verifies properties (i), (ii), (iii').

Further, the cited authors assume \(I\) continuous and commuting with \(S\) and \(T\). \(S(X) \subseteq I(X), T(X) \subseteq J(X)\).

Of course (4) is a consequence of (5) assuming

\[
f(t_1, t_2, t_3, t_4, t_5) = h. \max \{ t_1, t_2, (t_3 + t_4)/2, t_5 \}
\]

for any \(t_1, t_2, t_3, t_4, t_5 \geq 0\). However, (2) becomes (5) for \(I = J\) and moreover our assumptions of Theorem 2 are more general than those cited.

**Remark 3.** Imdad and Khan [8] proved a common fixed point theorem for three self-mappings \(S, I, J\) of \((X, d)\) satisfying the following condition

\[(6) \quad d(Sx, Sy) \leq f( d(Ix, Iy), d(Ix, Sx), d(Ix, Sy), d(Jy, Sx), d(Jy, Ty) ) \]

for all \(x, y\) in \(X\), where \(f \in F\). These authors assume \(I\) and \(J\) continuous, \(S\) commuting with \(I\) and \(J\) and \(S(X) \subseteq I(X) \cap J(X)\). Clearly (2) becomes (6) for \(S = T\) and therefore our Theorem 2 is a stronger result.

**Example 1.** Let \(X = [0, 1]\) with euclidean metric \(d\) and let \(S, T, I\) and \(J\) defined by

\[
Sx = x/(x + 2), \quad Tx = x/(x + 3), \quad Ix = x/2, \quad Jx = x/3
\]
for any $x$ in $X$. As shown in [16], $S$ weakly commutes with $I$. Since

$$d(TJx, JTx) = x/(x + 9) - x/(3x + 9) = 2x^2/(x + 9). (2x + 9)$$

$$\leq x^2/(3x + 9) = x/3 - x/(x + 3) = d(Jx, Tx)$$

for any $x$ in $X$, then $T$ weakly commutes with $J$ but $T$ does not commute with $J$ being $TJx \neq JTx$ for any non-zero $x$ in $X$. Let

$$f(t_1, t_2, t_3, t_4, t_5) = t_1/(6 + t_1) = g(t_1) \text{ for all } t_1, t_2, t_3, t_4, t_5 \geq 0.$$ It is immediately seen that $f$ enjoys properties (i), (ii) and (iii'). Further we have

$$T(X) = [0, 1/4] \subseteq [0, 1/2] = I(X), S(X) = [0, 1/3] = J(X)$$

and

$$d(Sx, Ty) = | 3x - 2y | / (x + 2). (y + 3) \leq$$

$$\leq | 3x - 2y | / (6 + | 3x - 2y |)$$

$$= g( d(Ix, Jy) )$$

for all $x, y$ in $X$. Being one of $S, T, I$ and $J$ continuous, then all the assumptions of Theorem 2 are verified resulting 0 the unique common fixed point of $S, T, I$ and $J$

The idea of this example appears in [17].

**Remark 4.** Fisher and Sessa [17], generalizing the results of [4], established a common fixed point theorem for four self-mappings $S, T, I$ and $J$ of $(X, d)$ satisfying the following condition

$$(7) \quad d(Sx, Ty) \leq \phi(\ max \{ d(Ix, Jy), d(Ix, Sx), d(Jy, Ty) \})$$

for all $x, y$ in $X$, where $\phi: R^+ \rightarrow R^+$ satisfies properties (i), (ii) and
(iii") $\phi(t) < t$ for any $t > 0$.

Obviously, by putting $f(t_1, t_2, t_3, t_4, t_5) = \phi \left( \max \{ t_1, t_2, t_3 \} \right)$ for any $t_1, t_2, t_3, t_4, t_5 \geq 0$, the condition (2) becomes (7). Now we give an example showing that our Theorem 2 is a more general than result of [17], even if one supposes $I = J$ = identity of $X$.

Example 2. Let $X = \{ A, B, C, D, E \}$ the subset of $R^2$, where

$A \equiv (-1, 0)$, $B \equiv (0, 0)$, $C \equiv (0, 1/2)$, $D \equiv (0, 1)$, $E \equiv (-1, 1)$, with euclidean metric $d$. Let $S$ and $T$ two selfmappings of $X$ defined as $SA = SB = SC = SD = C$, $SE = D$ and $TA = B$,

$TB = TC = TD = C$, $TE = D$.

Then it is not hard to verify that condition (2) is satisfied if we choose

$$f(t_1, t_2, t_3, t_4, t_5) = \frac{2t_1}{5} + \frac{t_2}{6} \left( t_3 + t_4 \right)/5$$

for any $t_1, t_2, t_3, t_4, t_5 \geq 0$. Condition (7) does not hold otherwise if there exists a function satisfying properties (i), (ii), (iii"), we should obtain for $x = E$ and $y = A$:

$$d(SE, TA) = d(D, B) = 1 \leq \phi \left( \max \{ d(A, E), d(E, D), d(A, B) \} \right) = \phi \left( 1, 1, 1 \right),$$

a contradiction to the required condition (iii").

The idea of this example appears in [11].

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SOME FIXED POINT THEOREMS FOR PAIRS OF MAPPINGS

By

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(Received : March 7, 1985)

In a recent paper [1] fixed point theorems were established for certain expansion mappings. An examination of the inequalities used in both [1] and this paper discloses the fact that further generalizations are doubtful. This paper establishes some fixed point theorems for pairs of mappings.

**Theorem 1.** Let \( f, g \) be surjective selfmaps of a complete metric space \((X, d)\). Suppose there exists a constant \( a > 1 \) such that

\[
(1) \quad d(fx, gy) \geq ad(x, y)
\]

for each \( x, y \) in \( X \). Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). Since \( f \) is surjective there exists a point \( x_1 \in f^{-1}x_0 \). Since \( g \) is surjective there exists a point \( x_2 \in g^{-1}x_1 \). Continuing in this manner one obtains a sequence \( \{x_n\} \) with \( x_{2n+1} \in f^{-1}x_{2n}, x_{2n+2} \in g^{-1}x_{2n+1} \).

Suppose \( x_{2n+1} = x_{2n} \) for some \( n \). Since \( d(x_{2n+1}, x_{2n}) = d(gx_{2n+2}, fx_{2n+1}) \), from (1), \( d(x_{2n+2}, x_{2n+1}) = 0 \). The condition \( x_{2n+1} = x_{2n} \) implies that \( x_{2n} \) is a fixed point of \( f \). Since also \( x_{2n+2} = x_{2n+1} \), \( x_{2n} \) is a fixed point of \( g \). Similarly, \( x_{2n+2} = x_{2n+1} \) leads to \( x_{2n+1} \) being a common fixed point of \( f \) and \( g \).

Assume \( x_n \neq x_{n+1} \) for each \( n \). From (1), \( d(x_n, x_{2n+1}) = \).
\[ d(fx_{2n+1}, gx_{2n+2}) \geq ad(x_{2n+1}, x_{2n+2}) \text{ and } d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) \geq ad(x_{2n+2}, x_{2n+3}). \] Therefore \( d(x_n, x_{n+1}) \geq ad(x_{n+1}, x_{n+2}) \), which implies \( \{x_n\} \) converges to a point \( x \in X \). Let \( y \in f^{-1}x \). The assumption \( x_n \neq x_{n+1} \) for each \( n \) implies that \( x_n \neq x \) for almost all \( n \). From (1),

\[ d(x_{2n+1}, x) = d(gx_{2n+2}, fy) \geq ad(x_{2n+2}, y). \]

Taking the limit as \( n \to \infty \) yields \( x = fx \). Let \( z \in g^{-1}x \). Then

\[ d(x_{2n}, x) = d(fx_{2n+1}, gz) \geq ad(x_{2n+1}, z), \]

and we obtain \( z = x \), and \( x \) is a common fixed point of \( f \) and \( g \).

Condition (1) forces uniqueness of the fixed point.

**Theorem 2.** Let \( f, g \) be surjective selfmaps of a complete metric space \( (X, d) \). Suppose there exist nonnegative functions \( p, q, r, s, t \), satisfying

\[
\begin{align*}
(2) & \quad \inf_{x, y \in X} (p(x, y) + q(x, y) + t(x, y)) > 1, \\
(3) & \quad \inf_{x, y \in X} \{(1 - q(x, y) + r(x, y)), (1 - p(x, y) + s(x, y))\} > 0, \\
(4) & \quad \sup_{x, y \in X} \{p(x, y), q(x, y)\} < 1,
\end{align*}
\]

and

\[
(5) \quad d(fx, gy) \geq p(x, y) d(x, fx) + q(x, y) d(y, gy) + r(x, y) d(x, gy) + s(x, y) d(y, fx) + t(x, y) d(x, y)
\]

for all \( x, y \in X \), \( x \neq y \). Then \( f \) and \( g \) have common fixed points.

**Proof.** Define \( \{x_n\} \) as in Theorem 1. Suppose \( x_{2n} = x_{2n+1} \) for some \( n \).
If $x_{2n+1} \neq x_{2n+2}$, then from (5),

$$
d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})
\geq ld(x_{2n+1}, x_{2n}) + qd(x_{2n+2}, x_{2n+1})
+ sd(x_{2n+2}, x_{2n}) + td(x_{2n+1}, x_{2n+2}),
$$

where $p, q, r, s,$ and $t$ are evaluated at $(x_{2n+1}, x_{2n+2})$.

Thus

$$
0 \geq (q + s + t) d(x_{2n+1}, x_{2n+2}).
$$

If $q + s + t = 0$ then $q + t = 0$ which, since $p < 1$, contradicts (2).

Therefore $x_{2n+1} = x_{2n+2}$ and $x_{2n}$ is a common fixed point of $f$ and $g$.

Similarly, $x_{2n+1} = x_{2n+2}$ for some $n$ leads to $x_{2n+1}$ being a common fixed point of $f$ and $g$.

Assume $x_{n} \neq x_{n+1}$ for each $n$. From (5),

$$
d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})
\geq pd(x_{2n+1}, x_{2n}) + qd(x_{2n+2}, x_{2n+1})
+ sd(x_{2n+2}, x_{2n}) + td(x_{2n+1}, x_{2n+2}),
$$

or

$$
(1 - p + s) d(x_{2n}, x_{2n+1}) \geq (q + s + t) d(x_{2n+1}, x_{2n+2}),
$$

where $p, q, r, s,$ and $t$ are evaluated at $(x_{2n+1}, x_{2n+2})$.

Again from (5),

$$
d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3})
\geq p'd(x_{2n+3}, x_{2n+2}) + q'd(x_{2n+2}, x_{2n+1})
$$

or

$$
(1 - p + s) d(x_{2n+1}, x_{2n+2}) \geq (q + s + t) d(x_{2n+2}, x_{2n+3}),
$$

where $p, q, r, s,$ and $t$ are evaluated at $(x_{2n+2}, x_{2n+3})$. 
or

\[(1 - q' + r') d(x_{2n+1}, x_{2n+2}) \geq (p' + r' + t') d(x_{2n+2}, x_{2n+3}),\]

where \(p', q', r', s\) and \(t'\) are evaluated at \((x_{2n+3}, x_{2n+2})\).

Inequalities (6) and (7), along with conditions (2) and (3), imply that \(\{x_n\}\) is Cauchy, hence convergent to some \(x\) in \(X\).

Without loss of generality we may assume that \(x_n \neq X\) for infinitely many \(n\) since, otherwise, \(f\) and \(g\) have a common fixed point. If there exists an infinite number of integers \(n\) such that \(x_{2n} \neq x\), define \(y \in g^{-1}x\). Then, from (5),

\[d(x_{2n}, x) = d(fx_{2n+1}, gy) \geq pd(x_{2n+1}, x_{2n}) + qd(y, gy) + rd(x_{2n+1}, gy)\]

\[+ sd(y, x_{2n}) + td(x_{2n+1}, y),\]

where \(p, q, r, s\) and \(t\) are evaluated at \((x_{2n+1}, y)\). The above inequality implies that

\[d(x_{2n}, x) \geq (q + s + t) \min \{d(y, x), d(y, x_{2n}), d(x_{2n+1}, y)\}\]

\[\geq (q+s+t) \min \{d(y, x), d(y, x_{n}), d(x_{2n+1}, y)\}.

Taking the limit as \(n \to \infty\) yields

\[0 \geq (q + s + t) \inf_{x, y \in X} d(x, y),\]

which implies that either \(x = y\) or \(\inf_{x, y \in X} (q + s + t) = 0\). However, the latter condition, along with (4), contradicts (2). Therefore \(x = y\).
If \( x_{2n+1} \neq x \) for all \( n \) sufficiently large, then \( x_{2n} = f x_{2n+1} = x \).

Taking the limit as \( n \to \infty \) yield \( x \) as a fixed point of \( f \).

In \( x_{2n+1} \neq x \) infinitely many \( n \), define \( f^{-1} x \).

Then, from (5), with \( p, q, r, s, \) and \( t \) evaluated \((z, x_{2n+2})\)

\[
\begin{align*}
d(x_{2n+1}, x) &= d(x_{2n+2}, fz) \\
&\geq pd(z, fz) + qd(x_{2n+2}, x_{2n+1}) + rd(z, x_{2n+1}) \\
&\quad + sd(x_{2n+2}, x) + td(z, x_{2n+2}) \\
&\geq \inf_{x, y \in X} \min \{d(z, x), d(z, x_{2n+1}), d(z, x_{2n+2})\}.
\end{align*}
\]

Taking the limit as \( n \to \infty \) yields \( 0 \geq \inf_{x, y \in X} \min \{(p+r+t) d(x, z), \}

which, in light of (4) and (2), implies \( x = z \).

**Theorem 3**: Let \( f, g \) be surjective continuous selfmaps of a complete metric space \( X \). If there exists a real number \( a > 1 \) such that

\[
(1) \quad d(fx, gy) \geq a \min \{d(x, fx), d(y, gy), d(x, y)\}
\]

for each \( x, y \in X \), then \( f \) or \( g \) has a fixed point or \( f \) and \( g \) have a common fixed point.

**Proof.** Define \( \{x_n\} \) as in Theorem 1. If \( x_n = x_{n+1} \) for any \( n \), then \( f \) or \( g \) has a fixed point.

Assume \( x_n \neq x_{n+1} \) for each \( n \). From (8),

\[
\begin{align*}
d(x_{2n}, x_{2n+1}) &= d(f x_{2n+1}, g x_{2n+2}) \\
&\geq a \min \{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\}.
\end{align*}
\]

and
\[ d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) \geq \]
\[ \geq a \min \{ d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1}) \}. \]

Thus, for each \( n \), \( d(x_n, x_{n+1}) \geq \min \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \} \),

which, since \( a > 1 \), implies that \( \{x_n\} \) is Cauchy, hence convergent, to some \( x \) in \( X \). The condition \( x_{2n} = fx_{2n+1}, x_{2n+1} = gx_{2n+2} \) and the continuity of \( f \) and \( g \) imply that \( s \) is a common fixed point of \( f \) and \( g \).

**REMARK 1.** Setting \( f = g \) in Theorem 1 and 3 yields Theorems 1 and 3 of [1]

**REMARK 2.** Setting \( f = g, s = t = 0, p, q, r \) constants in Theorem 2 yields Theorem 2 of [1].

**REFERENCE**

EFFECT OF VISCOSITY ON RAYLEIGH-TAYLOR INSTABILITY
IN THE PRESENCE OF A VERTICAL MAGNETIC FIELD

By

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(Received: April 5, 1984)

ABSTRACT

The character of equilibrium of a heavy, viscous, incompressible, finitely conducting and rotating fluid of variable density in the presence of a vertical magnetic field is investigated, when the lower bounding surface is rigid and the upper is free. The nature of the boundaries chosen alter the order of the dispersion relation compared to the cases where the boundaries are both free or both rigid. It is found that the stability criterion is independent of the effects of viscosity, finite resistivity and rotation. It is further investigated that the growth rates both increase or decrease with the increase in viscosity.

1. INTRODUCTION

The character of an incompressible fluid of variable density stratified in the vertical direction was first investigated by Rayleigh [3] and he found that the configuration is stable or unstable according as \( \frac{d\rho}{dz} \) is everywhere negative or anywhere positive. Chandrasekher [1] introduced the viscosity in the Rayleigh–Taylor instability and observed that in the stable case, the fluid oscillates about the mean position with an amplitude which decays exponentially at a rate which increases with increasing viscosity.
The hydromagnetic version of the Rayleigh-Taylor instability was further studied by Hide [2] for a fluid having exponentially varying density in the vertical direction. He included the effects of viscosity and finite resistivity in his problem. Sharma and Ariel [4] investigated the effect of finite resistivity of the medium on the equilibrium of a heavy, viscous, incompressible, rotating fluid of variable density in the presence of a vertical magnetic field. The present note investigates theoretically the same problem as considered by Sharma and Ariel [4], when the fluid is confined between two boundaries, the lower bounding surface being rigid and the upper free. Based on the existence of a variational principle we obtained the dispersion relation for a fluid having exponentially varying density. The object of this problem, under these boundary conditions, is to find out to what extent the instability of the configuration is affected by changing the viscosity of the medium.

2. PERTURBATION EQUATIONS

The linearised perturbation equations for the problem under consideration are (Sharma and Ariel [4], p. 104, eqns. (31), (32), (34) and (35))

\[
n[k^2 \varphi_0 w - D(\varphi_0 Dw) + \frac{KH_0(D^2 - k^2)Dh_z}{4\pi} - \frac{gk^2}{n}(D\varphi_0)w - \\
- D(2\varphi_0 \Omega \xi) + \mu_0(D^2 - k^2)^2 w + 2D\mu_0 (D^2 - k^2) Dw + 2\\n+ D^2\mu_0 (D^2 + k^2)w = 0, \tag{1}
\]

\[
[\eta \varphi_0 - \mu_0(D^2 - k^2) - D\mu_0 D] \zeta - \frac{KH_0 D\xi}{4\pi} 2\varphi_0 \Omega Dw, \tag{2}
\]

\[
[n - \eta (D^2 - k^2)]h_z = H_0 Dw, \tag{3}
\]

and \[n - \eta (D^2 - k^2)] \xi = H_0 D\xi, \tag{4}
\]

where
ρ₀ denotes the density before the system is disturbed,

\( w \) denotes the z-component of the velocity \( u \) at a fixed point,

\( g \) is the acceleration due to gravity, components \((0, 0, -g)\),

\( \mu_0 \) is the coefficient of viscosity, assumed to be variable in the undisturbed state,

\( K \) is the coefficient of magnetic permeability assumed to be constant,

\( H_0 \) is the magnetic field directed towards the z-axis in the undisturbed state,

\( n \) is the rate at which the system departs from equilibrium, \( h \) is the z-component of the magnetic field in the perturbed state,

\( k \) is the total wavenumber of the initial disturbance,

\( \zeta \) is the z-component of the vector curl \( u \),

\( \xi \) is the z-component of the vector curl \( h \),

\( D = \frac{d}{dz} \) and \( \eta = [4\pi K \rho]^{-1} \).

3. BOUNDARY CONDITIONS

The fluid is assumed to be confined to the planes \( z = 0 \) which is rigid and perfectly conducting and \( z = d \), which is free boundary.

The appropriate boundary conditions for the present problem are

(i) \( w(0) = D w(0) = h(0) = \zeta(0) = D \zeta(0) = 0 \),

(ii) \( w(d) = D^2 w(d) = D h(d) = D \xi(0) = \xi(d) = 0 \).

Multiplying the equation (1) for the characteristic value \( n \), by \( w \), and integrating across the vertical extent of the fluid we obtain on combining equations (2),(3) and (4), after a series of integrations by parts
\[ n(I_1 + I_4 + I_5 + I_7 + I_9) + I_3 + I_{10} - \frac{g k^2}{n} I_2 + \gamma k^2 (I_4 + 2I_5 + I_6 + I_9 + I_{10}) = 0, \quad (7) \]

the integrated parts vanish on account of boundary conditions,

where

\[ I_1 = \int_0^d \rho_0 [k^2 w^2 + (Dw)^2] \, dz, \quad (8) \]

\[ I_2 = \int_0^d D\rho_0 w^2 \, dz, \quad (9) \]

\[ I_3 = \int_0^d \mu_0 [k^2 w^2 + 2k^2 (Dw)^2 + (D^2 w)^2] + k^2 \int_0^d (D^2 \mu_0) \, dz, \quad (10) \]

\[ I_4 = \frac{K k^2}{4\pi} \int_0^d h^2 \, dz, \quad (11) \]

\[ I_5 = \frac{K}{4\pi} \int_0^d (Dh)^2 \, dz, \quad (12) \]

\[ I_6 = \frac{K}{4\pi k^2} \int_0^d (D^2 h)^2 \, dz, \quad (13) \]

\[ I_7 = \int_0^d \rho_0 \zeta^2 \, dz, \quad (14) \]

\[ I_8 = \int_0^d \mu_0 \left[ k^2 \zeta^2 + (D\zeta)^2 \right] \zeta \, dz, \quad (15) \]

\[ I_9 = \frac{K}{4\pi k^2} \int_0^d \xi^2 \, dz \quad (16) \]
\[ I_{10} = \frac{K}{4\pi k^2} \int_0^d (D\xi)^2 \, dz, \]  
(17)

4. THE CASE OF EXPONENTIALLY VARYING DENSITY

A case for which a simple analytical solution can be found is one in which the undisturbed density distribution is given by
\[ \rho_0(z) = \rho \exp \beta z, \]  
(18)
where \( \rho \) and \( \beta \) are constants.

Assuming \( v \), the coefficient of kinematic viscosity to be constant, we shall take
\[ \mu_0(z) = \nu \rho \exp \beta z. \]  
(19)

In order to ensure that the density variation within the fluid is small compared to the average density, we make an assumption that
\[ |\beta d| \ll 1. \]  
(20)

Let us assume the following trial functions which satisfy the boundary conditions (5) and (6):
\[ \begin{align*}
  w(z) &= W (\cos lz - \cos 3lz), \\
  h(z) &= X_1 \sin lz + X_3 \sin 3lz, \\
  \zeta(z) &= Z_1 \sin lz + Z_3 \sin 3lz, \\
  \xi(z) &= K_1 \cos lz = K_3 \cos 3lz,
\end{align*} \]  
(21)
where \( l = (\pi s/2d) \), \( s \) being an odd integer.

Substituting the values of \( w(z) \), \( h(x) \), \( \zeta(z) \) and \( \xi(z) \) equations (2), (3) and (4), we get
\[ [n + \gamma (l^2 + k^2)] Z_1 + \frac{kH_0 l}{4\pi \rho} K_1 = -2\Omega l W, \]  
(22)
Evaluating the integrals in equation (7) with the assumed form of \( w \), etc., and eliminating the constants from (22)-(27) we obtain the following dispersion relation between \( n \) and \( k \):

\[
\begin{align*}
\frac{n^2 (k^2 + 5l^2) - g\beta k^2 + nV (k^4 + 10l^2 k^2 + 41l^4)}{2} & + n^2 (l^2 + k^2) \left\{ \frac{n + \eta (l^2 + k^2)}{n + \eta (l^2 + k^2)} \right\} \frac{9(n^2 + 9k^2)}{n^2 + \eta (l^2 + k^2)} \\
& + \frac{9n + 9\eta (l^2 + k^2)}{n^2 + \eta (l^2 + k^2)} \frac{\eta (l^2 + k^2)}{n^2 + \eta (l^2 + k^2)} + 9l^2 V^2 
\end{align*}
\]

where \( V(= KH_0 / 4\pi \rho) \) denotes the Alfvén velocity.

It is convenient to discuss equation (28) in non-dimensional form, so that the important physical parameters of the problem may be brought out clearly. Let us choose a dimensionless growth rate \( \gamma \) and a dimensionless wavenumber \( x \) by measuring \( n \) and \( k \) in suitable units. We define:

\[
x = \frac{kd}{\pi s},
\]

(29)
and \( y = \frac{nd}{\pi s V} \). \hspace{1cm} (30)

From eqns. (28), (29) and (30), we have

\[
256y^6 + 128y^7 \left[ 4Ra + S(a^2 + 2a + 16) \right] + 64y^6 \left[ 2R^2 (a^2 - 16) + S(a^2 - 16) + 2RS(a^3 + 2a^2 + 16a + 64) + a + 36 + 4 (5A - 4Bx^2) \right] \\
+ 32y^5 \left[ 4R^3 (a^3 - 16a) + S^2 (a^4 - 256) + R^3S(5a^4 + 18a^3 + 64a^2 + 224a - 256) + 2RS^2 \left( 2a^4 + 9a^3 + 48a^2 + 112a \right) + 2R (a^2 + 46a) + 2S (5a^2 + 36a + 80) + 20A \{ 4Ra + S (a - 3, 2) \} - 16Bx^2 \left( 2Ra + 2Sa \right) \} + 16y^4 \left[ R^4 (a^4 - 256) 2R^3S(2a^5 + 8a^4 - 64a^2 - 512a - 1000) + 2R^2S^2 \left( 6a^5 + 19a^4 + 64a^3 - 512a + 256 \right) + 2S^3R \left( 2a^5 + 4a^4 - 512a - 1000 \right) + R^2 \left( 11a^5 + 66a^2 - 240a + 416 \right) + S^2 \left( 11a^3 + 4a^2 + 144a - 140 \right) + \right] 2RS \left( 22a^3 + 108a^2 + 96a + 460 \right) + 10a + 369 + 2A \{ 10R^2 (6a^2 - 32) + 2RS \left( 15a^2 - 54a + 18 \right) + 18 \} - 16Bx^2 \\
\{ R^2 (6a^2 - 32) + 8RSa^2 + S^2 (a^2 - 16) \} + 8y^3 \left[ R^4S (a^6 + 2a^5 - 16a^4 - 64a^3 - 256a^2 + 512a + 4096) + 4R^3 S^2 (2a^6 + 5a^5 - 32a^3 - 512a^2 - 744a) + R^2 S^3 (a^4 - 256) (6a^2 + 16a - 32) + RS^2 (36a^4 - 12a^3 + 208a^2 - 1344a - 256) + 2R^2 S (26a^4 + 59a^3 - 224a^2 - 368a - 3072) + 5R^3 (5a^4 + 10a^3 - 192a^2 - 160a + 1692) + R (60a^2 + 346a - 1952) + S (19a^2 + 328a - 1000) + 2A \{ R^2S (40a^2 - 472a + 584) + R^3 (40a^3 - 64a) + 54Ra \} - 16Bx^2 \left( 4R^3 (a^3 - 16a) + 2R^2S \right) + \right]
\]
\[(6a^3 - 32a) + 4RS^2 (a^3 - 16a) + R (30a - 32) + S (10a - 32) \} \] 
\[+ 4y^2 \left[ R^4 S^2 (a^2 - 16)^2 (a^3 + a^2 + 16a - 16) + 4R^3 S^3 (a^4 - 256) \right.\]
\[(a^2 - 16) (a + 2) + R^2 S^2 (31a^5 - 12a^4 - 512a^3 + 160a^2 - 1338a + \]
\[29696) + 2R^3 S (a^2 - 16) (10a^3 - 16a^2 - 256) + 2RS (39a^3 + 104a^2 + \]
\[1228a + 4730) + R^2 (a^2 - 16) (50a - 151) + 9 (a + 36) + \]
\[2A \{10R^4 (a^2 - 16)^2 + 2R^3 S (5a^4 - 12a^3 + 6a^2 + 192a - 1376) + \]
\[18R^2 (30a^2 - 16)\} - 16Bx^2 \{2R^3 S^2 (a^2 - 16) (3a^2 - 16) + 2RS (15a^2 - \]
\[64a + 80) + 8R^3 S (a^4 - 16a^2) + R^4 (a^2 - 16)^2 + R^2 (30a^2 - 64a - \]
\[160) + 9)\} + 2y \left[ 2R^4 S^3 (a^2 + 16) (a^2 - 16)^3 + R^3 S^2 (a^2 - 16) \right.\]
\[(5a^4 - 64a^3 + 160a^2 - 1024a + 3840) + R^2 S (a^2 - 16) (59a^2 - 160a + \]
\[224) + 45R (a^2 - 16) + 2A \{R^4 S (a^2 - 16)^2 (10a - 32) + 18R^3 (a^3 - \]
\[16a)\} - 16Bx^2 \{2R^3 S^2 a (a^2 - 16)^2 + 2R^2 S (11a^3 - 80a^2 + 80a + \]
\[256) + 2SR^4 a (a^2 - 16)^2 + 2S^2 R^2 a (a^2 - 16)^2 + R^3 (a^2 - 16) (10a - \]
\[32) + 18Ra\} \} - 16Bx^2 R^2 (a^2 - 16) \{R^2 S^2 (a^2 - 16)^2 + 2RS (5a^2 - \]
\[32a + 80) + 9)\} = 0, \quad (31)\]

where

\[R = (\eta \pi s/2dV), \quad (32)\]
\[S = (\nu \pi s/2dV), \quad (33)\]
\[B = (g \delta d^2/\pi^2 s^2 V^2), \quad (34)\]
\[A = (4 \Omega^2 d^2/\pi^2 s^2 V^2), \quad (35)\]
and \( a = 4x^2 + 5 \). \hspace{1cm} (36)

From the above equation we find that there are four parameters required to specify \( y \) for any given \( x \). These numbers \( R, S, B \) and \( A \) respectively represent measures of resistivity of the medium, coefficient of viscosity, buoyancy forces and coriolis forces in terms of magnetic field. The case of viscous, finitely conducting and non-rotating configuration has been discussed in detail by Sharma \([5]\), where the nature of the boundaries chosen gives rise to an additional monotonically decreasing mode. Another particular case of equation (31), when the viscosity is absent, has been treated by Sharma \([6]\). The dispersion relation (31) is an algebraical equation of degree eight in \( y \), hence it will have eight roots. To obtain the explicit expression for each value of \( y \) for general values of parameter is a task which is mathematically too involved. If we consider equation (31) in its general form, we observe that the stratification is stable or unstable thoroughly according as \( B \) is negative or positive. However, if we consider the fluid to be viscous but infinitely conducting \((R = 0)\), the configuration is rendered unstable for the wavenumber range

\[ x < \frac{1}{4} \left[ \frac{4}{4B-5} \right]^{1/2} \] \hspace{1cm} (37)

Thus the effect of resistivity is to cancel the stabilizing role of magnetic field altogether and render the system unstable for the whole wavenumber range.

We now examine the behaviour of growth rate with respect to viscosity analytically. To investigate this, we require to discuss the nature of the positive root of \( y \) in equation (31) in detail. We observe if \( B > 0 \), the absolute term in equation (31) is negative, therefore, the
equation (31) has at least one real positive root. Hence, the equilib­rium will be always unstable. The asymptotic behaviour of this root for \( X \to 0 \) and \( X \to \infty \) are

\[
y \to \frac{(9R^2S^2 + 10RS + 1)}{(9R^2S^2 + 10RS + 1) (41RS + S) + AR^2(9RS + 5)} \tag{38}
\]

\[
y \to \frac{B}{2Sx^2}. \tag{39}
\]

Now we study the behaviour of \( y \) (growth rate) on varying the value of \( S \) from equation (31). To find the role of \( S \), we examine the nature of \( dy/dS \) form (38), while keeping other nondimensional numbers \( R, A \) and \( B \) fixed. To bring out the peculiar features of the role of viscosity into focus, in the context of Rayleigh-Taylor instability, we divide our subsequent analysis of instability criterion into two cases by different range of \( R \) and \( A \).

Case I. \( AR^2 > 1 \). In this case we find that with increasing \( S \), the value of \( y \) decreases for all wavenumbers. Thus, more the viscosity of the medium, more it tends to stabilizes the configuration. This type of behaviour can also be observed even in the absence of rotation.

Case II. \( AR^2 < 1 \). In this case a peculiar tendency is exhibi ed by \( S \), when (a) \( S < \tilde{S} \), (b) \( S > \tilde{S} \), where

\[
\tilde{S} = \frac{1}{18R^2} \left\{ -10R + \left[ 100R^2 - \frac{1}{82} \left( 82 - 9AR^3 - (82A^2R^4 + 5248AR^2)^{1/2} \right)^{1/2} \right] \right\}. \tag{40}
\]

So long as \( S < \tilde{S} \), an increase in the value of \( S \) leads to the decre­ase in the value of \( y \) for all wavenumbers. Thus, we can say that viscosity has a stabilizing influence on the configuration. When \( S \) becomes smaller than \( \tilde{S} \), this behaviour is reversed for small values of
x. Now an increase in the value of $S$ leads to an increase in the value of $y$. This particular behaviour is however, marked only for small $x$. After reasonably large values of $x$, once again the uniform pattern can be observed, namely that with increasing $S$, more stability is imparted to the system. Thus, we conclude that viscosity has a destabilizing influence for small wavenumbers of disturbance but it has a stabilizing influence for large wavenumbers.

Lastly, when $B<0$ we find that all roots of $y$ are either real and negative or there are complex roots with negative real parts. The system is therefore stable in each case. Hence, the potentially stable configuration remains stable whether the effects of viscosity, resistivity and rotation are included or not.

REFERENCES


[4] B. M. Sharma and P. D. Ariel, The character of the equilibrium of a heavy, viscous, incompressible, finitely conducting rotating fluid in the presence of a vertical magnetic field,

SOME INTEGRALS AND SERIES OF THE PRODUCT OF TWO MULTIVARIABLE $H$-FUNCTIONS

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(Received: December 5, 1983; Revised: July 5, 1984)

ABSTRACT

In the present paper, we evaluate three definite integrals which are then used to establish two Fourier sine series and one infinite series for the multivariable $H$-function due to H. M. Srivastava and R. Panda [10].

1. INTRODUCTION

The multivariable $H$-function defined by Srivastava and Panda [10, p. 271, Eq. (4.1)] (see also Prasad and Singh [7] and Srivastava and Panda [11]) will be represented here in the manner already detailed by Srivastava, Gupta and Goyal [9, p. 251, Eq. (C.1)]. The object of this paper is to evaluate three definite integrals which are then used to establish a number of expansion formulas involving these multivariable $H$-functions.

2. Integral Formulas

In this section, we evaluate the following definite integrals

$$
\int_0^{\pi/2} \left( \sin x \right)^{n-1} \left( \cos x \right)^{n-1} e^{\omega(u + v)x} H_{P, Q} \left[ P, Q', 1 \right] \; dx
$$

where $n, \omega, u, v, P, Q, P', Q'$ are appropriate real numbers, and $H_{P, Q}$ is the multivariable $H$-function.
\[
(1, N^{(r)})
\]
\[
[P^{(r)}, Q^{(r)} + 1]
\]
\[
\left[ Z_1 (\sin x e^{\omega x})^{\sigma_1} ; \ldots ; Z_r (\sin x e^{\omega x})^{\sigma_r} \right]
\]
\[
\{(e_p; E_p^{(r)} ; \ldots ; E_{p}^{(r)}) \} : \{(C_p', \gamma_p')\} ; \ldots ;
\]
\[
\{(f_q; F_q^{(r)} , \ldots , F_q^{(r)})\} : \{(D_q', \delta_q') \} \{(D_q' , \delta' \delta')\} ; \ldots ;
\]
\[
\{(C^{(r)} p^{(r)} ; \gamma^{(r)} p^{(r)})\}
\]
\[
(D_0^{(r)} , \delta_0^{(r)} ) , \{(D^{(r)} Q^{(r)} ; \delta^{(r)} Q^{(r)} )\}
\]
\[
H \left[ x_1 (\sin x e^{\omega x}) h_1 ; \ldots ; x_r (\sin x e^{\omega x}) h_r \right]
\]
\[
\left\{ \begin{array}{l}
\{(a_p; a_p', \ldots , a_p^{(r)})\} : \{(A'_r X'_r , \eta' X'_r )\} ; \ldots ;
\{(b_q; b_q', \ldots , b_q^{(r)})\} : \{(B'_r Y'_r , \xi' Y'_r )\} ; \ldots ;
\{(A^{(r)} X^{(r)} ; \eta^{(r)} X^{(r)} )\}
\{(B^{(r)} Y^{(r)} ; \xi^{(r)} Y^{(r)})\}\end{array} \right\}
\]
\[
\frac{e^{\omega u \pi/2}}{\delta_0 \cdots \delta_0^{(r)}} \prod_{ \nu_1 , \ldots , \nu_r = 0}^{\infty} \phi(\rho_{\nu_1} , \ldots , \rho_{\nu_r})
\]
\[
\frac{\omega}{\sum_{i=1}^{r} \sigma_i \rho_{\nu_i}} \prod_{i=1}^{r} \{ \theta_i (\rho_{\nu_i}) \}
\]
\[
\frac{(-1)^{\nu_1} \overline{Z_1 \rho_{\nu_1}}}{\nu_1 !} \left\{ 0, 1 : (V', W') ; \ldots ; (V^{(r)}, W^{(r)}) \right\}
\]
\[
H \left[ x_1 e^{\omega h_1} \pi/2 , \ldots , x_r e^{\omega h_r} \pi/2 \right]
\]
\[
\left(1 - u - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} ; h_1 , \ldots , h_r \right)
\]
\[
\{(b_q; b_q', \ldots , b_q^{(r)})\}.
\]
\[
\{(a_i; \alpha_i, \ldots, \alpha^{(r)}_i)\} : \{(A^{(r)} X^{(r)}, \eta^{(r)} X^{(r)})\} ; \ldots \]

\[
(1-u-v- \sum_{i=1}^{r} \sigma_i \rho_{v_i} ; h_1, \ldots, h_r) : (B^{(r)} Y^{(r)}, \xi^{(r)} Y^{(r)}) ; \ldots \]

\[
\left\{ \left( \begin{array}{c} A^{(r)} X^{(r)} \\ \eta^{(r)} X^{(r)} \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} B^{(r)} Y^{(r)} \\ \xi^{(r)} Y^{(r)} \end{array} \right) \right\} \quad (2.1)
\]

provided that \( \omega = \sqrt{-1} \), \( h_1, h_2, \ldots, h_r > 0 \), \( \Re (u + \sum_{i=1}^{r} (\sigma_i \rho_{v_i} + h_i \alpha_i)) > 0 \), \( \Re (v) > 0 \), \( |\arg x_i| < \frac{1}{2} U_i \pi, U_i > 0 \),

\[|\arg Z_i| < \frac{1}{2} V \pi, V_i > 0 \quad (i = 1, \ldots, r),\]

where \( \alpha_i, U_i \) and \( V_i \) are as follows:

\[
\alpha_i = \min \Re \left( B^{(i)} / \xi^{(i)} \right), \quad j = 1, \ldots, V^{(i)}, i = 1, \ldots, r, \quad (2.2)
\]

\[
U_i = - \sum_{j=1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{V^{(i)}} \xi_j^{(i)} - \sum_{j=V^{(i)}+1}^{Y^{(i)}} \xi_j^{(i)} + \sum_{j=1}^{W^{(i)}} \gamma_j^{(i)} - \frac{X^{(i)}}{j=W^{(i)}+1} \gamma_j^{(i)} > 0, \quad i = 1, \ldots, r, \quad (2.3)
\]

\[
\text{and}
\]

\[
V_i = - \sum_{j=1}^{P} E_j^{(i)} - \sum_{j=1}^{Q} F_j^{(i)} + \delta_0 Q^{(i)} - \sum_{j=1}^{N^{(i)}} \delta_j^{(i)} + \sum_{j=1}^{\gamma_j^{(i)}} \chi_j^{(i)}
\]

\[
- \sum_{j=N^{(i)}+1}^{P^{(i)}} \gamma_j^{(i)} > 0, \quad i = 1, \ldots, r. \quad (2.4)
\]

\[\int_{0}^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{\omega(u+v)x} H^{0,0}(1+N') \quad \ldots\]

\[P, Q : [P', Q' + 1] ; \ldots\]
(1, N^{(r)})

\[ [P^{(r)}, Q^{(r)} + 1] \left[ Z_1 (\tan x)^{\sigma_1}, \ldots, Z_r (\tan x)^{\sigma_r} \right] \]

\[
\begin{align*}
&\{(e_p; E_p', \ldots, E^{(r)}_p)\} : \\
&\{(f_Q; F_Q', \ldots, F^{(r)}_Q)\}:
\end{align*}
\]

\[
\{(C^{(r)}_p, \gamma_p')\} \ldots \{(C^{(r)}_p, \gamma_p')\}
\]

\[
\{(D'^0, \delta'^0), \{(D'^0, \delta'^0)\} ; \ldots ; (D^{(r)}_0, \delta^{(r)}_0) \{(D^{(r)}_0, \delta^{(r)}_0)\} \}
\]

\[ H \]

\[
H \int_{p \cdot q : [X', Y'] ; \ldots ; [X^{(r)}, Y^{(r)}]} \left[ x_1 (\tan x)^{h_1}, \ldots, x_r (\tan x)^{h_r} \right] \]

\[
= e^{\omega u \pi/2} \Gamma(u + v) \delta_0' \ldots \delta_0^{(r)} \sum_{v_1, \ldots, v_r = 0} \phi(p_v_1, \ldots, p_v_r)
\]

\[
e^{\omega (\sum_{i=1}^r \sigma_i p_{v_i}) \pi/2} e_{v_i} \left\{ b_i (p_{v_i}) \left( \frac{-1}{v_i} \right)^{v_i} Z_i p_{v_i} \right\}
\]

\[
1, 1 : (V', W') ; \ldots ; (V^{(r)}, W^{(r)})
\]

\[
H \int_{p+1, q+1 : [X', Y'] ; \ldots ; [X^{(r)}, Y^{(r)}]} \left[ x_1 e^{\omega h_1 \pi/2}, \ldots, x_r e^{\omega h_r \pi/2} \right] \]

\[
(1 - u - \sum_{i=1}^r \sigma_i p_{v_i} ; h_1 ; \ldots, h_r), \{(a_p; a'_p, \ldots, a^{(r)}_p)\}; \{(A^{(r)}_X', \eta^{(r)}_X)\} ;
\]

\[
\{(b_q; \beta'_q, \ldots, \beta^{(r)}_q)\}, (\nu - \sum_{i=1}^r \sigma_i p_{v_i} ; h_1 ; \ldots, h_r), \{(B^{(r)}_Y', \xi^{(r)}_Y)\} ;
\]
\begin{align}
\cdots; \{(A^{(r)}X^{(r)}, \gamma^{(r)}X^{(r)})\} \cdots; \{(B^{(r)}Y^{(r)}, \xi^{(r)}Y^{(r)})\} \right] \omega = \sqrt{-1}, \\
\text{provided that } h_1, h_2, \ldots, h_r > 0, \ Re (u + \sum_{i=1}^{r} (\sigma_i \rho_{\gamma_i} + h_i \alpha_i)) > 0, \\
\Re (v - \sum_{i=1}^{r} (\sigma_i \rho_{\gamma_i} + h_i \beta_i)) > 0, \ | \arg \chi_i | < \frac{1}{2} U_i \pi, \ U_i > 0, \\
\Re (v - \sum_{i=1}^{r} (\sigma_i \rho_{\gamma_i} + h_i \beta_i)) > 0, \ | \arg Z_i | < \frac{1}{2} V_i \pi V_i > 0 (i = 1, \ldots, r), \text{ where } \alpha_i, U_i \text{ and } V_i \text{ are given by the Equation (2.2), (2.3) and (2.4) respectively and} \\
\beta_i = \max \Re ((A_j^{(i)} - 1)/\gamma_0^{(i)}), j = 1, \ldots, W^{(i)}, i = 1, \ldots, r. \\
\int_{0}^{\infty} x^p e^{-x} L^{(s)}(x) H \right] p, q : [P', Q' + 1]; \cdots; [P^{(r)}, Q^{(r)} + 1] \\
\left[ Z_1(x)^{q_1}, \ldots, Z_r(x)^{q_r} \right] \left[ \{(e_P; E_P', \ldots, E^{(r)}P') \}; \{(C^{(r)}P', \gamma^{(r)}P') \}; \cdots; \{(D'o, \delta'o), (D'o', \delta'o') \}; \cdots; \{(C^{(r)}P^{(r)}, \gamma^{(r)}P^{(r)}) \}; \{(D^{(r)}Q^{(r)}, \delta^{(r)}Q^{(r)}) \} \right] \\
\left[ C_1 \left( x^{\mu_1} \right); \cdots; C_r \left( x^{\mu_r} \right) \right] \\
\left[ \{(a_o; \alpha_o', \ldots, \alpha_o^{(r)}) \}; \{(A'X', \gamma'X') \}; \cdots; \{(A^{(r)}X^{(r)}, \gamma^{(r)}X^{(r)}) \}; \cdots; \{(B'^{r}Y'^{r}, \xi'^{r}Y'^{r}) \}; \cdots; \{(B^{(r)}Y^{(r)}, \xi^{(r)}Y^{(r)}) \} \right] dx.
\end{align}
\[= \frac{1}{k! \delta_0 \ldots \delta_0^{(r)}} \sum_{v_1, \ldots, v_r} \phi(\rho_{v_1}, \ldots, \rho_{v_r})\]

\[
\frac{r}{\prod_{i=1}^{r} \{t_i(\rho_{v_i}) \frac{(-1)^{v_i}}{v_i!} Z_i^{\rho_{v_i}} \}}\]

\[H^p + 2, q + 1 : [X', Y'] ; \ldots ; [X^{(r)}, Y^{(r)}] \left[ C_1, \ldots, C_r \right]\]

\[
(\sigma - \rho - \sum_{i=1}^{r} \sigma \rho_{v_i} ; \mu_1 ; \ldots ; \mu_r), \{(\alpha_p; \alpha'_p, \ldots, \alpha^{(r)}_p)\},
\]

\[
(- \sigma - \rho - \sum_{i=1}^{r} \sigma \rho_{v_i} + k; \mu_1 ; \ldots, \mu_r),
\]

\[
\{(b_q; \beta'_q, \ldots, \beta^{(r)}_q)\} ; \{(B'_r, \xi'_r)\} ; \ldots ; \{(A'_rX', \eta'_rX')\} ; \ldots ; \{(A^{(r)}rX^{(r)}, \eta^{(r)}rX^{(r)})\} \]

\[
\{(B^{(r)}rY^{(r)}, \xi^{(r)}rY^{(r)})\} \right\},
\]

provided that \(\Re(\rho + 1 + \sum_{i=1}^{r} (\sigma \rho_{v_i} + \mu \alpha)) > 0\),

\(| \text{are } C_i | < \frac{1}{2} U_i \pi, U_i > 0, | \arg Z_i | < \frac{1}{2} V_i \pi, V_i > 0 (i = 1, \ldots, r),\)

where \(\alpha_i, \beta_i\) and \(V_i\) are given by the Equations (2.2), (2.3) and (2.4) respectively.

**Proof.** To prove (2.1), we write the series expansion of

\[H^{(1)} [Z_1 (\sin x e^{\sigma X})^1, \ldots, Z_r (\sin x e^{\sigma X})^r \text{ with the help of the result}]

given by Saxena [8] and Mukherjee and Prasad [6, p.6], as follows:

\[ H^{(1)}[Z_1 (\sin x e^{\omega x})^{\sigma_1}, \ldots, Z_r (\sin x e^{\omega x})^{\sigma_r}] \]

\[ = \frac{1}{\delta^{0(1)} \ldots \delta^{0(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \phi(\rho_{\nu_1}, \ldots, \rho_{\nu_r}) \]

\[ \prod_{i=1}^{r} \{ \theta_1(\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i !} Z_i^{\rho_{\nu_i}} (\sin x e^{\omega x})^{\sigma_i(\rho_{\nu_i})} \}, \rho_{\nu_i} = \frac{D^{0(i)} + \nu_i}{\delta^{0(i)}}, \]

\[ \omega = \sqrt{-1}, \]

(2.8)

where

\[ \theta_1(s_i) = \prod_{j=1}^{N(i)} \Gamma(1 - C_j(s_i) + \gamma_j(s_i)) \]

\[ \prod_{j=1}^{P(i)} \Gamma(1 - D_j(s_i) + \delta_j(s_i)) \]

\[ \prod_{j=N(i)+1}^{P(i)} \Gamma(C_j(s_i) - \gamma_j(s_i)) \]

\[ i = 1, \ldots, r, \]

(2.9)

and

\[ \phi(s_1, \ldots, s_r) = \left[ \prod_{j=1}^{P} \Gamma(e_j - \sum_{i=1}^{r} E_j(s_i)) \prod_{j=1}^{Q} \Gamma(1 - f_j + \sum_{i=1}^{r} F_j(s_i)) \right]^{-1} \]

(2.10)

where \((\sigma_i) > 0, |\arg Z_i| < 1/2 V_i \pi, V_i > 0 (i = 1, \ldots, r),\)

where \(V_i\) is given by Equation (2.4); take the contour integral form for

\[ H^{(2)}[x_1(\sin x e^{\omega x})^{h_1}, \ldots, x_r (\sin x e^{\omega x})^{h_r}] = \frac{1}{(2\pi \omega)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \]
\[
\phi_r(s_r)\Psi(s_1, \ldots, s_r) x_1^{s_1} (\sin x e^{\omega x})^{h_1 s_1} \ldots x_r s_r (\sin x e^{\omega x})^{h_r s_r} ds_1 \ldots ds_r, \quad \omega = \sqrt{-1}, 
\]

where

\[
\phi_i(s_i) = \prod_{j=1}^{V(i)} \frac{\Gamma(B_i^{(i)} - \xi_i^{(i)} s_i)}{\prod_{j=V(i)+1}^{V(i)} \Gamma(1 - B_i^{(i)} + \xi_i^{(i)} s_i)} \prod_{j=1}^{\bar{W}(i)} \frac{\Gamma(1 - A_j^{(i)} + \eta_j^{(i)} s_i)}{\prod_{j=\bar{W}(i)+1}^{\bar{W}(i)} \Gamma(A_j^{(i)} - \eta_j^{(i)} s_i)}
\]

\[i = 1, \ldots, r,
\]

\[
\Psi(s_1, \ldots, s_r) = \left[ \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_i^{(i)} s_i) \right]^{-1}
\]

\[
\prod_{j=1}^{q} \Gamma(a_j - \sum_{i=1}^{r} \alpha_i^{(i)} s_i)
\]

in the integrand of left hand side of (2.1) and change the order of integration, which is justifiable under the given conditions. The left-hand side of (2.1) reduces to

\[
\frac{1}{\delta_0 \ldots \delta_0^{(r)}} \sum_{v_1, \ldots, v_r = 0}^\infty \phi(\rho, \ldots, \rho) \prod_{i=1}^{r} \{ \theta_i(\rho) \}^{-\frac{v_i}{v_i !}} Z_1 \rho v_i
\]

\[
\frac{1}{(2\pi \omega)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \Psi(s_1, \ldots, s_r) x_1^{s_1} \ldots x_r^{s_r}
\]

\[
\sum_{i=1}^{r} (\sigma_i \rho v_i + h_i s_i) - 1
\]

\[
\int_0^{\pi/2} (\sin x)^{u + \sum_{i=1}^{r} (\sigma_i \rho v_i + h_i s_i) - 1} (\cos x)^{v-1}
\]
\[
\omega(u + v + \sum_{i=1}^{r} (\sigma_i \varphi_{v_i} + \mu_i s_i)) \quad \left\{ \begin{array}{l}
edx \\ dx \end{array} \right\} ds_1 \ldots ds_r. \quad (2.14)
\]

Now, evaluating the inner integral in (2.14) with the help of the result [4, p. 80, Eq. (7)]

\[
\int_{0}^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{\omega(u+v)x} \, dx = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u+v)} e^{\omega u \pi/2},
\]

\[Re(u) > 0, \quad Re(v) > 0, \quad \omega = \sqrt{-1},\]

we arrive at the right hand side of (2.1) stated as above.

The proof of (2.5) is similar to the proof of (2.1).

To prove (2.7), we write the series expansion of \( H^{(1)}[Z_1(x)^{\sigma_1}, \ldots, Z_r(x)^{\sigma_r}] \) from the Equation (2.8) and contour integral form of \( H^{(2)}[C_1(x)^{\mu_1}, \ldots, C_r(x)^{\mu_r}] \) from the Equation (2.11) in the left-hand side of (2.7) and change the order of integration, which is justifiable under given conditions. The left-hand side of (2.7) reduces to

\[
\left\{ \begin{array}{l}
\frac{1}{\delta_0 \ldots \delta_0^{(r)}} \sum_{v_1, \ldots, v_r=0}^{\infty} \phi(\varphi_{v_1}, \ldots, \varphi_{v_r}) \prod_{i=1}^{r} \left\{ \begin{array}{l}
\theta_1(\varphi_{v_i}) (-1)^{v_i} v_i!
\end{array} \right\} Z_i^{\varphi_{v_i}}
\\ \frac{1}{(2\pi \omega)^{r}} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \Psi(s_1, \ldots, s_r) C_1^{s_1} \ldots C_r^{s_r}
\\ + \sum_{i=1}^{r} (\sigma_i \varphi_{v_i} + \mu_i s_i)
\\ e^{-\sigma L(\varphi)(x)} dx \right\} ds_1 \ldots ds_r.
\]

(2.16)
Now, on evaluating the inner integral of (2.16) with the help of the known result [3, p. 292, Eq. (1)]

$$
\int_0^\infty x^{\beta-1} e^{-x} L_b^{(2)}(x) \, dx = \frac{\Gamma(\alpha - \beta + k + 1) \Gamma(\beta)}{k! \Gamma(\alpha - \beta + 1)}, \quad Re(\beta) > 0,
$$

we arrive at the right-hand side of (2.7).

3. INFINITE SERIES

In this section, we establish an infinite series for the product of two multivariable H-functions. The series is

$$
\left[ Z_1(x) Z_2(x) \right] \sum_{t=0}^{\infty} \frac{x^{2a_1 - 2 - t}}{t!} \frac{W_t}{H} \frac{0, 0 : (1, N') ; \ldots ; (1, N^{(r)})}{P, Q : [P', Q' + 1]; \ldots; [P^{(r)}, Q^{(r)} + 1]}
$$

$$
\left[ \{C_1(x)^{-2a_1} ; \ldots ; C_r(x)^{-2a_1} \} \right]
$$

$$
\left[ \{A_{1}^{(r)} X^{(r)}, Y^{(r)} \} ; \ldots ; \{A_{1}^{(r)} X^{(r)}, Y^{(r)} \} \right]
$$

$$
\left[ \{B_{1}^{(r)} Y^{(r)}, \xi^{(r)} Y^{(r)} \} ; \ldots ; \{B_{1}^{(r)} Y^{(r)}, \xi^{(r)} Y^{(r)} \} \right]
$$
\[\begin{align*}
&= (x^2 - Wx)^{a_1-1} \frac{1}{\delta_0' \ldots \delta_0^{(r)}} \phi_{v_1, \ldots, v_r} = 0 \cdot \phi_{v_1, \ldots, v_r} \\
&= \prod_{i=1}^{r} \left\{ \theta_i \left( \frac{1}{v_i} \right)^{v_i} \frac{1}{v_i !} Z_i^{v_i} (x)^{2E_1^{(i)}} \right\} \\
&\quad \left\{ (a_p; \alpha_p', \ldots, \alpha_p^{(r)}) : \{ (A'X', \eta'X') \} , \\
&\quad \quad \{ (b_q; \beta_q', \ldots, \beta_q^{(r)}) : \{ (B'Y', \xi'Y') \} : \\
&\quad \quad \quad \quad \{ (a_0', \alpha_0', \ldots, \alpha_0^{(r)}) : \ldots, (a_p; \alpha_p', \ldots, \alpha_p^{(r)}) \} \right\}.
\end{align*}\]

provided that \(| w/x | < 1, | \arg C_i | < \frac{1}{2} U_i \pi, U_i > 0, | \arg Z_i | < \frac{1}{2} V_i \pi, V_i > 0\), where \(U_i\) and \(V_i\) are given by the Equations (2.3) and (2.4) respectively, and \((a_i; a_i', \ldots, a_i^{(r)})_{2r}\) stands for the set of parameters \((a_0; a_0', \ldots, a_0^{(r)}) \ldots, (a_p; \alpha_p', \ldots, \alpha_p^{(r)})\).

**Proof:** To prove the result (3.1), we write the series expansion of \(H^{(1)} [Z_1 (x)^{2E_1'}, \ldots, Z_r (x)^{2E_1^{(r)}}] \) with the help of equation (2.8) and write \(H^{(2)} [C_1 (x)^{-2a_1'}, \ldots, C_r (x)^{-2a_1^{(r)}}] \) into its contour form, change the order of integration and summation and use the formula \(1F_0 (a; -x) = (1 - x)^{-a} \), and finally interpret the result in the light of multivariable \(H\)-function.

**PARTICULAR CASES**

1. On taking \(r = 1, P = Q = 0 = P' = Q' = D_0', \delta_0' = 1 = E_1' \)
$Z_1 \to 0$ in $H^{[1]} [Z_1 (x)^{2E_1'}, \ldots, Z_r (x)^{2E_1^{(r)}}]$ in the Equation (3.1). we arrive at the result recently studied by Maurya [5, p. 255].

(2) On taking $r = 2$ in the particular case (1), we arrive at the result given by Chaurasia [2].

Acknowledgement

Our thanks are due to Professor H. M. Srivastava, University of Victoria, B. C., Canada, for his valuable suggestion in preparation of this paper.

REFERENCES


FINITE INTEGRALS INVOLVING THE PRODUCTS OF THE $H$-FUNCTIONS OF TWO AND MORE VARIABLES

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(Received: December 13, 1984)

ABSTRACT

In this paper we evaluate two finite integrals associated with the product of the $H$-function of two variables and Srivastava and Panda's multivariable $H$-function. These integrals are quite general in character and a number of interesting (known or new) integrals can be deduced as particular cases. Some special cases are also discussed briefly.

1. INTRODUCTION

The $H$-function of two variables occurring in this paper is due to Mittal and Gupta [6]. The details of this function can be found in a recent book by Srivastava, Gupta and Goyal [11].

The following expansion formula ([4], p. 17, Eq. (2.1)) for a special $H$-function of two variables will be required in the sequel:

\begin{equation}
H^*[ax, bx] = \sum_{0, n_1', 1, n_2': 1, n_3'} \left( p_1', q_1': p_2', q_2' + 1 ; p_3', q_3' + 1 \right) \frac{\Gamma(p_1') \Gamma(q_1') \Gamma(p_2') \Gamma(q_2') \Gamma(p_3') \Gamma(q_3')}{\Gamma(p_1 + q_1 + p_2 + q_2 + p_3 + q_3)}
\end{equation}

\begin{align*}
(ax, U_j, \mu_j, U_j)_{1-p_1'} & : (g_3, G_j)_{1-p_2'} ; (k_j, K_j)_{1-p_3'} \\
(bx, \eta_j, V_j)_{1-q_1'} & : (0, 1), (h_j, H_j)_{1-q_2'} ; (0, 1), (l_j, L_j)_{1-q_3'}
\end{align*}
where

\begin{align}
(1.2) \quad h_M &= \sum_{N=0}^{M} \phi'(M - N, N) \theta_1'(M - N) (b/a)^N \binom{M}{N} \\
(1.3) \quad \phi'(\xi, \lambda) &= \sum_{j=1}^{n_1'} \Gamma(1 - u_j + \mu_j \xi + U_j \lambda) \\
& \quad \left[ \prod_{j=n_1'+1}^{p_1'} \Gamma(u_j - \mu_j \xi - U_j \lambda) \right]^{-1} \\
& \quad \left[ \prod_{j=1}^{q_1'} \Gamma(1 - v_j + \gamma_j \xi + V_j \lambda) \right]^{-1} \\
(1.4) \quad \theta_1'(\xi) &= \prod_{j=1}^{n_2'} \Gamma(1 - g_j + G_j \xi) \\
& \quad \left[ \prod_{j=n_2'+1}^{p_2'} \Gamma(g_j - G_j \xi) \right]^{-1} \\
& \quad \left[ \prod_{j=1}^{q_2'} \Gamma(1 - h_j + H_j \xi) \right]^{-1}
\end{align}

and with \( \theta_2'(\lambda) \) defined analogously to \( \theta_1'(\xi) \) in terms of the parameter sets \((k_j, K_j)_{1, p_3'} \), \((1_j, L_j)_{1, q_3'} \).

The \( H \)-function of several complex variables (or the multivariable \( H \)-function) was introduced and studied systematically in a series of papers by Srivastava and Panda (see, e.g., [12] and [13]; see also [11]). We shall employ the following contracted notation of Srivastava and Panda ([13], p. 130, Eq. (1.3)):

\begin{align}
(1.5) \quad H[z_1, \ldots, z_r] &\equiv H_{0, n : m_1, n_1, \ldots, m_r, n_r} \left[ \begin{array}{c}
z_1 \\
p, q : p_1, q_1, \ldots, p_r, q_r \\
\end{array} \right] \\
&\left[ \begin{array}{c}
z_r \\
\end{array} \right]
\end{align}
(a_j; \alpha_j^{'}, \ldots, \alpha_j^{'(r)})_1, p_1; \ldots; (c_j^{'(r)}, \gamma_j^{'(r)})_1, p_r,
(b_j; \beta_j^{'}, \ldots, \beta_j^{'(r)})_1, q_1; \ldots; (d_j^{'(r)}, \delta_j^{'(r)})_1, q_r

to denote the \textit{H}-function of \( r \) complex variables \( z_1, \ldots, z_r \). [See\ Srivastava, Gupta and Goyal ([11], p. 251, Eq. (4.1)) for details of this function.]

With a view to facilitating the derivation of our main integral (2.1) in the next section, we give here an elementary integral contained in the following

\textbf{Lemma. If}

\begin{align*}
(i) \quad & \min_{1 \leq j \leq r} \left\{ \sigma_j, \rho_j, \text{Re} (\alpha), \text{Re} (\beta) \right\} > 0, \\
& \text{Re} (\alpha) + \sum_{j=1}^{r} \sigma_j \xi_j > 0, \text{Re} (\beta) + \sum_{j=1}^{r} \rho_j \xi_j > 0,
\end{align*}

where

\begin{equation}
(1.6) \quad \xi_j = 1 \leq j \leq m_k \left[ \text{Re} (d_j^{(k)} / \delta_j^{(k)}) \right] \quad \forall \; k \in \{1, \ldots, r\},
\end{equation}

and

\begin{align*}
(ii) \quad & U_k > 0, \; | \arg z_k | > \frac{1}{2} \pi U_k \pi,
\end{align*}

where

\begin{align*}
(1.7) \quad U_k &= - \sum_{j=n+1}^{p} a_j^{(k)} - \sum_{j=1}^{q} \beta_j^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} \\
& - \sum_{j=n_k+1}^{p_k} \gamma_j^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} \delta_j^{(k)} \quad \forall \; k \in \{1, \ldots, r\},
\end{align*}
Then

\[ (1.8) \quad \int_0^{\pi/2} \exp \{ i(\alpha + \beta) \theta \} \sin^{\alpha-1} \theta \cos^{\beta-1} \theta \, d\theta \]

\[ = \exp(i\pi\alpha/2) \quad z_1 \exp \{ i(\sigma_1 + \rho_1) \theta \} \sin^{\sigma_1} \theta \cos^{\rho_1} \theta, \ldots, \]

\[ z_r \exp \{ i(\sigma_r + \rho_r) \theta \} \sin^{\sigma_r} \theta \cos^{\rho_r} \theta \]

\[ = \exp (i\pi\alpha/2) \quad H_{\alpha,\beta}^{(1)} [z_1 \exp (i\pi\alpha/2), \ldots, z_r \exp (i\pi\sigma_r/2)] \]

where

\[ (1.9) \quad H_{\alpha,\beta}^{(1)} [z_1, \ldots, z_r] \equiv H_{\alpha,\beta} \quad 0, n + 2 : m_1, n_1 ; \ldots ; m_r, n_r \]

\[ \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \begin{bmatrix} (1 - \alpha \cdot \sigma_1, \ldots, \sigma_r), (1 - \beta \cdot \rho_1, \ldots, \rho_r) \\ (a_j, a_j', \ldots, a_j^{(r)})_{1,p} : (c_j, c_j')_{1,p_1}, \ldots, (c_j^{(r)}, c_j^{(r)})_{1,p_r} \\ (b_j, b_j', \ldots, b_j^{(r)})_{1,q} : (d_j, d_j')_{1,q_1}, \ldots, (d_j^{(r)}, d_j^{(r)})_{1,q_r} \end{bmatrix} \]

**Proof.** To establish (1.8), we first write the Mellin-Barnes contour integral ([11], p. 251, Eq. (4.1)) and change the order of integrations therein. We then apply MacRobert's result ([5], p. 450, Eq. (4)):

\[ (1.10) \quad \int_0^{\pi/2} \exp \{ i(\alpha + \beta) \theta \} \sin^{\alpha-1} \theta \cos^{\beta-1} \theta \, d\theta \]

\[ = \exp (i\pi\alpha/2) \quad B(\alpha, \beta), \]

\{Re(\alpha) > 0, Re(\beta) > 0\},

in order to evaluate the \(\theta\)-integral, and interpret the resulting contour
integral as the multivariable $H$–function; the integral (1.8) follows at once.

2. THE MAIN INTEGRALS

We shall establish the following two general and finite integrals:

\[ \frac{\pi}{2} \int_0^\infty \exp \left\{ i(a + \beta) \theta \right\} \sin^{a-1} \theta \cos^{b-1} \theta \]

\[ H^* \left[ a \exp \left\{ i(\sigma + \rho) \theta \right\} \sin^{\sigma} \theta \cos^{\rho} \theta \right] \]

\[ b \exp \left\{ i(\sigma + \rho) \theta \right\} \sin^{\sigma} \theta \cos^{\rho} \theta \]

\[ H \left[ z_1 \exp \left\{ i(\sigma_1 + \rho_1) \theta \right\} \sin^{\sigma_1} \theta \cos^{\rho_1} \theta \right] \ldots , \]

\[ z_r \exp \left\{ i(\sigma_r + \rho_r) \theta \right\} \sin^{\sigma_r} \theta \cos^{\rho_r} \theta \]

\[ d\theta \]

\[ = \sum_{M=0}^{\infty} h_M \frac{(-a)^M}{M!} \exp \left\{ i \pi (a + \sigma M)/2 \right\} \]

\[ H \left[ z_1 \exp \left( i \pi \sigma_1/2 \right) , \ldots , z_r \exp \left( i \pi \sigma_r/2 \right) \right] \]

The integral (2.1) is valid if the following (sufficient) conditions are satisfied:

(i) The sets of conditions (i) and (ii) mentioned with the Lemma hold true

(ii) $\rho > 0$, $\sigma > 0$, $U \geq 0$, and $V \geq 0$, where

\[ U = 1 + \sum_{j=1}^{q_1'} \eta_j + \sum_{j=1}^{q_2'} H_j - \sum_{j=1}^{p_1'} \mu_j - \sum_{j=1}^{p_2'} G_j \]
(2.3) \( V = 1 + \sum_{j=1}^{q_1'} V_j + \sum_{j=1}^{q_3'} L_j - \sum_{j=1}^{p_1'} U_j - \sum_{j=1}^{p_3'} K_j \)

where each of the equalities holds when the variables are suitably constrained.

(iii) The series occurring on the right-hand side of (2.1) is absolutely convergent.

(2.4) \[ \int_0^1 \frac{x^{a-1} (t - x)^{\beta-1} (1 + Bx)^{-\alpha-\beta}}{(1 + Bx)^{\sigma+\rho}} \cdot \left[ H^* \frac{a x^\sigma (t - x)^\rho}{(1 + Bx)^{\sigma+\rho}}, \frac{b x^\sigma (t - x)^\rho}{(1 + Bx)^{\sigma+\rho}} \right] \]

\[ H \left[ z_1 \left( \frac{x}{1+Bx} \right)^{\sigma_1}, ..., z_r \left( \frac{x}{1+Bx} \right)^{\sigma_r} \right] dx \]

\[ = \frac{t^{\alpha+\beta-1}}{(1+Bl)^x} \sum_{M=0}^{\infty} h_M \frac{(-a)^M}{M!} \left( \frac{t^{\sigma+\rho}}{(1+Bl)^{\sigma+\rho}} \right)^M \]

\[ H \left[ \frac{1}{a+\sigma M, \beta+\rho M} \left[ z_1 \frac{t^{\sigma_1+\rho_1}}{(1+Bl)^{\sigma_1}}, ..., z_r \frac{t^{\sigma_r+\rho_r}}{(1+Bl)^{\sigma_r}} \right] \right] \]

The integral formula (2.4) is valid under the following conditions:

(i) \( Bl > -1 \)

(ii) conditions (i) and (ii) given with the Lemma hold true

(iii) The series occurring on the right-hand side of (2.4) is absolutely convergent.
In (2.1) and (2.4), $H^*$-function, $H^{(1)}_{\alpha, \beta}$-function and $h_M$ are defined by (1.1), (1.9) and (1.2), respectively.

**Derivation of the integrals.** To establish the integral (2.1), we use the series expansion (1.1) for $H^*$-function occurring in the integrand, change the order of integration and summation (which is justified under the conditions given with (2.1) and evaluate the resulting $\theta$-integral with the help of (1.8), we easily arrive at the right hand side of (2.1).

The proof of the integral formula (2.4) is similar to that of the integral (2.1) with the only difference that here we use a known integral due to Agal and Koul ([1], p. 14, Eq (2.1)) instead of (1.8).

**3 Special Cases** The multivariable $H$-function is an extension of the $G$-function of several complex variables, and includes Fox's $H$-function of one and two variables, Meijer's $G$-function of one and two variables, the generalized Lauricella function of Srivastava and Daoust [16], Appell functions, the Whittaker functions, etc. (cf.[11]). Therefore, our integrals can be suitably applied to evaluate the corresponding integrals involving these functions simply by specializing the parameters of the multivariable $H$-functions.

On the other hand, on specialization of the parameters of the $H^*[ax, bx]$ suitably, several integrals involving the various products of elementary functions of one or two variables and the multivariable $H$-function can be easily obtained. Thus, for example, if we put $n_1' = p_1'$, $n_2' = p_2'$, $n_3' = p_3'$ in (2.1) and use the known result ([11], p. 88, Eq. (6. 4. 2)), we get
(3.1) \[
\int_0^{\pi/2} \exp \{i(\alpha + \beta) \theta\} \sin^{\alpha-1} \theta \cos^{\beta-1} \theta, \]

\[S[a \exp \{i(\sigma + \rho) \theta\} \sin^\sigma \theta \cos^\rho \theta, b \exp \{i(\sigma + \rho) \theta\} \sin^\sigma \theta \cos^\rho \theta] \]

\[H[z_1 \exp \{i(\sigma_1 + \rho) \theta\} \sin^{\sigma_1} \theta \cos^{\rho_1} \theta, \ldots, \]

\[z_r \exp \{i(\sigma_r + \rho_r) \theta\} \sin^{\sigma_r} \theta \cos^{\rho_r} \theta \] \] \[= \sum_{M=0}^{\infty} h' M \frac{\alpha^M}{M!} \exp \{i \pi (\alpha + \sigma M)/2\} \]

\[H^{(1)}_{\alpha + \sigma M, \beta + \rho M} [z_1 \exp (i \pi \sigma_1/2), \ldots, z_r \exp (i \pi \sigma_r/2)] \]

where

(3.2) \[S[x, y] \equiv S \]

\[p_1', \ p_2'; \ p_3' \quad \left[ (u_j; \mu_j, U_j)_{1,p_1'}, \right. \]

\[q_1', q_2'; q_3' \quad \left[ (v_j; \eta_j, V_j)_{1,q_1'}, \right. \]

\[(g_j, G_j)_{1,p_2'}, (k_j, K_j)_{1,p_3'}, (l_j, L_j)_{1,q_2'}; x \ y \] \]

stands for the (Srivastava and Daoust) generalized Kampé de Fériet function ([9], p. 199, Eq. (2.1)),

(3.3) \[h_M' = \sum_{N=0}^{M} \phi''(M - N, N) \theta_1''(M - N) \theta_2''(N) (b/a)^N \left( \begin{array}{c} M \\ N \end{array} \right), \]

\[\phi''(\xi, \lambda) = | \phi'(\xi, \lambda) | \left| n_1' = p_1' \right., \]

\[\theta_1''(\xi) = | \theta_1'(\xi) | \left| n_2' = p_2' \right. \]

and
\[ \theta_2''(\lambda) = \left. \theta_2'(\lambda) \right|_{n_3'} = p_3'. \]

The condition of validity for the integral are the same as mentioned with the main integral (2.1).

Again, setting \( n_1' = p_1' = q_1' = 0 \) in (2.1) and using the known result ([10], p. 90, Eq. (6.4.15)), we arrive at the following interesting integral:

\[
\int_0^{\pi/2} \exp \{ i(\alpha + \beta) \theta \} \sin^{\alpha-1} \theta \cos^{\beta-1} \theta \\
\times \left[ \begin{array}{c}
H_{n_2'}^{1, n_2'} \left[ a \exp \{ i(\sigma + \rho) \theta \} \sin^\sigma \cos^\rho \theta \right] (g_1, G_1)_{1, p_2'} \\
(0, 1), (l_1, H_1)_{1, q_2'}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
H_{n_3'}^{1, n_3'} \left[ b \exp \{ i(\sigma + \rho) \theta \} \sin^\sigma \cos^\rho \theta \right] (k_1, K_1)_{1, p_3'} \\
(0, 1), (l_2, L_1)_{1, q_3'}
\end{array} \right]
\]

\[
H \left[ z_1 \exp \{ i(\sigma_1 + \rho_1) \theta \} \sin^{\sigma_1} \theta \cos^{\rho_1} \theta , \ldots , \\
z_r \exp \{ i(\sigma_r + \rho_r) \theta \} \sin^{\sigma_r} \theta \cos^{\rho_r} \theta \right] d \theta
\]

\[
= \sum_{M=0}^{\infty} h_M'' \left( -\frac{a}{M!} \right)^M \exp \{ i \pi (\alpha + \sigma M)/2 \}
\]

\[
H^{(1)}_{\alpha + \sigma M, \beta + \rho M} \left[ z_1 \exp \left( i \pi \sigma_1/2 \right) , \ldots , z_r \exp \left( i \pi \sigma_r/2 \right) \right]
\]

where

\[
(3.5) \quad h_M'' = \sum_{N=0}^{M} \theta_1' (M - N) \theta_2 (N) (b/a)^N (M/N)
\]
Similar type of integrals can be obtained from the integral (2.4) also.

Lastly, if we put $n_3' = p_3' = q_3' = 0$ in (3.4) and let $b \to 0$ therein, we arrive at the known result due to Garg ([3], p. 136, Eq. (2.1)). The results established earlier by Rathie ([8], p. 237, Eq. (2.3)), Vasishat and Goyal [14], p. 13, Eq. (1.1)), and others, can also be deduced as special cases of (2.1). Also, the known integrals due to Garg ([2], p. 23, Eq. (2.1)), Agal and Koul ([1], p. 14, Eq. (2.1)), Prasad and Singh ([7], p. 126, Eq. (2.1), and Srivastava and Panda ([13], p. 131, Eq. (2.2)), follow easily as special cases of our second integral (2.4).

**Acknowledgements**

The author is grateful to Dr. S. P. Goyal for his generous help and guidance in the preparation of this paper. He is also thankful to Prof. M. C. Gupta and Prof. H. M. Srivastava (Canada) for their very valuable suggestions and encouragement.

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CERATIN APPLICATIONS OF JACKSON'S SUMMATION FORMULA

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(Received: January 17, 1983; Revised: March 12, 1984)

ABSTRACT

In this note we show that the $q$-Watson and $q$-Whipple summations due to Andrews are the particular cases of the well known Jackson's formula for summing $\phi_7$. We also obtain the identities of Rogers-Ramanujan type related to the modulus 23.

1. INTRODUCTION

If we let

$$
|q| < 1, [a; q]_n = (1-a)(1-aq) \cdots (1-aq^{n-1}), [a; q]_0 = 1,
$$

(1.1)

$$
[a; q]_\infty = \prod_{j=0}^{\infty} (1 - aq^j)
$$

(1.2)

then the basic hypergeometric series may be defined by

$$
\Phi_{p+r} \left[ \begin{array}{c}
  a_1, \ldots, a_{p+1} ; q; x \\
  b_1, \ldots, b_{p+r}
\end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1; q]_n \cdots [a_{p+1}; q]_n (-x^n q^n)^{n(n+1)/2}}{[q; q]_n [b_1; q]_n \cdots [b_{p+r}; q]_n}.
$$

(1.3)

where the series converges for all positive integral values of $r$ and for
all $x$, except that, when $r = 0$, it converges only for $|x| < 1$. Further, we shall simply write

$$
\prod_{j=0}^{\infty} \frac{(a - a_j q^j) \ldots (1 - a_r q^r)}{(1 - b_1 q^j) \ldots (1 - b_s q^r)}
$$

On the other hand, the well-poised series

$$
\,_{p+3} \Phi_{p+2} \left( \begin{array}{c} a, q \sqrt{a}, -q \sqrt{a}, b_1, \ldots, b_p; \ b; z \\ a, -\sqrt{a}, aq/b_1, \ldots, aq/b_p \end{array} \right)
$$

will be abbreviated by

$$
\,_{p+2} W_{p+2} [a; b_1, \ldots, b_p; q; z].
$$

F. H. Jackson studied the concept of the basic number at length. He gave the basic analogues of most of the special summation theorems. In particular, he proved the basic analogues of Dougall’s theorem as early as 1909. But he did not give the $q$–analogues of Watson’s and Whipple’s summation formulas. Andrews [1] used Watson’s $q$–analogue of Whipple’s theorem [10] and obtained the $q$–analogues of the terminating versions of Watson’s and Whipple’s summation theorems. We show here that the $q$–Watson and $q$–Whipple formulas of Andrews [1] are special cases of Jackson’s theorem [7; 3. 1. 1] for $\,_{8} \Phi_{7}$. We also obtain a natural extension of a formula of Verma and Jain [8; 1. 3] in the form.

$$
\,_{12} W_{11} \left[ a; a^2 q/b^2, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, q^{-m}, -q^{-m}, q; q \right] = \frac{[a^2 q^2; q^2]_m [b^2 q^2/c^2; q^2]_m [b^2 q^2/d^2; q^2]_m [a^2 q^2/c^2 d^2; q^2]_m}{[b^2 q^2; q^2]_m [a^2 q^2/c^2; q^2]_m [a^2 q^2/d^2; q^2]_m [b^2 q^2/c^2 d^2; q^2]_m}
$$
which is a $q$–analogue of a known formula [2; 8. 11] (on replacing $a$ by $-a$). In fact, writing the series in reverse order on both the sides of (1.1) one may get a $q$–analogue of a well–known formula [2; 8. 1] which will be different in nature from a transformation of Bailey [3]. Similarly, we also obtain the $q$–analogues of the formulas [2; 8. 21, 8. 31, 8. 41] which are different in nature than the known ones [3] and [5; 4. 6]. Lastly, as applications of (1.1), we deduce identities of Rogers–Ramanujan type related to the modulus 23.

2. SPECIALIZATIONS OF JACKSON’S FORMULA

For $c = -d = \sqrt{a}$, Jackson’s summation formula [7; 3. 3. 1. 1]

$$\p{8}{7} [a; b, c, d, a^2 q^{1+n}/bcd, q^{-n}, q; q]$$

reduces to

$$\frac{[aq; q]_n [aq/bc; q]_n [aq/bd; q]_n [aq/cd; q]_n}{[aq/b; q]_n [aq/c; q]_n [aq/d; q]_n [aq/bcd; q]_n}$$

(2.1)

(2.2) is a $q$–analogue of the terminating version of Dixon’s theorem [7; 2. 3. 3. 6].

Transforming the $\phi_3$ on the left–hand side of (2.2) by the formula [6]
\[4 \Phi_3 \left[ a, b, c, q^{-n}; q; q \right] = \frac{[e/c; q]_n [de/ab; q]_n}{[e; q]_n [de/abc; q]_n}\]

where \(abcq^{1-n} = def\) (with \(d \rightarrow aq^{1+n}, e \rightarrow aq/b, f \rightarrow bq^{-n}, c \rightarrow b, b \rightarrow -aq^{1+n}/b\)) we get

\[4 \Phi_3 \left[ d/a, d/b, c, q^{-n}; q; q \right] = \frac{[d/abc; q]_n}{[d/e; q]_n [de/abc; q]_n} (2.3)\]

(2.4) is q–Whipple’s theorem due to Andrews [1].

Next, transforming the \(4 \Phi_3\) on the left-hand side of (2.2) by (2.3) (with \(a \rightarrow -aq^{1+n}/b, c \rightarrow a, d \rightarrow aq/b, e \rightarrow aq^{1+n}, f \rightarrow bq^{-n}\)) we get a known formula [4; 3. 17]. On the other hand transforming the \(4 \Phi_3\) (with \(b \rightarrow aq^{1+n}/b, c \rightarrow b, d \rightarrow aq/b, e \rightarrow aq^{1+n}, f \rightarrow bq^{-n}\)) one may obtain another known formula [4; 3. 19].

Now, in the non-terminating version of (2.1) i.e. [7; p. 248(15)], setting \(c = -d = \sqrt{a}\), we get a non-terminating version of (2.2) in the form

\[4 \Phi_3 \left[ a, b, -aq/bf, f; q; q \right] + \Pi \left[ a, f, b/a, bq/f, -aq/bf, -b^2f/a; q \right] = \Pi \left[ b/a, -b, q, q \right] \cdot 4 \Phi_3 \left[ b^2/a, b, -q/f, bf/a; q ; q \right] = \Pi \left[ b^2f^2/a, aq^2/f^2, aq; q^2 \right] (2.5)\]
(2.5) for \( a = q^{-n} \) reduces to

\[
\Phi_3 \begin{bmatrix} q^{-n}, b, -q^{-n}/bf, f; q \end{bmatrix} = \frac{[bf; q]_n (-f)^n q^{n(n-1)/2}}{[b; q]_n [f; q]^n}.
\]

Transforming the \( \Phi_3 \) on the left hand side by (2.3) (with \( a \rightarrow -q^{-n}/bf, c \rightarrow f, d \rightarrow q^{-n}/b, e \rightarrow -bf, f \rightarrow q^{-n}/f \)) we get \( q \)-Watson's summation formula due to Andrews [1].

3. PROOF OF (1.4)

Multiply to the following form of (2.1)

\[
W_7 [a; a^2q/b^2, bq^n, -bq^n, -q^{-n}, q^{-n}, q; q]
\]

\[
= \frac{b^{2n} [a^2q^2; q^2]_n [a^2q^2/b^2; q^2]_n [-b^2/a; q]_{2n}}{(qa^2)^n [-aq; q]_{2n} [b^4/a^2; q^2]_n [b^2/a^2; q^2]_n}
\]

by

\[
\frac{[b^2; q^2]_n [b^2/a^2; q^2]_n [c^2; q^2]_n [d^2; q^2]_n [b^2q^2; q^4]_n [q^{-2m}; q^2]_n}{[q^2; q^2]_n [a^2q^2; q^2]_n [b^2q^2/c^2; q^2]_n [b^2q^2/d^2; q^2]_n [b^2; q^4]_n}.
\]

and summing with respect to \( n \) from 0 to \( m \), we get

\[
W_9 [b^2, -b^2/a, -b^2q/a, a^2q^2/b^2, c^2, d^2, a^2b^2q^2+2m/c^2d^2, q^{-2m}; q^2; b^2q/a^2]
\]

\[
= \sum_{r=0}^{m} \frac{[a; q]_r [aq^2; q^2]_r [a^2q/b^2; q]_r [b^2q^2; q^2]_{2r} [c^2; q^2]_r}{[q; q]_r [a; q^2]_r [b^2/a; q]_r [a^2q^2; q^2]_{2r} [b^2q^2/c^2; q^2]_r}.
\]
\[
\frac{[d^2; q^2], [a^2b^2d^2q^2+2m/c^2d^2; q^2], [q^{-2m}; q^2], q^r b^{2r}}{[b^2q^2/d^2; q^2], [c^2d^2q^{-2m}/a^2; q^2], [b^2q^2+2m; q^2], a^{2r}}.
\]

\[8W_7 [b^2q^2; b^2/a^2, c^2q^2r, d^2q^2r, a^2b^2q^2+2m+2r/c^2d^2, q^{-2m+2r}; q^2; q^2]\]
on summing the inner \(8W_7\) by (2.1), we have (1.1) on some simplification.

In (1.1) setting \(c^2 = -aq^2, d = -aq\) and then summing the resulting \(6\Phi_6\) on the right hand side, we get (on replacing \(b\) by \(b\sqrt{q}\))

\[(1 = \sqrt{-1})\]

\[
10W_9 [a; a^2/b^2, iq\sqrt{a}, -iq\sqrt{a}, bq^m, -bq^m, -q^{-m}, q^{-m}; q; q]
\]

\[= \frac{[a^2q^2; q^2]_m [a^2/b^2; q^2]_m [-b^2q/a; q]_2 b^{2m}}{[b^2q^2/a^2; q^2]_m [b^2/a^2; q^2]_m [-a; q]_2 a^{2m} q^m}\]

\[(3.2)\]

(3.2) is a \(q\)-analogue of the formula \([2; \S 8 (B)]\).

Next, we prove the following three transformations

\[
12W_11 [a; a^2/b^2, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, -q^{-m}, q^{-m}; q; q^2]
\]

\[= \frac{[a^2b^2; q^2]_m [b^2q^2/c^2; q^2]_m [b^2q^2/d^2; q^2]_m [a^2q^2/c^2d^2; q^2]_m}{[b^2q^2; q^2]_m [a^2q^2/c^2; q^2]_m [b^2q^2/c^2d^2; q^2]_m}.
\]

\[
.10W_9 [b^2; a^2/b^2, -b^2/a, -b^2q/a, c^2, d^2, a^2b^2q^{2+2m}/c^2d^2, q^{-2m}; q^2; b^2/q^3]
\]

\[= \frac{a^2q^2/c^2d^2; q^2]_m}{[a^2q^2/c^2; q^2]_m [b^2q^2/c^2d^2; q^2]_m}
\]

\[(3.3)\]

\[
14W_13 [a; a^2/b^2, iq\sqrt{a}, -iq\sqrt{a}, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, -q^{-m}, q^{-m}; q; q]
\]
\[
= \frac{[a^2q^2; q^2]_m [b^2q^2/c^2; q^2]_m [b^2q^2/d^2; q^2]_m [a^2q^2/c^2d^2; q^2]_m}{[b^2q^2; q^2]_m [a^2q^2/c^2; q^2]_m [a^2q^2/d^2; q^2]_m [b^2q^2/c^2d^2; q^2]_m}.
\]

\[
10W_9 [b^2; -b^2q/a, -b^2q^2/a, a^2/b^2, c^2, d^2, a^2b^2q^2+2m/c^2d^2, q^{-2m}; q^2; b^2q/a^2]
\]

(3.4)

and

\[
14W_{13} [a; a^2/b^2q, iq\sqrt{a}, -iq\sqrt{a}, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, \\
-\sqrt{q^m}, q^{-m}; q; q^2]
\]

\[
= \frac{[a^2q^2; q^2]_m [b^2q^2/c^2; q^2]_m [b^2q^2/d^2; q^2]_m [a^2q^2/c^2d^2; q^2]_m}{[b^2q^2; q^2]_m [a^2q^2/c^2; q^2]_m [a^2q^2/d^2; q^2]_m [b^2q^2/c^2d^2; q^2]_m}.
\]

\[
10W_9 [b^2; a^2/b^2q^2, -b^2q/a, -b^2q^2/a, c^2, d^2, a^2b^2q^2+2m/c^2d^2, q^{-2m}; \\
q^2; \frac{b^2q^2}{a^2}]
\]

(3.5)

**Proof of (3.3)**: We have a summation formula [9; 3.1]

\[
8W_7 [a; a^2/b^2, bq^n, -bq^n, -q^{-n}, q^{-n}; q; q^2]
\]

\[
= \frac{[a^2q^2; q^2]_n [a^2/b^2; q^2]_n [-b^2/a; q]_m b^{2n} q^n}{[b^2/a^2; q^2]_n [b^4q^2/a^2; q^2]_n [-aq; q]_m a^{2n}}.
\]

(3.6)

**Proof of (3.3)** follows on the lines of the proof of (1.1) on using (3.6) instead of (4.1).

However, from (3.3) one may obtain the summation formula

\[
10W_9 [a; a^2/b^2q, iq\sqrt{a}, -iq\sqrt{a}, bq^n, -bq^n, -q^{-n}, q^{-n}; q; q^2]
\]
Proofs of (3.4) and (3.5) follow on the lines of the proof of (1.1) on using (3.2) and (3.7) respectively instead of (3.1).

(3.3)-(3.5) are \( q \)-analogues of the formulae of Bailey \([2; 8.31, 8.21, 8.41]\).

4. FURTHER TRANSFORMATIONS

We begin this section by proving the following transformations:

\[
\sum_{n, m, r=0}^{\infty} \frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 - 4p(n+m+r)}}{[q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-aq; q]_2r [-a^2q^2; q^2]_{2m+2r}} \;
\]

\[
= \sum_{j=0}^{P} \frac{q^{-4n} q^4 (-)^j a^4 j q^{2j(s+1)}}{[q^4; q^4]_j} \sum_{n=0}^{\infty} \frac{[a; q]_n (1-aq^{2n})^n (-)^n a^{11n}}{[q; q]_n (1-a)}
\]

\[
q^{n(23n-1-8p+16j)/2}
\]

and

\[
\sum_{n, m, r=0}^{\infty} \frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 - 4p(n+m+r) + 4(n+2m+2r)}}{[q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-aq; q]_{2r}} \;
\]

\[
= \sum_{j=0}^{P} \frac{q^{-4n} q^4 (-)^j a^4 j q^{2j(s+1)}}{[q^4; q^4]_j} \sum_{n=0}^{\infty} \frac{[a; q]_n (1-aq^{2n})^n (-)^n a^{11n}}{[q; q]_n (1-a)}
\]

\[
q^{n(23n-1-8p+16j)/2}
\]
\[ \sum_{j=0}^{p} \frac{[q^{-4p}; q^4]_j (-)^j a^4 j q^{2(j+3)}}{[q^4; q^4]_j} \cdot \sum_{n=0}^{\infty} \frac{[aq; q]_n (1-a^4 q^{8n+4})}{[q; q]_n} \cdot a^{11n} (-)^n q^{n(23n + 15 - 8p + 16j)/2} \]

(4.2)

**Proof of (4.1)** (1.1) for \( d^2 = -q^{-2m} \), \( b \to \infty \), \( c \to \infty \) yields

\[ \sum_{r=0}^{m} \frac{[a; q]_r (1-aq^{2r}) (-)^r a^{3r} q^{r(7r-1)/2}}{[q; q]_r (1-a) [a^4 q^4; q^4]_{m+r} [q^4; q^4]_{m-r}} = \frac{1}{[-a^2 q^2; q^2]_{2r}} \sum_{r=0}^{m} \frac{a^{2r} q 2r^2}{[q^2; q^2]_r [-a; q]_{2r} [q^4; q^4]_{m-r}} \]

(4.3)

Multiply to (4.3) by \( \frac{[x; q^4]_m [y; q^4]_m [q^{-4n}; q^4]_m q^{4m}}{[xyq^{-4n}/a^4; q^4]_m} \) and sum with respect to \( m \) from 0 to \( n \), we get on using \( q \)-analogue of Saalschutz summation theorem [7; 3. 3. 2. 2] and then letting \( x, y \to \infty \):

\[ \sum_{r=0}^{n} \frac{[a; q]_r (1-aq^{2r}) (-)^r a^{7r} q^{r(15r-1)/2}}{[q; q]_r (1-a) [a^4 q^4; q^4]_{n+r} [q^4; q^4]_{n-r}} = \sum_{r, m \geq 0} \frac{a^{4m+6r} q^{4(m+r)^2 + 2r^2}}{[q^2; q^2]_r [q^4; q^4]_m [-a; q]_{2r} [-a^2 q^2; q^2]_{2m+2r}} \cdot \frac{1}{[q^4; q^4]_{n-m-r}} \]

(4.4)

Using (4.4) in the formula [8, 2 15]

\[ [aq; q]_{\infty} \sum_{n=0}^{\infty} a^n q^{n(n-p)} p_n = \sum_{j=0}^{p} \frac{[q^{-p}; q]_j (-a)^j q^{j(j+1)/2}}{[q; q]_j} \]
\[ \sum_{n=0}^{\infty} a^n q^{n(n-p+2)} \beta_n \]

where

\[ \beta_n = \sum_{r=0}^{n} \frac{a_r}{[q; q]_{n-r} [aq; q]_{n+r}} \]

for evaluating \(<\beta_n>\) with \(a\) and \(q\) replaced by \(a^4\), \(q^4\) respectively and

\[ a_r = \frac{[aq; q]_r (1-aq^{2r}) (-)^r a^{3r} q^{r(15r-1)/2}}{[q; q]_r (1-a)} \]

we get (4.1).

**Proof of (4.2):** However (4.3) may be written in the form (see [8] for details)

\[ \sum_{r=0}^{m} \frac{[aq; q]_r (1-a^4 q^{3r+4}) (-)^r a^{3r} q^{r(7r-1)/2}}{[q; q]_r [q^4; q^4]_{m-r} [a^4 q^8; q^4]_{m+r}} \]

\[ = \frac{(1-a^4 q^4)}{[-a^2 q^2; q^2]_{2m}} \sum_{r=0}^{m} \frac{a^{2r} q^{2r^2}}{[q^2; q^2]_r [-aq; q]_{2r} [q^4; q^4]_{m-r}} \]

(4.5)

Proof of (4.2) follows on the lines of the proof of (4.1) on using (4.5) instead of (4.3).

**Identities of Roger S–Ramanujan Type Related to Modulus 23**

(4.1) for \(p = 0\), \(a = 1\), \(q\, q^2, q^3, q^4\) and \(q^5\) yields the following identities

\[ \frac{[q^4; q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r=0}^{\infty} \frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2}}{[q^4; q^4]_r [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q]_{2m+2r}} \]

\[ = \prod_{n \equiv 0, 11, 12 \pmod{23}} (1-q^n)^{-1} \]

(4.6)
\[
\frac{[q^4, q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r=0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4n + 8m + 10r} [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+1} \\
= \Pi (1-q^n)^{-1} \\
\text{if } n \equiv 0, 1, 22 \pmod{23}
\]

(4.7)

\[
\frac{[q^4, q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r=0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 2r} [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q^2]_{2m+2r+1} \\
= \Pi (1-q^n)^{-1} \\
\text{if } n \equiv 0, 10, 13 \pmod{23}
\]

(4.8)

\[
\frac{[q^4, q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r=0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4n + 8m + 10r} [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+1} \\
= \Pi (1-q^n)^{-1} \\
\text{if } n \equiv 0, 2, 21 \pmod{23}
\]

(4.9)

\[
\frac{[q^4, q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r=0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 20r + 8n + 16m + 9} [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+2} \\
= \Pi (1-q^n)^{-1} \\
\text{if } n \equiv 0, 10, 13 \pmod{23}
\]

(4.10)
\[
\frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 8m + 4n + 12r}(1 + q^2 + q^{2r+1})}{[q^4; q^4]^n [q^4; q^4]^m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+1}}
\]

\[
= \prod_{n \equiv 0, 3, 20 \pmod{23}} (1-q^n)^{-1} \tag{4.11}
\]

Next, (4.2) for \(a = 1\) and \(p = 0, 1\) gives

\[
\frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4n + 8m + 8r}}{[q^4; q^4]^n [q^4; q^4]^m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q^2]_{2m+2r}}
\]

\[
= \prod_{n \equiv 0, 4, 19 \pmod{23}} (1-q^n)^{-1} \tag{4.12}
\]

Furthermore (4.1) for \(a = 1\), \(p = 1\) gives the following identity (on using (4.12))

\[
\frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4m + 4r}}{[q^4; q^4]^n [q^4; q^4]^m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q^2]_{2m+2r}}
\]

\[
= \prod_{n \equiv 0, 8, 15 \pmod{23}} (1-q^n)^{-1} \tag{4.13}
\]
(4.1) for \( a = q, p = 1 \) reduces to

\[
[q^4; q^4]_\infty \sum_{n, m, r = 0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4n + 6r} [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+1} = \Pi (1 - q^n)^{-1},
\]

\( n \equiv 0, 5, 18 \pmod{23} \)  \( -q^2 \equiv 0, 3, 20 \pmod{23} \) (4.15)

Lastly, (4.11) for \( a = q^3, p = 1 \) gives the following identity (on using (4.15))

\[
[q^4; q^4]_\infty \sum_{n, m, r = 0}^{\infty} q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2 + 4m + 6r} (1 + q^{4m+4n+4r+2}) [q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q^2]_{2m+2r+1} = \Pi (1 - q^n)^{-1},
\]

\( n \equiv 0, 6, 17 \pmod{23} \) (4.16)

Acknowledgements

I would like to express my gratitude to Professor H. M. Srivastava for his suggestions and for redrafting this paper in its present form.

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UNIFORM SPACE VALUED UNIFORMLY ALMOST PERIODIC FUNCTIONS DEPENDING ON PARAMETERS

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(Received: May 31, 1984; Revised: November 2, 1984)

ABSTRACT

Sharma, Reddy and Funakoshi [3] studied some fundamental properties of uniform space valued almost periodic functions depending on parameters. The purpose of this paper is to generalize some results given in [1] on uniform spaces. For uniform spaces and the related nations, see [2]

1. INTRODUCTION AND DEFINITIONS

Let $T$ and $E$ be uniform spaces. We shall study here only, $E$-valued functions defined on $T \times \mathbb{R}$, where $\mathbb{R}$ is the real line, and assume that any function in theorems or definition is continuous.

Definition 1: Let $f(t, x)$ be a real or complex function defined for all real values of $x$. A number $\tau$ is called a $U$ translation number of $f(t, x)$ if

$$
\forall u, b. \quad (f(t, x + \tau), f(t, x)) \in U,
-\infty < x < \infty
$$

Whenever $t \in T$ and $x \in \mathbb{R}$.

We denote the set of all $U$ translation numbers of a function $f(t, x)$ by $E \{U, f(t, x)\}$. 
**Definition 2**: A continuous function \( f(t, x) \) is called uniformly almost periodic in \( x \), uniformly with respect to \( t \in T \), if to each entourage \( U \) of \( R \) the set \( E(U, f(t, x)) \) is relatively dense.

**2. MAIN RESULTS**

We first state and prove

**Theorem 1.** If \( T \) is a compact uniform space and \( f(t, x) \) is an uniformly almost periodic function in \( x \) uniformly with respect to \( t \in T \), when the set \( \{ f(t, x) | t \in T, x \in R \} \) is precompact.

**Proof.** Let \( V \) be an entourage of \( E \). Then there is a symmetric entourage \( U \) of \( E \) with \( U^2 \subseteq V \). Let \( (t_0, x_0) \) be an arbitrary element of \( T \times R \). By the definition of uniformly almost periodicity of \( f(t, x) \), there is a \( U \) translation number \( \tau \) of the set \( E(U, f(t_0, x_0)) \). Obviously, the set

\[
S = \{ f(t, x) | t \in T, x \in [0, k(U)] \}
\]

where \( k(U) \) is a positive number, is compact. So there are \( a_1, ..., a_n \) \( \in S = \bigcup_{i=1}^{n} U(a_i) \) for which we can find \( a_m \) \( m \in \{1, ..., n\} \) such that

\[
f(t_0, x_0 + \tau) \in U(a_m), \text{ since } x_0 + \tau \in [0, k(U)],
\]

or, equivalently,

\[
(f(t_0, x_0 + \tau), a_m) \in U.
\]

Also, since \((f(t_0, x_0 + \tau), f(t_0, x_0)) \in U\),

it follows from the above that

\[
(f(t_0, x_0), a_m) \in U^2 \subset V(a_m).
\]

Therefore, we have
\[{f(t, x) | t \in T, x \in R} \subset \bigcup_{i=1}^{n} V(a_i)\]

which proves the above theorem.

Next we state

**Theorem 2.** If \(T\) is a compact uniform space and \(f(t, x)\) is an uniformly almost periodic function in \(x\) uniformly with respect to \(t \in T\), then the function \(f(t, x)\) is uniformly continuous on \(T \times R\).

**Proof.** Let \(V\) be an entourage of \(E\). Then there exists a symmetric entourage \(U\) of \(E\) with \(U^3 \subset V\). Let \(k(U)\) be the length associated with \(U\) such that \((f(t_1, y_1), f(t_2, y_2)) \in U\) for any \((t_1, y_1), (t_2, y_2)\) belonging to the interval \(T \times [0, 1 + k(U)]\) if only \((t_2, t_1) \in W, |y_2 - y_1| < \delta\) where \(\delta\) is a positive number \(\delta = \delta(U) < 1\). Let now \((t_1, x_1), (t_2, x_2)\) be any two points of \(T \times R\) with \((t_2, t_1) \in W\), where \(W\) is an entourage of \(T\) and \(|x_1 - x_2| < \delta\). There exists a \(U\) translation number \(\tau\) of the set \(E\{U, f(t, x)\}\) such that both the numbers \((t_1, x_1 + \tau), (t_2, x_2 + \tau)\) belong to the interval \(T \times [0, 1 + k(U)]\). we have then

\[(f(t_1, x_1 + \tau), f(t_2, x_2 + \tau)) \in U.\]

On the other hand

\[(f(t, x + \tau), f(t, x)) \in U, \text{ for any } x \in R.\]

Thus

\[(f(t_1, x_1), f(t_2, x_2)) \in U^3 \subset V.\]

Therefore, \(f(t, x)\) is uniformly continuous on \(T \times R\).

This proves the theorem.
3. FURTHER RESULTS

Theorem 3. If \( \{ f_n(t, x) \} \) is a sequence of uniformly almost periodic functions in \( x \) uniformly with respect to \( t \in T \) which is uniformly convergent on \( T \times \mathbb{R} \) to a function \( f(t, x) \) is also uniformly almost periodic in \( x \) uniformly with respect to \( t \in T \).

Proof. Let \( V \) be an entourage of \( E \). Then there exists a symmetric entourage \( U \) of \( E \) with \( U^3 \subset V \). Thus we can choose an \( n_0 \) such that

\[
(f(t, x), f_{n_0}(t, x)) \in U
\]

for every \( (t, x) \in T \times \mathbb{R} \). Non let \( \tau \) be a translation number of \( E\{U, f_{n_0}(t, x)\} \). Then

\[
(f(t, x + \tau), f(t, x)) \in U^3 \subset V,
\]

which shows that

\[
E\{U, f(t, x)\} \supset E\{U, f_{n_0}(t, x)\}.
\]

Thus \( E\{U, f(t, x)\} \) is relatively dense. Consequently, \( f(t, x) \) is uniformly almost period.

Theorem 4. The sum of two uniform space valued uniformly almost periodic functions \( f_1(t, x), f_2(t, x) \) depending on parameters is again uniformly almost periodic function depending on parameters.

Proof. Let \( \tau \) be any \( U \) translation number of the set

\[
E \{U, f_1(t, x)\}, E\{U, f_2(t, x)\}.
\]

then

\[
(f_1(t, x + \tau) + f_2(t, x + \tau) - f_1(t, x) - f_2(t, x)) \in U,
\]

which shows that \( \tau \) is a \( U \) translation number of the set
Thus

\[ E\{U, f_1 (t, x) + f_2 (t, x)\} \supset E\{U, f_1 (t, x)\}. E\{U, f_2 (t, x)\}. \]

Therefore, \( E\{U, f_1 (t, x) + f_2 (t, x)\} \) is relatively dense. Consequently, \( f_1 (t, x) + f_2 (t, x) \) is uniformly almost periodic depending on parameters, which proves the theorem.

**REFERENCES**


THE HARMONIC CESÁRO PRODUCT SUMMABILITY OF THE DERIVED FOURIER SERIES

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(Received: March 5, 1984; Revised [Final]: April 15, 1985)

1. DEFINITIONS AND NOTATIONS

If \( \sigma_n \) denotes the \((C, 1)\) Transform of partial sum \( S_n \) of an infinite series \( \sum_0^\infty U_n \) and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\sigma_{n-k}}{k} = S,
\]

where \( S \) is finite, then the series \( \sum_0^\infty U_n \) is said to be summable \((H, 1)\) \((C, 1)\) to \( S \), or symbolically

\[
\sum_0^\infty U_n = S \ [((H, 1) (C, 1)]
\]

Let \( f(x) \) be a function integrable in the sense of Lebesgue over the interval \((- \pi, \pi)\) and periodic with period \( 2\pi \). Let the Fourier series associated with \( f(x) \) be

\[
\frac{1}{2} a_0 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x)
\]
The derived series of the Fourier series (1.2) is

\[ \sum_{n=1}^{\infty} n (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} n B_n(x) \]

We shall use, for fixed \( x \) and \( s \) the following notations:

\[ \phi(t) = f(x + t) + f(x - t) - 2s \]
\[ g(t) = \frac{f(x + t) + f(x - t)}{4 \sin t/2} \]

2. INTRODUCTION

Siddiqi [3] and Hille and Tamarkin [1] have, respectively, established the following theorems on the harmonic summability of the Fourier series:

Theorem A (Siddiqi [3]). If

\[ \Phi(t) = \int_0^t |\Phi(u)| \, du = O(t/\log 1/t), \text{ as } t \to 0 \]

then \( \sum_{0}^{\infty} A_n(x) = S(H. 1) \).

Theorem B (Hille and Tamarkin [1]). If

\[ \Phi(t) = \int_0^t |\Phi(u)| \, du = O(t), \text{ as } t \to 0, \]

and

\[ \int_{\pi/n}^{\eta} \frac{|\Phi(t) - \Phi(t+\pi/n)|}{t} \log 1/t \, dt = O(\log n), \text{ as } n \to \infty, \]
where $\eta$ is a positive constant, then

$$\sum_{0}^{\infty} A_n(x) = S(H, 1).$$

Since the derived Fourier series is neither $(C, 1)$ summable nor $(H, 1)$ under the hypothesis of Theorems A and B, respectively. Therefore it is natural to expect the extensions of Theorems A and B to the product summability $(H, 1)$ $(C, 1)$ of the derived Fourier series under analogous conditions with $\phi(t)$ replaced by $g(t)$, with this point of view, we prove here

**Therefore 1. If**

$$G(t) = \int_{0}^{t} |g(u)| du = o(t | \log 1/t), \text{ as } t \to 0,$$

then the derived Fourier series (1.3) is summable $(H, 1)$ $(C, 1)$ to zero, at the point $x$.

**Theorem 2. If**

$$G(t) = \int_{0}^{t} |g(u)| du = o(t), \text{ as } t \to 0,$$

and

$$\int_{\pi/n}^{\eta} \frac{|g(t) - g(t + \pi/n)|}{t} \log 1/t dt = o(\log n), \text{ as } n \to \infty.$$

Where $\eta$, is a positive constant, then the derived Fourier Series (1.3) is summable $(H, 1)$ $(C, 1)$ to zero, at the point $x$.

**3. Proof of Theorem 1.**

Following Sachan [2] the $(C, 1)$ transform $\{\sigma_{n+1}\}$
of the sequence \( \{ S_{n+1} \} \) of partial sums of the derived Fourier series is given by

\[
\sigma_{n+1} = \frac{2}{\pi} \left[ \frac{\delta}{0} g(t) \left[ \frac{2(1 - \cos(n+1)t)}{(n+1)t^2} + \frac{\sin(n+1)t}{(n+1)t} \right] \right] dt + o(1)
\]

We choose \( \delta > 0 \) so small that for \( t \in (0, \delta) \) the condition (2.3) is satisfied.

Denoting the harmonic transform of \( \sigma_{n+1} \), i.e., the \((H.1), (C.1)\) transform of \( S_{n+1} \) by \( H_n \), we have, by the regularity of the method of summation,

\[
H_n = \frac{1}{\pi \log n} \left[ \frac{\delta}{0} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1)} \right]
\]

\[
\left[ \frac{2(1 - \cos rt)}{rt^2} + \frac{\sin rt}{rt} - \frac{\sin rt}{t} \right] dt + o(t)
\]

\[
= \frac{2}{\pi \log n} \left[ \frac{\pi/n}{0} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1)} \right]
\]

\[
\left[ \frac{2(1 - \cos rt)}{rt^2} + \frac{\sin rt}{rt} - \frac{\sin rt}{t} \right] dt
\]

\[
+ \frac{2}{\pi \log n} \left[ \frac{\delta}{\pi/n} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1)} \left[ \frac{2(1 - \cos rt)}{rt^2} + \frac{\sin rt}{rt} \right] dt \right]
\]

\[
- \frac{2}{\pi \log n} \left[ \frac{\delta}{\pi/n} g(t) \sum_{r=1}^{n} \frac{\sin rt}{(n-r+1)t} \right] dt + o(t)
\]
\[(3.1) \quad = 2/\pi \left[ P + Q - R \right] + O(1), \quad \text{say}\]

Since for \( t > 0 \),
\[
\frac{2(1 - \cos rt)}{t^2} + \frac{\sin rt}{rt} - \frac{\sin tr}{t} = O(r)
\]

We obtain
\[
|P| = \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r(n-r+1)} \int_{0}^{\pi/n} |g(t)| O(r) \, dt
\]
\[
= \frac{1}{\log n} O(n \log n) \int_{0}^{\pi/n} |g(t)| \, dt
\]

\[(3.2) \quad = o(1), \quad \text{as} \quad n \to \infty.
\]

By an integration by parts, we obtain
\[
|Q| = \frac{1}{\log n} \left| \sum_{r=1}^{n} \frac{1}{r(n+1-r)} \int_{\pi/n}^{\delta} g(t) \left[ \frac{2(1 - \cos rt)}{t^2} + \frac{\sin rt}{t} \right] \, dt \right|
\]
\[
= O \left( \frac{1}{(n+1)\log n} \right) \sum_{r=1}^{n} \left\{ \frac{1}{r} + \frac{1}{n+1-r} \right\} \int_{\pi/n}^{\delta} \frac{|g(t)|}{t^2} \, dt
\]

\[(3.3) \quad = o(1), \quad \text{as} \quad n \to \infty.
\]

Since for \( t \in (\pi/n, \delta) \),
\[
\sum_{1}^{n} \frac{\sin rt}{r(n-r+1)} = O(\log 1/t)
\]

By an integration by parts, we get
\[
|R| = O \left( \frac{1}{\log n} \right) \int_{\pi/n}^{\delta} \frac{|g(t)|}{t} \log (1/t) \, dt
\]
\[
= O \left( \frac{1}{\log n} \left\{ \left( \frac{G(t)}{t} \log \frac{1}{t} \right) \frac{1}{\pi/n} + \int_{\pi/n}^{\delta} G(t) \left( \frac{1 + \log \frac{1}{t}}{t^2} \right) dt \right\} \right)
\]

\[
= O \left( \frac{1}{\log n} \left[ o(1) + o(1) \left\{ \int_{\pi/n}^{\delta} \left( \frac{1}{t \log \frac{1}{t}} + \frac{1}{t} \right) dt \right\} \right] \right)
\]

\[
= o(1) + O \left( \frac{1}{\log n} \left[ o(\log \log n) + o(\log n) \right] \right)
\]

(3.4) \quad = o(1), \text{ as } n \to \infty.

Finally, from (3.1), (3.2), (3.3) and (3.4), we find

\[ H_n = o(1), \text{ as } n \to \infty \]

This completes the proof of Theorem 1.

4. Proof of Theorem 2

It may be noted that from the proof of Theorem 1 that \( P, Q = o(1) \) under the hypothesis (2.5). Thus to prove Theorem 2, it is sufficient to show that \( R = o(1) \), under the hypothesis (2.5) and (2.6).

We write

\[
R = \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{g(t)}{t} \sin(n+1)t \sum_{r=1}^{n} \frac{\cos rt}{r} dt
\]

\[
- \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{g(t)}{t} \cos(n+1)t \sum_{r=1}^{n} \frac{\sin rt}{r} dt
\]

\[ = R_1 - R_2, \text{ say} \]

(2.6) and an integration by parts give
\[ |R_2| = \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{|g(t)|}{t} O(1) \, dt \]

\[ = o(1), \text{ as for (3.3)}. \]

Further, we write

\[ 2 R_1 = \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{g(t)}{t} \sin (n+1)t \log (1/2 \sin t/2) \, dt \]

\[ = \frac{1}{\log n} \left[ \int_{\pi/n}^{\delta} \frac{g(t)}{t} \sin nt \log 1/t \, dt + o(1) \right] \]

\[ = \frac{1}{\log n} \left[ \int_{0}^{\pi/n} \frac{g(t)}{t} \log (1/t + \pi/n) \sin nt \, dt + o(1) \right] \]

\[ = \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{g(t) - g(t + \pi/n)}{n} \log (1/t) \sin nt \, dt \]

\[ + \frac{1}{\log n} \int_{\pi/n}^{\delta} \frac{g(t + \pi/n)}{t} \left[ \log 1/t - \log 1/t + \pi/n \right] \sin nt \, dt \]

\[ + \frac{1}{\log n} \int_{\pi/n}^{\delta} g(t + \pi/n) \left[ 1/t - \frac{1}{(t + \pi/n)} \right] \log (1/t + \pi/n) \sin nt \, dt \]

\[ - \frac{1}{\log n} \int_{0}^{\pi/n} \frac{g(t + \pi/n)}{t + \pi/n} \log (1/t + \pi/n) \sin nt \, dt \]

\[ + \frac{1}{\log n} \int_{\delta - \pi/n}^{\delta} \frac{g(t)}{t} \log 1/t \sin nt \, dt + o(1) \]
\[ L_1 + L_2 + L_3 - L_4 + L_5 + o(1) \]

By (2.6), we get
\[ |L_1| = o(1), \quad \text{as } n \to \infty \]

An integration by parts, we have
\[
|L_2| = \frac{1}{\log n} \int_{\pi/n}^{\delta-\pi/n} \left| \frac{g(t+\pi/n)}{t} \right| \log (1+\pi/nt) \, dt
\]
\[
= O \left( \frac{1}{\log n} \right) \int_{\pi/n}^{\delta-\pi/n} \left| \frac{g(t+\pi/n)}{t} \right| \, dt
\]
\[
= o(1), \quad \text{as } n \to \infty. \]

Again, on integration by parts, we obtain
\[
|L_3| < \int_{\pi/n}^{\delta-\pi/n} \left| \frac{g(t+\pi/n)}{t^2} \right| \, dt
\]
\[
= O(1/n) \int_{\pi/n}^{\delta-\pi/n} \left| \frac{g(t+\pi/n)}{t^2} \right| \, dt
\]
\[
= O(1/n) \left[ \left\{ o(t+\pi/n) \frac{1}{t^2} \right\} \int_{\pi/n}^{\delta-\pi/n} + \int_{\pi/n}^{\delta-\pi/n} o(t+\pi/n) \frac{1}{t^3} \, dt \right]
\]
\[
= O(1/n) \left[ \left\{ o(1/t) \right\} \int_{\pi/n}^{\delta-\pi/n} + \int_{\pi/n}^{\delta-\pi/n} o(1/t^2) \, dt \right]
\]
\[
= o(1), \quad \text{as } n \to \infty. \]

Similarly, we find
\[
|L_4| \leq \frac{1}{\log n} \int_{\pi/n}^{\delta-\pi/n} \left| \frac{g(t)}{t} \right| \log 1/t \, dt
\]
\[ = O(n) \int_{\pi/n}^{2\pi/n} |g(t)| \, dt \]

\[ = o(1), \quad \text{as} \ n \to \infty \]

and that

\[ |L_5| \leq \left| \frac{\log (\delta - \pi/n)}{(\delta - \pi/n) \log n} \right| \int_{\delta - \pi/n}^{\delta} |g(t)| \, dt \]

\[ = o(1), \quad \text{as} \ n \to \infty \]

Thus, collecting the results, we obtain

\[ R = o(1), \quad \text{as} \ n \to \infty, \]

and the proof of Theorem 2 is completed.

Acknowledgements

The authoress is extremely grateful to Dr. P. D. Kathal and Prof. M. P. Schan for their kind interest and valuable suggestions in the preparation of this paper. She is also thankful to prof. H. M. Srivastava for his generous encouragement and very special suggestions for the improvement of the paper.

REFERENCES


Statement of ownership and other particulars
about the journal

JÑĀNABHĀ

1. Place of publication
   D. V. Postgraduate College
   Orai–285 001, U. P., India

2. Periodicity of its publication
   Annual

3. Printer’s name
   Anil Kumar Gupta
   Nationality
   Indian
   Address
   Navin Printing Press, 98– Gopal Ganj
   Orai–285 001, U. P.

4. Publisher’s name
   Dr. R. C. Singh Chandel
   Nationality
   Indian
   Address
   D. V. Postgraduate College
   Orai–285 001, U. P., India

5. Editor’s name
   Dr. R. C. Singh Chandel
   Nationality
   Indian
   Address
   D. V. Postgraduate College
   Orai–285 001, U. P., India

6. Name and address of the
   individuals who own the journal
   and partners or share holders
   holding more than one percent
   of the total capital
   
   Vijnana Parishad of India,
   Address : D. V. Postgraduate College
   Orai–285 001, U. P., India

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