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TIME DEPENDENT SLOW FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN WAVY WALLS

By

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ABSTRACT

Slow flow of a viscous incompressible fluid through rough parallel plates, under the application of periodic pressure gradient, has been considered. The expressions presented for the physical quantities involved are true for all values of the frequency of the pressure gradient. The roughness is assumed to be small. The effect of the frequency and time on the velocity profiles is studied numerically for the particular cases of sinusoidal roughness when the phase differences are zero and π.

1. INTRODUCTION

The exact solutions of Navier-Stokes' equations for a viscous incompressible fluid with axially parallel flow through a tube of circular cross-section, under the influence of periodic pressure gradient have been studied by Sexl [8] and Uchida [9], while those of co-axial circular cylinders by Verma [10]. Hepworth and Rice [4,5] have also studied the flow between parallel plates and circular rectangular tubes with arbitrary time varying pressure gradient. The slow viscous flow between rotating concentric infinite cylinders with axial roughness

was studied by Citron [1]. Verma and Gaur [11] have discussed the slow flow of a viscous incompressible fluid through a circular tube with axial roughness, under the influence of periodic pressure gradient while Gaur [2] has studied between two wavy walls and Gaur and Mehta [3] between two coaxial circular cylinders. Recently, Mehta and Gaur [6] have studied the effect of frequency and time for the slow flow through a rough circular tube.

In the present paper, the slow flow of a viscous incompressible fluid between wavy walls with roughness along their length has been investigated. The roughness of the walls is assumed to be small as compared to the distance between the walls. The expressions for the longitudinal and the transverse velocity and the pressure have been obtained by using Fourier transform technique. Two particular cases of sinusoidal roughness have been studied numerically.

2. Formulation of Problem

Under the assumption of slow motion, the Navier–Stokes' equations and the equation of continuity, in rectangular co–ordinate system, for a viscous incompressible fluid, in the non–dimensional form, are given by

$$(1)\quad \frac{\partial U}{\partial T} = -\frac{\partial P}{\partial X} + \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}\right),$$

$$(2)\quad \frac{\partial W}{\partial T} = -\frac{\partial P}{\partial Z} + \left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2}\right),$$

and

$$(3)\quad \frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0,$$

where

$$U = \frac{au}{\nu},\ W = \frac{aw}{\nu},\ X = x/a,\ Z = z/a,\ P = \frac{pa^2}{\nu^2\rho} \text{ and } T = \frac{t\nu}{a^2}.$$

The plates are symmetrically placed on the two sides of the x–axis, with a distance $2a$ between them and z is measured at right angles to the plates. U, V, P and T are non–dimensional longitudinal velocity, transverse velocity, pressure and time respectively.

Under the assumption of slow motion,

(4) $\nabla^2 P = 0.$

Using (4) in (1), we have

(5) $$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)^2 U = \frac{\partial}{\partial T}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}\right).$$

The boundary conditions are

(6) $U = 0 = W$ at $Z = 1 + \varepsilon N_1(X)$

and $Z = -1 + \varepsilon N_2(X),\ T > 0, X > 0,$

where $\varepsilon << $ is the roughness parameter and $N_1(X)$ and $N_2(X)$ are arbitrary functions of X.

Let

(7) $P(X, Z, T) = P_0(X, T) + P_1(X, Z, T),$

$W(X, Z, T) = W_1(X, Z, T),$

and

$U(X, Z, T) = U_0(Z, T) + U_1(X, Z, T),$

where P_1, U_1 and W_1 are the variations caused by the roughness and

P_0 and U_0 are the quantities for the case of smooth plates.

Let

$$(8)\quad -\frac{\partial P_0}{\partial X} = K \cos nT = Re\,[Ke^{inT}],$$

and

$$(9)\quad U_0 = Re\,[f(Z)\,e^{inT}],$$

where Re means 'the real part of' and K is a constant.

The solution for U_0 is given by

$$(10)\quad U_0(Z, T) = \mathrm{Re}\,[\frac{K}{m^2}\left(1 - \frac{\cosh mZ}{\cosh Z}\right) e^{inT}],$$

where $m = \sqrt{(in)}$.

Form (1), (2), (3), (5) and (7), we have

$$(11)\quad \frac{\partial U_1}{\partial T} = -\frac{\partial P_1}{\partial T} + \left(\frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2}\right),$$

$$(12)\quad \frac{\partial W_1}{\partial T} = -\frac{\partial P_1}{\partial Z} + \left(\frac{\partial^2 W_1}{\partial X^2} + \frac{\partial^2 W_1}{\partial Z^2}\right),$$

$$(13)\quad \frac{\partial U_1}{\partial X} + \frac{\partial W_1}{\partial Z} = 0,$$

and

$$(14)\quad \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)^2 U_1 = \frac{\partial}{\partial T}\left(\frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2}\right),$$

under the boundary conditions

(15) $U_1 = -U_0$, $W_1 = 0$ at $Z = 1 + \varepsilon N_1(X)$ and

$Z = -1 + \varepsilon N_2(X)$, $T > 0, X > 0$,

and

$U_1 = 0$ at $-1 + \varepsilon N_2(X) < Z < 1 + \varepsilon N_1(X)$, $T \geqslant 0, X > 0$.

3. METHOD OF SOLUTION

Following (9), we assume

(16) $U_1(X, Z, T) = Re\,[U_1(X, Z)\, e^{inT}]$

and therefore equation (14) reduces to

(17) $\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)^2 U_1 = in\left(\frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2}\right).$

Let us suppose that

(18) $\left(\frac{\partial^2 U_1}{\partial X^2} + \frac{\partial^2 U_1}{\partial Z^2}\right) = f(X, Z)$

and

(19) $F(\xi, Z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(X, Z)\, sin\,(\xi X)\, dX,$

where $F(\xi, Z)$ is the Fourier sine transform of $f(X, Z)$.

Using equation (18) in (17), taking the Fourier sine transform and solving the resulting equation, we get

(20) $U(\xi, Z) = A(\xi)\, e^{-bZ} + B(\xi)\, e^{bZ} + C(\xi)\, e^{-\xi Z} + D(\xi)\, e^{\xi Z}$

where $U(\xi\ Z)$ is the Fourier sine transform of $U_1(X, Z)$ and $A(\xi)$,

$B(\xi)$, $C(\xi)$ and $D(\xi)$ are constants of integration and $b = (\xi^2 + in)^{1/2}$.

Taking inverse Fourier sine transform of equation (20), we get

$$(21)\ U_1(X,Z,T) = Re\ (2/\pi)^{\frac{1}{2}} \int_0^\infty [A(\xi)\, e^{-bZ} + B(\xi)\, e^{bZ} + C(\xi)\, e^{-\xi Z}$$

$$+ D(\xi)\, e^{\xi Z}\ ]\ sin\ (\xi X)\ e^{inT}\ d\xi.$$

Using (21), in (13), we get

$$(22)\ W_1\ (X,Z,T) = Re - (2/\pi)^{\frac{1}{2}} \int_0^\infty [-\ \xi/b\ A(\xi)\ e^{-bZ}$$

$$+ \xi/b\ B(\xi)\ e^{bZ} - C\ (\xi)\ e^{-\xi Z} + D(\xi)\ e^{\xi Z}\ cos\ (\xi X)\ e^{int}\ d\xi.$$

Using equation (21) and (22) in (11) and (12), we get

$$(23)\ P_1(X,Z,T) = Re\ (2\pi)^{\frac{1}{2}}\ in \int_0^\infty 1/\xi\ [C(\xi)\ e^{-\xi Z} + D(\xi)\ e^{\xi Z}] cos(\xi X)$$

$$e^{inT}\ d\xi + C,$$

where C is a constant of integration.

Let us assume

$$(24)\ A(\xi) = A_0(\xi) + \varepsilon\ A_1(\xi) + \cdots$$

and similar expansions for $B(\xi)$, $C(\xi)$ and $D(\xi)$.

Using the boundary conditions (15) in equations (21) and (22) and equating the co-efficients of ε^0 and ε and inverting the resulting equations by Fourier sine and cosine integral theorems, we get

$$(25)\ A_0(X) = B_0(X) = C_0(X) = D_0(X) = 0,$$

$$(26)\ A_1(X) = \frac{Kb'\,g(n)}{4}\left[\frac{1}{G(X)}\{\bar{N}_1(X) - \bar{N}_2(X)\} \times \right.$$

$$\left. \sinh X - \frac{1}{H(X)}\{\bar{N}_1(X) + \bar{N}_2(X)\}\cosh X\right],$$

$$B_1(X) = \frac{Kb'\,g(n)}{4}\left[\frac{1}{G(X)}\{\bar{N}_1(X) - \bar{N}_2(X)\} \times \right.$$

$$\left. \sinh X + \frac{1}{H(X)}\{\bar{N}_1(X) + \bar{N}_2(X)\}\cosh X\right],$$

$$C_1(X) = \frac{KXg(n)}{4}\left[-\frac{1}{G(X)}\{\bar{N}_1(X) - \bar{N}_2(X)\} \times \right.$$

$$\left. \sinh b' + \frac{1}{H(X)}\{\bar{N}_1(X) + \bar{N}_2(X)\}\cosh b'\right],$$

and

$$D_1(X) = \frac{KXg(n)}{4}\left[-\frac{1}{G(X)}\{\bar{N}_1(X) - \bar{N}_2(X)\} \times \right.$$

$$\left. \sinh b' - \frac{1}{H(X)}\{\bar{N}_1(X) + \bar{N}_2(X)\}\cosh b'\right],$$

where $G(X) = b' \cosh b' \sinh X - X \cosh X \sinh b'$,

$H(X)\ b' \sinh b' \cosh X - X \cosh b' \sinh X$,

$b' = (X^2 + in)^{1/2}$ and $g(n)$ is a known function of n, given by

$g(n) = (1/\sqrt{in}).\ \tanh\,(i^{1/2}\,n^{1/2})$. $\bar{N}_1$ and $\bar{N}_2$ are Fourier sine transforms of N_1 and N_2, respectively.

Thus the complete expressions for the velocities and pressure are given by

$$(27)\quad U(X, Z, T) = U_0(Z, T) + U_1(X, Z, T)$$

$$= Re\left[\frac{K}{in} \left(1 - \frac{\cosh (i^{1/2} n^{1/2} Z)}{\cosh (i^{1/2} n^{1/2})} e^{inT} \right) \right.$$

$$+ Re\left[K/2\, g(n)\, \epsilon\, (2/\pi)^{1/2} \right.$$

$$\int_0^\infty \left[\frac{b \sinh \xi \cosh bZ - \xi \sinh b \cosh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \right.$$

$$\{ \bar{N}_1 (\xi) + \bar{N}_2 (\xi) \}$$

$$\left. + \frac{b \cosh \xi \sinh bZ - \xi \cosh b \sinh \xi Z}{b \cosh \xi \sinh b - \xi \sinh b \cosh \xi} . \{ \bar{N}_1 (\xi) + \bar{N}_2 (\xi) \} \right]$$

$$\sin (\xi X)\; e^{inT}\, d\xi \Big],$$

(28) $$W (X, Z, T) = W_1 (X, Z, T) = Re\left[- K/2\, g(n)\, \epsilon\, (2/\pi)^{1/2} \right.$$

$$\int_0^\infty \xi \left[\frac{\sinh \xi \sinh bZ - \sinh b \sinh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \{ \bar{N}_1 (\xi) + \bar{N}_2 (\xi) \} \right.$$

$$\left. + \frac{\cosh \xi \cosh bZ - \cosh b \cosh \xi Z}{b \cosh \xi \sinh b - \xi \cosh b \sinh \xi} \{ \bar{N}_1 (\xi) + \bar{N}_2 (\xi) \} \right]$$

$$\cosh (\xi X)\; e^{inT}\, d\xi \Big],$$

and

(29) $$P(X, Z, T) = P_0 (X, T) + P_1 (X, Z, T)$$

$$= C - K X \cos nT - Re\left[K/2\, g(n)\, \varepsilon\, (2/\pi)^{1/2}\, in. \right.$$

$$\int_0^\infty \left[\frac{\sinh b \cosh \xi Z}{b \sinh \xi \cosh b - \xi \sinh b \cosh \xi} \{ \bar{N}_1 (\xi) - \bar{N}_2 (\xi) \} \right.$$

$$+\frac{\cosh b \sinh \xi Z}{b \cosh \xi \sinh b - \xi \cosh b \sinh \xi}\{\bar{N}_1(\xi) + \bar{N}_2(\xi)\}\,]$$

$$\cosh(\xi X)\, e^{inT}\, d\xi\,],$$

where C is a constant of integration.

4. Particular Cases (Sinusoidal Roughness)

Case (i) Let

(30) $N_1(X) = \sin X/l = -N_2(X)$

where $2\pi l$ is the wavelength of the roughness at the walls. *i. e.* the phase difference in the roughness of the walls is π. We may formally write

(31) $\bar{N}_1(\xi) = -\bar{N}_2(\xi) = (\pi/2)^{1/2}.\,\delta(\xi - 1/l),$

where $\bar{N}_1(\xi)$ and $\bar{N}_2(\xi)$ are the Fourier sine transforms of $N_1(X)$ and $N_2(X)$, respectively, and δ is the Dirac delta function.

Substituting (31) in (27) and (28) and using the property of Dirac delta function (Sneddon [7]), the longitudinal and the transverse velocities for $l = 1$ and for different values of n are obtained as follows : For $n = 1$

(32) $U(X, Z, T) = [\, U_0(Z, T)_{n=1} - Re\,[K\varepsilon\,[\,(1.07626 + 2.46087\, i)$

$\cosh\{Z(1+i)^{1/2}\} - (0.68334 + 2.61039\, i)\cosh Z]\,\sin X.\, e^{iT}\,],$

(33) $W(X, Z, T) = Re\,[K\varepsilon\,[\,(1.07626 + 2.46087\, i)$

$$\frac{\sinh\{Z(1+i)^{1/2}\}}{(1+i)^{1/2}}$$

$$- (0.68334 + 2.61039\, i)\, \sinh Z\ ]\ \cos X\, e^{iT}].$$

For $n = 2$

$$(34)\quad U(X, Z, T) = [\, U_0(Z, T)]_{n=2} - Re\, [K\epsilon\, [(0.76328 + 0.84048\, i)$$

$$\cosh \{Z\, (1+2\, i)^{1/2}\} - (0.47817 + 1.05276\, i) \cosh Z)\, \sin X.\ e^{2iT}],$$

$$(35)\quad W(X, Z, T) = Re\, [\, K\epsilon\, [\, (0.76328 + 0.84048\, i)$$

$$\frac{\sinh \{Z\, (1 + 2i)^{1/2}\}}{(1+2i)^{1/2}}$$

$$- (0.47817 + 1.05276\, i\,)\, \sinh Z\ ]\ \cos X.\ e^{2iT}].$$

For $n = 4$

$$(36)\quad U(X, Z, T) = [\, U_0(Z, T)\,]_{n=4}$$

$$- Re\, [\, K\epsilon\, [(0.35991 + 0.16441\, i), \cosh \{Z\, (1+4i)^{1/2}\}$$

$$- (0.21517 + 0.36588\, i) \cosh Z]\, \sin X\, e^{4iT}],$$

$$(37)\quad W(X, Z, T) = Re\, [K\epsilon\, [\, (0.35991 + 0.16441\, i)$$

$$\frac{\sinh \{Z\, (1+4i)^{1/2}\}}{(1+4i)^{1/2}} - (0.21517 + 0.36588\, i) \sinh Z)\, \cosh X.\ e^{4iT}].$$

Case (ii)

Let

$$(38)\quad N_1(X) = N_2(X) = \sin (X/l),$$

i. e. the phase difference in the roughness of the two walls is zero.

The expression for U and W for $l = 1$, now are

For $n = 1$,

(39) $U(X, Z, T) = [U_0(Z, T]_{n=1}$

$- Re\ [K \epsilon\ [(0.11265 + 1.20146\ i).\ sinh\ \{Z(1+i)^{1/2}\}$

$- (0.11925 + 1.05053\ i)\ sinh\ Z]\ sin\ X.\ e^{iT}],$

(40) $W(X,Z, T) = Re\ [\ K \epsilon\ [(0.11265 + 1.20146\ i)\ \dfrac{cosh\ \{Z(1+i)^{1/2}\}}{(1+i)^{1/2}}$

$- (0.11925 + 1.05053\ i)\ cosh\ Z]\ cos\ X.\ e^{iT}].$

For $n = 2$,

(41) $U(X, Z, T) = [U_0(Z, T)\ ]_{n=2}$

$- Re\ [K \epsilon\ [(0\ 20451 + 0.59854\ i)\ sinh\ \{Z(1 + 2i)^{1/2}\}$

$- (0.08542 + 0.48298\ i)\ sinh\ Z]\ sin X\ \ e^{2iT}],$

(42) $W(X, Z, T) = Re\ [K \epsilon\ [(0.20451 + 0.59854\ i)\ \dfrac{cosh\ \{Z(1+2i)^{1/2}\}}{(1+2i)^{1/2}}$

$- (0.08542 + 0.48298\ i)\ cosh\ Z]\ cos\ X\ e^{2iT}].$

For $n = 4$

(43) $U(X, Z, T) = [U_0(Z, T)_{n=4}$

$- Re\ [K \epsilon[(0.19472 + 0.20719\ i).\ sinh\ \{Z(1+4i)^{1/2}\}$

$- (0.04171 + 0\ \ 21354\ i)\ sinh\ Z]\ sin\ X\ e^{4iT}].$

(44) $W(X, Z, T) = Re\ [\ K \epsilon[(0.19472 + 0.20719\ i)$

$\dfrac{cosh\ \{Z\ (1 + 4i)^{1/2}\}}{(1+4i)^{1/2}}$

$- (0.04171 + 0.21354\ i)\ cosh\ Z]\ cos\ X.\ e^{4iT}].$

5. NUMERICAL DISCUSSIONS

The longitudinal velocity profiles for the wavy walls (phase difference of the sinusoidal roughness of the walls being π) at different cross—sections for $\varepsilon = 0.1$, $l = 1$ and for different values of n and T are shown by Figures 1, 2 and 3. It is observed that the longitudinal velocity decreases as the distance between the walls increases and vice versa. For $T = 0$ and $T = 1$, the increase in the frequency of the applied pressure gradient decreases the magnitude of the longitudinal velocity. For $T = 2$, the magnitude of the longitudinal velocity is quite small for $n = 1$, which further decreases when n becomes 2, but again increases as it becomes 4.

The transverse velocity profiles for the same values of parameters as above, at the plane $X = 0$ are depicted in Figures 4, 5 and 6. For $X = \pi/2$ and $3\pi/2$, the transverse velocity vanishes identically for all values of Z and for $X = \pi$, it is the image of the curves drawn for $X = 0$. For $T = 0$ and $T = 1$, the increase in the frequency of the applied pressure gradient decreases the transverse velocity. For the forward pressure gradient, the resulting effect of the longitudinal and the transverse velocities is that the direction of the flow is towards the boundary when the width between the plates increases and vice versa.

Figure 7 depicts the transverse velocity profiles for the same values of ε, l, n and T, at the plane $X = 0$ when the phase difference of the sinusoidal roughness of the walls is zero. It is noted that the transverse velocity is optimum at the mid plane and for $T = 0$ and $T = 1$, it decreases with the increase in the frequency of applied pressure gradient. The disturbances in the flow become more rapid with the increase in the roughness parameter ε.

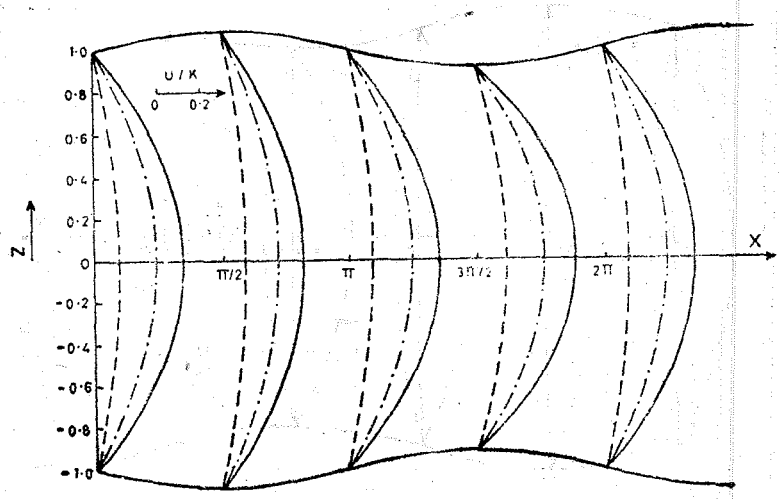

Fig. 1 : The longitudinal velocity profiles at different sections of a roughness wave taking $l = 1$, $\varepsilon = 0.1$, and $T = 0$ for $n = 1$ (—), 2(-.-.-.-) and 4(- - - -).

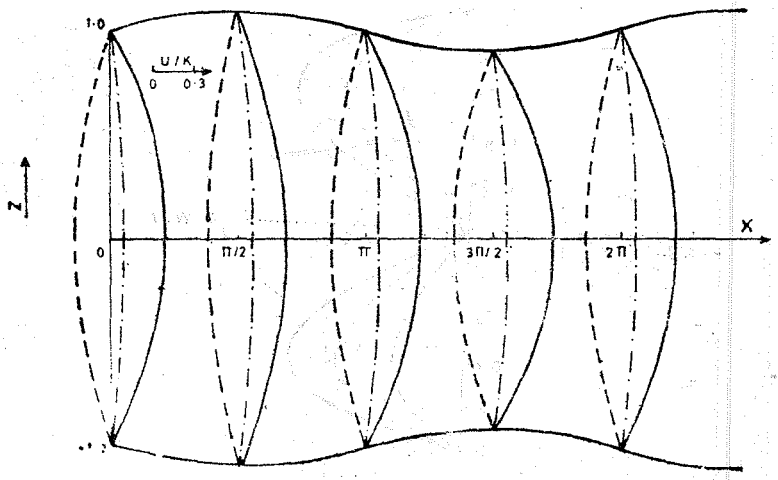

Fig. 2 : The longitudinal velocity profiles at different sections of the roughness wave taking $l = 1$, $\varepsilon = 0.1$, and $T = 1$ for $n = 1$ (—), 2(-.-.-.-) and 4(- - - -).

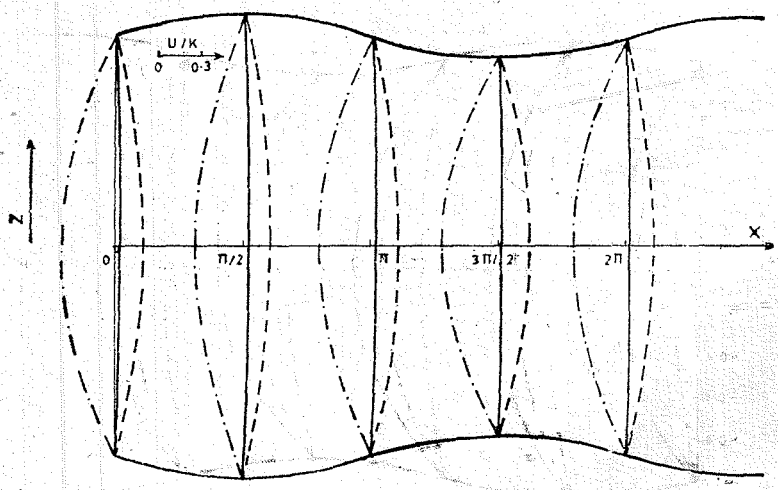

Fig. 3 : The longitudinal velocity profiles at different sections of a roughness wave taking $l = 1$, $\varepsilon = 0.1$ and $T = 2$ for $n = 1$ (—), 2(–.–.–.–) and 4(– – – –).

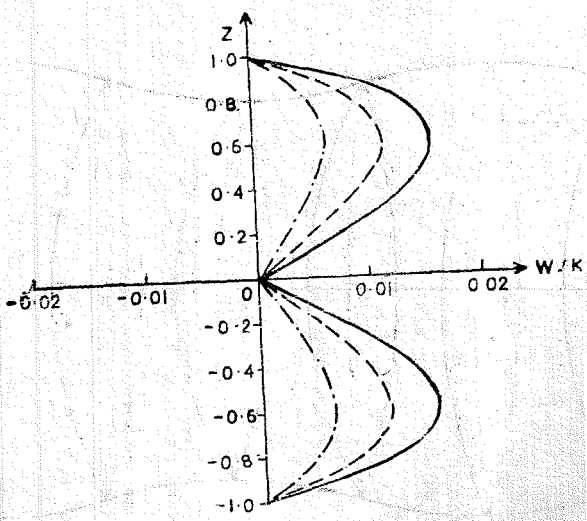

Fig. 4 : The transverse velocity profiles of a roughness wave taking $l = 1$, $\varepsilon = 0.1$ and $T = 0$ at $X = 0$ plane for $n = 1$ (—), 2(– – –) and 4(–.–.–.).

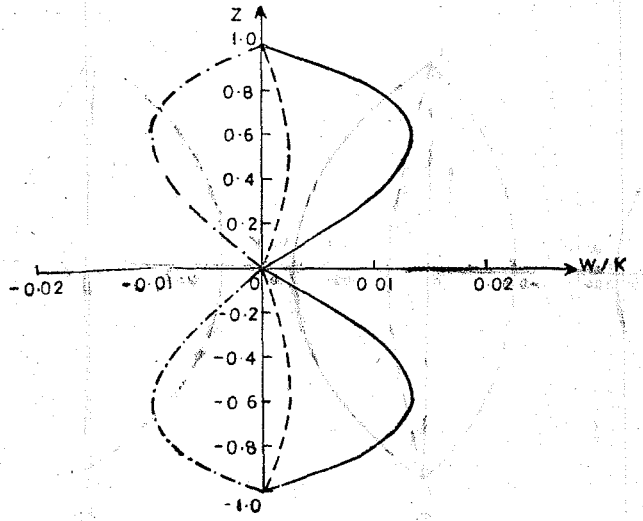

Fig. 5 : The transverse velocity profiles of a roughness wave taking $l = 1$, $\varepsilon = 0.1$ and $T = 1$ at $X = 0$ plane for $n = 1$ (—), 2(- - - -) and 4(-.-.-.).

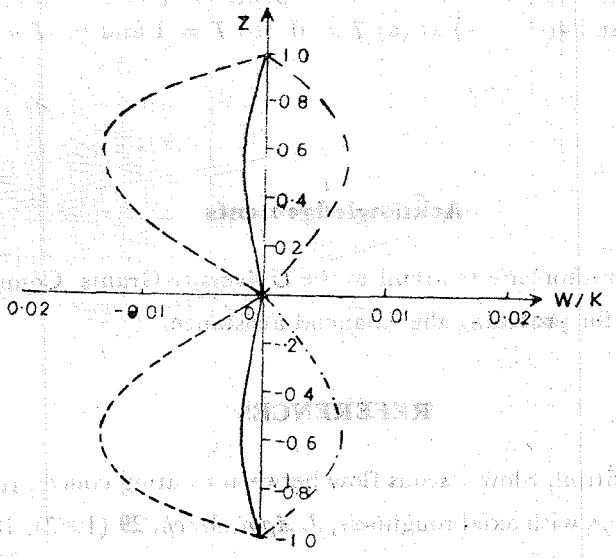

Fig. 6 : The transverse velocity profiles of a roughness wave taking $l = 1$, $\varepsilon = 0.1$ and $T = 2$ at $X = 0$ plane for $n = 1$ (—), 2(- - - -) and 4(-.-.-.-.).

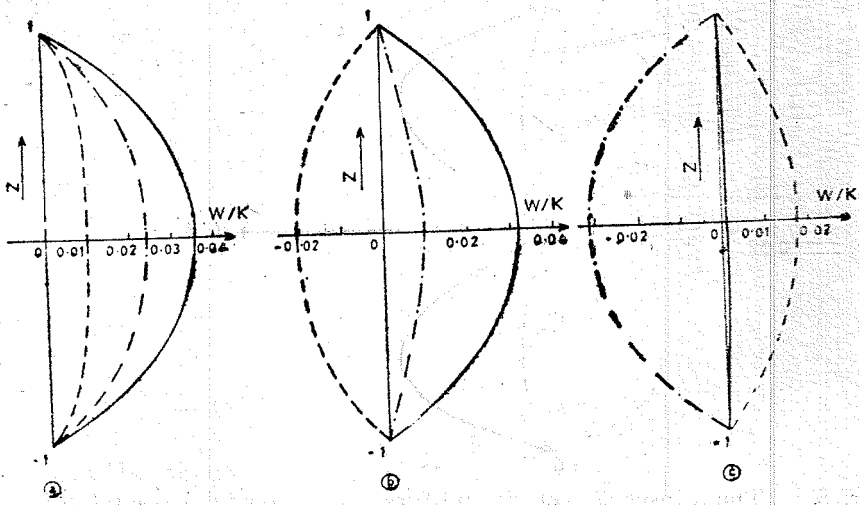

Fig. 7 : The transverse velocity profiles of a roughness wave taking $l = 1$, $\varepsilon = 0.1$ at $X = 0$ plane for $n = 1$ (—), 2(-.-.-.-) and 4(- - - -) at (a) $T = 0$, (b) $T = 1$ and (c) $T = 2$.

Acknowledgements

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RELATIONS BETWEEN THE HANKEL AND MULTIVARIABLE H-FUNCTION OPERATORS

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ABSTRACT

The object of this paper is to obtain relations between the Hankel operator and the multivariable H-function operator. Several results of S.L.Kalla and R.K. Saxena[4],and of R.K. Saxena and R.K. Kumbhat [5], become special cases of our main relations.

1. Introduction

The multivariable H-function due to H. M. Srivastava and R. Panda [8, *p*. 130, *Eq*. (1. 1)] is defined and represented as follows (See also Srivastava, Gupta and Goyal [7, p 251, *Eq*. (C. 1)]:

$$H\begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} = H^{o,\,n;\,m_1,\,n_1\,;\ldots;\,m_r,\,n_r}_{p,\,q;\,p_1,\,q_1\,;\ldots;\,p_r,\,q_r}\begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix}\begin{matrix} (a_j;\,\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,\,p}: \\ (b_j;\,\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,\,q}: \end{matrix}$$

$$\left.\begin{matrix} (c_j^{(1)},\gamma_j)_{1,\,p_1}\,;\ldots;\,(c_j^{(r)},\gamma_j^{(r)})_{1,\,p_r} \\ (d_j^{(1)},\delta_j^{1})_{1,\,q_1}\,;\ldots;\,(d_j^{(r)},\delta_j^{(r)})_{1,\,q_r} \end{matrix}\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(s_1)\ldots\phi_r(s_r)\,\Psi(s_1,\ldots,s_r)$$

$$\cdot\, z_1^{s_1} \ldots z_r^{s_r}\, ds_1 \ldots ds_r \tag{1.1}$$

where $w = \sqrt{-1}$,

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_i^{(i)} s_i)}$$

$$(i = 1, \ldots, r),$$

and

$$\Psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i) \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i)}.$$

For further details and asymptotic expansions of the above function (1.1), refer to [7] and [8].

Banerji and Sethi [1] have defined the multivariable H-function operator as follows:

$$\underset{(\lambda_n)}{\overset{\delta,\beta}{P_x}} f(x) = \xi x^{-\delta-\xi\beta-1} \int_0^x t^{\delta} (x^{\xi} - t^{\xi})^{\beta} f(t)\, H \begin{bmatrix} \lambda_1 U \\ \vdots \\ \lambda_n U \end{bmatrix} dt \tag{1.2}$$

$$\underset{(\lambda_n)}{\overset{\alpha,\beta}{S_x^{\xi}}} f(x) = x^{\alpha} \int_x^{\infty} t^{-\alpha-\xi\beta-1} (t^{\xi} - x^{\xi})^{\beta} f(t)\, H \begin{bmatrix} \lambda_1 V \\ \vdots \\ \lambda_n V \end{bmatrix} dt \tag{1.3}$$

where

$$U = \left(\frac{t^{\xi}}{x^{\xi}}\right)^{m_i} \left(1 - \frac{t^{\xi}}{x^{\xi}}\right)^{n_i},$$

$$V = \left(\frac{x^{\xi}}{t^{\xi}}\right)^{m_i} \left(1 - \frac{x^{\xi}}{t^{\xi}}\right)^{n_i},$$

and ξ, m_i, n_i are positive numbers. These operators exist under the following set of conditions:

(i) $1 \leqslant p_i, q_i < \infty, p_i^{-1} + q_i^{-1} = 1$

(ii) $Re\ (\delta + \xi \sum_1^r m_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{q_i}$

(iii) $Re\ (\beta + \xi \sum_1^r n_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{q_i}$

(iv) $Re\ (\alpha + \xi \sum_1^r m_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{p_i}$

(v) $f(x) \varepsilon L_{p_i}(0, \infty)$

$(j = 1, 2, \ldots, m_r; \quad i, \ldots, r)$

The conditions (v) ensure that both P and S exist and also that both belong to $L_{p_i}(0, \infty)$.

The operator of the Hankel transformation is given by (see Sneddon [6])

$$S'_{\eta,\alpha} f = 2^{\alpha} x^{-\alpha} \int_0^{\infty} y^{1-\alpha} f(y)\ J_{2\eta+\alpha}(xy) dy, \qquad \ldots (1.4)$$

where $J_n(z)$ denotes the Bessel function defined by

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \; z^{2k+n}}{2^{2k+n} \; k! \; \Gamma(1+n+k)} \qquad \ldots(1.5)$$

We also need [2, *p*. 10, *Eq*. (17)]

$$\int_0^x t^{m-1} \, (x^{\xi} - t^{\xi})^{n-1} \, dt = \frac{x^{m+n\xi+1} \; \Gamma(\frac{m+1}{\xi}) \; \Gamma(n+1)}{\Gamma(\frac{m+1}{\xi} + n+1)} \qquad \ldots(1.6)$$

$(Re \; \xi > 0, \; Re \; n > 0, \; Re \; m > 0)$

and [3, *p*. 295, *Eq*. (3)]

$$\int_x^{\infty} t^{\mu-1} \, (t^p - x^p)^{\nu-1} \, dt = x^{\mu+p\nu-p} \, B(1 - \nu - \frac{\mu}{p}, \; \nu), \qquad \ldots(1.7)$$

where $B(\alpha, \beta)$ is the Beta function.

2. The Main Results

The following results will be established here :

$$\underset{(\lambda_n)}{\overset{\delta, \beta}{P_x}} \; S'_{\eta, \alpha} f = x^2 \, H \begin{matrix} 0, n+2 : m_1, n_1 \; ; \ldots ; \; m_r, n_r \\ p+2, q+1 : p_1, q_1 \; ; \ldots ; \; p_r, q_r \end{matrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}$$

$$(a_j ; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}), \; \left(\frac{\delta + 2k + 2\eta + 2}{\xi} ; \; m_1, \ldots, m_r \right),$$

$$(b_j ; \beta_j^{(1)}, \ldots, \beta_j^{(r)}), \; \left(\frac{\delta + \xi\beta + 2\xi + 2k + 2\eta + 2}{\xi} ; \right.$$

$(\beta + 2 ; n_1, \ldots, n_r) :$

$m_1 + n_1, \ldots, m_r + n_r) :$

$$\left.\begin{matrix}(c_j^{(1)},\ \gamma_j^{(1)})_{1,p_1}\ ;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,p_r}\\ (\delta_j^{(1)},\ \gamma_j^{(1)})_{1,q_1}\ ;\ldots;\ (\delta_j^{(r)},\ \gamma_j^{(r)})_{1,q_r}\end{matrix}\right] S'_{\eta,\alpha} f \qquad \ldots(2.1)$$

$$\underset{(\lambda_n)}{\overset{\alpha,\ \beta}{S_x}} \cdot S'_{\theta,\ \varepsilon} f = x^{2\varepsilon} H \begin{matrix} 0,\ n+2 : m_1, n_1\ ;\ldots;\ m_r, n_r \\ p+2,\ q+1 : p_1, q_1\ ;\ldots;\ p_r, q_r \end{matrix} \left[\begin{matrix}\lambda_1\\ \\ \lambda_n\end{matrix}\right|$$

$$(a_j;\ \alpha_j^{(1)},\ldots,\alpha_j^{(r)}),\ \left(\frac{\alpha+2k+2\theta+2\varepsilon}{\xi};\ m_1,\ldots,m_r\right),\ (\beta+1;\ n_1,\ldots,n_r):$$

$$(b_j;\ \beta_j^{(1)},\ldots,\beta_j^{(r)}),\ \left(\frac{\alpha+\xi+\xi\beta-2k-2\theta}{\xi};\ m_1+n_1,\ldots,m_r+n_r\right):$$

$$\left.\begin{matrix}(c_j^{(1)},\ \gamma_j^{(1)})_{1,p_1}\ ;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,p_r}\\ (\delta_j^{(1)},\ \gamma_j^{(1)})_{1,q_1}\ ;\ldots;\ (\delta_j^{(r)},\ \gamma_j^{(r)})_{1,q_1}\end{matrix}\right] S'_{\theta,\ \varepsilon} f \qquad \ldots(2.2)$$

Proof of (2.1). Consider

$$I \equiv \underset{(\lambda_n)}{\overset{\delta,\ \beta}{P_x}} S'_{\eta,\alpha} f = x^{-\delta-\xi\beta-1} \int_0^x t^{\delta} (x^{\xi} - t^{\xi})^{\beta} H \begin{bmatrix}\lambda_1 U\\ \vdots \\ \lambda_n U\end{bmatrix} dt$$

$$\cdot \int_0^{\infty} 2^{\alpha}\, t^{-\alpha}\, y^{1-\alpha} f(y)\, J_{2\eta+\alpha}(ty)\, dy,$$

which, on replacing the H-function by (1.1) and the Bessel function by (1.5), becomes

$$I = 2^{\alpha}\, \xi x^{-\delta-\xi\beta-1} \int_0^{\infty} y^{1-\alpha} f(y)\, dy \int_0^x t^{\delta-\alpha} (x^{\xi} - t^{\xi})^{\beta}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (ty)^{2k+2\eta+\alpha}}{2^{2k+2\eta+\alpha}\, k!\, \Gamma(1+2\eta+\alpha+k)} \frac{1}{(2\pi w)^r}$$

$$\int_{L_1} \cdots \int_{L_r} \Pi\, \phi_i\,(s_i)\ \Psi\,(s_1, \ldots, s_r)$$

$$\cdot \left(\frac{t^{\xi}}{x^{\xi}}\right)^{m_i \Sigma s_i} (\lambda_i)^{s_i} \left(1 - \frac{t^{\xi}}{x^{\xi}}\right)^{n_i \Sigma s_i} ds_1 \ldots ds_r\, dt.$$

On changing the order of integration (which is justified under the conditions stated with the definitions of the operators), we get

$$I = 2^{\alpha}\, \xi x^{-\delta-\xi\,\beta-1} \int_0^{\infty} y^{1-\alpha} f(y)\, dy\ \frac{1}{(2\pi w)^r}$$

$$\int_{L_1} \cdots \int_{L_r} \Pi \phi_i\,(s_i)\ \Psi\,(s_1, \ldots, s_r)$$

$$\cdot\ (\lambda_i)^{s_i}\ \frac{1}{x^{\xi \Sigma s_i\,(m_i+n_i)}}\, ds_1 \ldots ds_r \sum_{k=0}^{\infty} \frac{(-1)^k\, y^{2k+2\eta+\alpha}}{2^{2k+2\eta+\alpha}\, k!\, \Gamma(1+2\eta+\alpha+k)}$$

$$\cdot \int_0^x t^{(\delta+2k+2\eta+\xi\, m_i\, \Sigma\, s_i+1)-1}\ (x^{\xi}-t^{\xi})^{(\beta+\eta_i\, \Sigma\, s_i+1)-1}\ dt.$$

Evaluating the t–integral, using (1.6), we get

$$I = 2^{\alpha}\, \xi\, x^{-\delta-\xi\beta-1} \int_0^{\infty} y^{1-\alpha} f(y)\, dy\ \frac{1}{(2\pi w)^r}$$

$$\int_{L_1} \cdots \int_{L_r} \Pi \phi_i\,(s_i)\ \Psi\,(s_1, \ldots, s_r)$$

$$\cdot\ (\lambda_i)^{s_i}\ \frac{1}{x^{\xi\, \Sigma s_i (m_i+n_i)}} \sum_{k=0}^{\infty} \frac{(-1)^k\, y^{2k+2\eta+\alpha}}{2^{2k+2\eta+\alpha}\, k!\, \Gamma(1+2\eta+\alpha+k)}$$

$$\cdot\ \xi^{-1}\ x^{\delta+2k+2\eta+\xi\, m_i\, \Sigma\, s_i+1+\xi\beta+\xi n_i\, \Sigma\, s_i+1+1}$$

$$\frac{\Gamma\left(\dfrac{\delta+2k+2\eta+\xi\, m_i\, s_i+1+1}{\xi}\right)\Gamma(\beta+\eta_i\, \Sigma\, s_i+1+1)}{\Gamma\left(\dfrac{\delta+2k+2\eta+\xi\, m_i\, \Sigma\, s_i+1+1+\xi\beta+\xi n_i\, \Sigma\, s_i+\xi+\xi}{\xi}\right)}\, ds_1 \ \ ds_r$$

$$= x^2\, H^{\,0,\, n+2:\, m_1,\, n_1\, ;\ldots;\, m_r,\, n_r}_{\,p+2,\, q+1:\, p_1,\, q_1\, ;\ldots;\, p_r,\, q_r}\left[\begin{matrix}\lambda_1\\ \\ \lambda_n\end{matrix}\right|$$

$$(a_j;\ \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}),\ \left(\frac{\delta+2k+2\eta+2}{\xi};\ m_1, \ldots, m_r\right), (\beta+2;\ n_1, \ldots, n_r):$$

$$(b_j;\ \beta_j^{(1)}, \ldots, \beta_j^{(r)}),\ \left(\frac{\delta+\xi\beta+2\beta+2k+2\eta+2}{\xi};\ m_1+n_1, \ldots, m_r+n_r\right):$$

$$\left.\begin{matrix}(c_j^{(1)},\ \gamma_j^{(r)})_{1,p_1}\ ;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,p_r}\\ (\delta_j^{(1)},\ \gamma_j^{(1)})_{1,q_1}\ ;\ldots;\ (\delta_j^{(r)},\ \gamma_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$\cdot\ 2^{\alpha}\, x^{-\alpha} \int_0^{\infty} y^{1-\alpha}\, J_{2\eta+\alpha}\,(xy)\, dy$$

This completes the proof of (2.1).

Similarly, by using (1.7), we get the result (2.2).

3. SPECIAL CASES

Setting $n=2$ in (2.1) and (2.2) we obtain the relationships between the operators studied in [5]. The results of [4] can be also obtained by further specializing our results.

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FIXED POINT THEOREM FOR MAPPING WITH A GENERALIZED CONTRACTIVE ITERATE IN 2-METRIC SPACES

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The concept of 2 metric spaces has been investigated by S. Gähler in a series of papers ([3] to [6]) and other related works have been done by himself and others for extensive bibliography, we mention the work of K. Iséki [8].

The notion of contraction type mapping and fixed point theorem in 2-metric spaces has been introduced by K. Iséki [7]. Dass and Gupta [1] generalized Banach contraction principle for mapping of T satisfying

$$d(T_x, T_y) \leqslant \alpha \frac{d(y, Ty)\,[1+d(x, T_x)]}{[1+d(x, y)]} + \beta d(x, y)$$

for all $x, y \in X$, $\alpha, \beta > 0$, $\alpha + \beta < 1$.

The object of this paper is to establish a fixed point theorem for functions satisfying generalized contractive type condition which generalized the result of Guseman [2].

Lemma. *Let X be 2-metric space, and $T : X \to X$ be a mapping. Let $B \subset X$ with $T(B) \subset B$. If there exist a $u \in B$ and positive integer $n(u)$*

such that $T^{n(u)}(u) = u$, *and (for some non-negative constants* α, β *with* $\alpha+\beta < 1$) *let*

$$(1)\quad d(T^{n(u)}(x), T^{n(u)}(u), a) \leqslant \alpha \frac{d(u, T^{n(u)}(u), a)\,[1+d(x, T^{n(u)}(x), a)]}{[1+d(x, u, a)]} + \beta\, d(x, u, a)$$

for all $x, a \in X$. *Then* x *is the unique fixed point of* T *in* B, *and* $T^{n(u)}(y_0) \to u$ *for each* $y_0 \in B$.

Proof By (1), u is the unique fixed point of $T^{n(u)}$ in B then

$T(u) = T(T^{n(u)}(u)) = T^{n(u)}(T(u)) \Rightarrow Tu = u$. But u is the unique fixed point of T in B. Let $y_0 \in B$, and note that $T(B) \subset B \Rightarrow \{T^n(y_0);\ n \geqslant 1\} \subset B$.

Let $r(y_0) = max\,\{d(T^n(y_0), u, a);\ 1 \leqslant m < n(u) - 1\}$ and for all n sufficiently large with $n = r.\,n(u) + s$, where $n(u) = p$, $n = rp + s$, with $r > 0$, $0 \leqslant s \leqslant p-1$, such that $T^p(u) = (u)$.

Then

$$d(T^n(y_0), u, a) \leqslant d(T^n(y_0), u, T^p(u)) + d(T^n(y_0), T^p(u), a) + d(T^p(u), u, a).$$

T.ɜ is can be written as

$$d(T^n(y_0), u, a) = d(T^n(y_0), T^p(u), a)$$

$$= d(T^{rp+s}(y_0), T^p(u), a)$$

$$= d(T^p\, T^{(r-1)p+s}(y_0), T^p(u), u)$$

$$d(u, T^p(u), a)\,[1+d(T^{(r-1)p+s}(y_0),$$

$$\leqslant \alpha \frac{T^{rp+s}(y_0), a)\,]}{[1+d(T^{(r-1)p+s}(y_0), u, a)\,]}$$

$$+ \beta\, d(T^{(r-1)p+s}(y_0), u, a)$$

$$= \beta\, d(T^{(r-1)p+s}(y_0, u, a) < \beta^2\, d(T^{(r-1)p+s}(y_0, u, a) < \ldots$$

$$< \beta^r\, d(T^s(y_0, u, a)$$

$$< \beta^r\, r(y_0)$$

and it follows that $T^n(y_0) \to u$ is finite.

Theorem. *Let (X, d) be a comlete 2-metric space and let T be a mapping of X into itself. Suppose there exists a $B \subset X$, such* that

(a) $T(B) \subset B$.

(b) $\alpha + \beta < 1$ *and each $y \in B$, there is an integer $n(y) \geqslant 1$, with*

$$d(T^{n(y)}(x), T^{n(y)}(y), a) \leqslant \frac{\alpha d(y, T^{n(y)}(y), a)\,[1+d(x, T^{n(y)}(x), a)\,]}{[1+d(x, y, a)\,]}$$

$$+ \beta\, d(x, y, a)$$

for all $x \in B$.

(c) *For some $x_0 \in B$, $Cl\,\{T^n(x_0);\ n \geqslant 1\} \subset B$.*

Then there is a unique $u \in B$, such that $T(u) = u$ Furthermore, if

$$d(T^{n(y)}(x), T^{n(y)}(y), a) \leqslant \frac{\alpha\, d(y, T^{n(y)}(y), a)\,[1+d(x, T^{n(y)}(x), a)\,]}{[1+d(x, y, a)\,]}$$

$$+ \beta\, d(x, y, a)$$

for all a. $x \in X$, then *y is the unique fixed point in X, and $T^n(x_0) \to u$ for each $x_0 \in X$.*

Similarly,

$$d(x_3, x_2, a) = d(T^{m_1}(T^{m_2}(x_1), x_1, a)$$

$$\leqslant \beta\, d(T^{m_2}(x_1), (x_1), a)$$

$$\leqslant \beta\beta\, d(T^{m_2}(x_0), x_0, a).$$

Proceeding in this way we can get

$$d(x_{i+1}, x_i, a) \leqslant \beta^i d(T^{m_i}(x_0), x_0, a),\ i \geqslant 1$$

$$\leqslant \beta^i r(x_0) \qquad , i \geqslant 1.$$

By routine calculation one can show that the following inequality holds

$$d(x_j, x_i, a) \leqslant \sum_{l=1}^{j-1} d(x_{l+1}, x_l, a),\ j > i$$

$$\leqslant \frac{\beta^i}{1-\beta}\, r(x_0),\ j > i.$$

It follows that the sequence $\{x_i\}$ is a Cauchy, using completeness and (c), we have $x_i \to u \in B$. There is an integer $n(u) \geqslant 1$, such that

$$d(T^{n(u)}(y), T^{n(u)}(u), a) \leqslant \alpha\, \frac{d(u, T^{n(u)}(u), a)\,[1 + d(y, T^{n(u)}(y), a)]}{[1 + d(y, u, a)]}$$

$$+ \beta d(y, u, a)$$

for each $y \in B$.

It implies that $T^{n(u)}(x_i) \to T^{n(u)}(u)$.

Then $d(T^{n(u)}(u), a) = \lim_{i \to \infty} d(T^{n(u)}(x_i), u, a)$.

However,

$$d(T^{n(u)}(x_i), x_i, a) = d(T^{n(u)} T^{m_{i-1}}(x_{i-1}), T^{m_{i-1}}(x_{i-1}), a)$$
$$\leqslant \beta d(T^{n(u)}(x_{i-1}), x_{i-1}, a) \text{ from (A)}$$
$$\leqslant \beta^i d(T^{n(u)}(x_0), (x_0), a) \to 0 \text{ as } i \to \infty.$$

Hence $T^{n(u)}(u) = u$. By the lemma, u is the unique fixed point of T in B, and $T^n(y_0) \to u$ for each $y_0 \in B$.

The last assertion of the theorem follows directly from the lemma.

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A FIXED POINT THEOREM FOR L-CONTRACTIONS IN GENERALIZED METRIC SPACES

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ABSTRACT

The aim of this note is to extend a result concerning the existence of fixed-point for contractive mappings in generalized metric spaces to a more general class of mappings.

Before stating our result, we introduce, for sake of completeness, some standard notions [1].

Let E be a Banach space. A subset K of E is called a *cone* if it is closed, convex, $tK \subset K$ for $t \in R^+$ and $K \cap (-K) = 0$.

Given a cone K in E we define a *partial ordering* in E by writing $x < y$ if and only if $y - x \in K$.

Moreover, K is called *normal* if there exists $\sigma > 0$ such that $0 < x < y$ implies $\| x \| \leqslant \sigma \| y \|$.

We shall assume in the following that K is a *normal cone.*

Definition. A topological space X is said to be a *generalized metric space* if there exists a function d: $X \times X \to K$ such that:

(i) $d(x, y) = 0 \in K \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) < d(x, z) + d(z, y)$

("$<$" denotes the partial ordering induced by K).

Let $l: K \to R^+$ be a sublinear positively homogenous functional (that is, if $u, v \in R^+$ then

$l(u + v) \leqslant l(u) + l(v)$ and $l(tu) = t\, l(u)$ for $t \geqslant 0$) such that $l^{-1}(0) = 0$. Then, if $d^*: X \times X \to R^+$ is the function defined by

$$d^*(x, y) = l(d(x, y)), \tag{1}$$

we have that (x, d^*) is a metric space.

We shall say that a generalized metric space is a *complete generalized metric space* if it is complete in the metric defined by (1).

In a complete generalized metric space (x, d) the following result was proved in [1, Theorem 6. 2].

Let $T: X \to X$ be a map such that

$$d(T(x), T(y)) < Ed(x, y), \tag{2}$$

where L is a positive ($L(K) \subset K$) and bounded linear operator in E with spectral radius, $r(L)$, less than 1. If x is complete generalized metric space with $l(u) = \| u \|$, than T has a unique fixed point which is the limit of the successive approximations

$$x_{n+1} = Tx_n \qquad n = 0, 1, 2$$

for any initial value $x' = x_0 \in X$.

In the Theorem below we shall show that if $T: X \to X$ is not a

necessarily continuous map which satisfies a relaxed condition than (2) then the same result holds.

More precisely we have the following

Theorem. *Let X be a generalized metric space which is complete in the metric* (1) *with* $l(u) = \| u \|$ ($\| u \|$ stands for the norm in E).

If $T: \to X$ *satisfies*

$$d(T(x), T(y)) < L[\, d(T(x), x) + d(T(y), y)\,], \tag{3}$$

where L is a bounded positive linear operator in E with spectral radius smaller than $\frac{1}{2}$, *then the equation*

$$x = Tx$$

has a unique solution in x, which is the limit of the successive approximations

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2$$

for any initial approximation $x' \in X$.

Proof. Since L is sublinear, it follows from (3) for $x = Tx'$ and $y = x'$

$$d(T(Tx'), T(x')) < Ld(T^2(x'), T(x')) + Ld(T(x'), x').$$

That is

$$(I - L)\, d(T^2(x'), T(x')) < Ld(T(x'), x\ ).$$

Being $r(L) < \frac{1}{2} < 1$, we have that $(I - L)$ is invertible, hence

$$d(T^2(x'), T(x')) < (I - L)^{-1} Ld(Tx', x'). \tag{4}$$

Clearly we have that

$$d(T^{n+1}(x'), T^n(x')) < (I - L)^{-n} L^n d(T(x'), x'). \tag{5}$$

Indeed, for $n = 1$, it has been already proved.

Assume that (5) is true for $n = 1$, then

$$d(T^{n+1}(x'), T^n(x')) = d(T(T^n(x'), T(T^{n-1}(x')) <$$

$$< Ld(T^{n+1}(x'), T^n(x')) + Ld(T^n(x'), T^{n-1}(x')).$$

That is

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < Ld(T^n(x'), T^{n-1}(x)).$$

By the inductive assumption

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < L(I - L)^{-(n-1)} L^{n-1} d(T(x'), x').$$

Since

$$L(I - L)^{-(n-1)} = (I - L)^{-(n-1)} L,$$

we obtain

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < (I - L)^{-(n-1)} L^n d(T(x'), x').$$

Thus (5) is proved.

Furthermore, we have

$$d(T^{n+m}(x'), T^n(x')) < d(T^{n+m}(x'), T^{n+m-1}(x') + d(T^{n+m-1}(x'), T^{n+m-2}(x')) +$$

$$+ \dots + d(T^{n+1}(x'), T^n(x')) <$$

$$< \{(I - L)^{-(n+m-1)} L^{n+m-1} + (I - L)^{-(n+m-2)} L^{n+m-2} + \dots +$$

$$+ (I - L)^{-n} L^n\} d(T(x'), x') <$$

$$< (I-L)^{-n}L^n \left(\sum_{m=1}^{\infty} (I-L)^{-(m-1)} L^{m-1} (d(T(x'), x'))\right) = (I-L)^{-1}L)^n u'$$

where u' is the unique solution of

$$u = (I-L)^{-1}Lu + d(T(x'), x').$$

Indeed, from the spectral mapping theorem, we have that the spectral operator $(I-L)^{-1}L$ is such that $r((I-L)^{-1}L) < 1$.

Hence

$$d(T^{n+m}(x'), T^n(x')) < ((I-L)^{-1}L)^n u'$$

Being K *normal*, we have

$$\| d(T^{n+m}(x'), T^n(x') \| \leqslant \sigma \| ((I-L)^{-1} L)^n u' \| . \tag{6}$$

Since the right-hand side of (6) is going to zero when $n \to +\infty$, we obtain that $\{T^n(x')\}$ is a Cauchy-sequence with respect to the metric d^*. Being (X, d^*) complete, we denote by x^* the lemit of $\{T^n(x')\}$. Then the following inequalitiesholds

$$d(T(x^*), x^*) < d(T(x^*), T^n(x') + d(T^n(x'), x^*) <$$

$$< (I-L)^{-1} Ld(x^*, T^{n-1}(x')) + d(T^n(x'), x^*).$$

Finally, using the normality of K,

$$\| d(T(x^*), x^*) \| \leqq \sigma (\| (I-L)^{-1}L \| . \| d(x^*, T^{n-1}(x') \| +$$

$$+ \| d(T^n(x'), x^*) \|). \tag{7}$$

Letting $n \to +\infty$ in (7), we obtain

$$T(x^*) = x^*.$$

At last, we would like to add in passing that the Theorem would be still true if the map T is a "*generalized contraction*" map (cfr. [2 Ch. 1] for an extensive bibliography).

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INTEGRALS INVOLVING A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

H. M. Srivastava and N. P. Singh [4] recently established integrals involving certain products of a general class of polynomials $S_n^m(x)$ and the multivariable H-function. The aim of this paper is to further extend these results and establish certain integrals (both single and multiple) involving the products of $S_n^m(x)$ and the multivarible H-functions.

1. INTRODUCTION

Srivastava [2, p. 1, eqn. (1)] introduced a general class of polynomials $S_n^m(x)$ defined by means of the following equation :

$$(1.1) \quad S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \; n = 0, 1, 2, \ldots,$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ are arbitrary constants.

On specializing these arbitrary coefficients $A_{n,k}$, $S_n^m(x)$ yields a number of known polynomials as special cases. These include, among

othors, the Jacobi polynomials, the Hermite polynomials, the Laguerre polynomials, the Gould-Hopper polynomials, Brafman polynomials, etc. (see [4]). Srivastava and Singh [4] recently established integrals involving the products of $S_n^m(x)$ and certain multivariable H-functions. In this paper we further extend these results and establish integrals (both, single and multiple) involving the products of $S_n^m(x)$ and multivariable H-functions. Our results are thus capable of yielding a number of integrals involving products of various polynomials with the multivariable H functions and other special functions to which the multivariable H-functions reduce.

In the sequel, we also require the following general class of polynomials of r variables :

$$(1.2)\quad R_n^{m_1,\ldots,m_r}(x_1,\ldots,x_r)$$

$$= \sum_{k_1,\ldots,k_r=0}^{J \leqslant n} (-n)_J\, A(n;k_1,\ldots,k_r) \prod_{i=1}^{r} \{x_i^{k_i}/k_i!\},$$

where $J = \sum_{i=1}^{r} (m_i . k_i)$, $m_i \geqslant 1$, $i = 1,\ldots,r$, and the coefficients

$A(n;k_1,\ldots,k_r)$, are arbitrary constants.

The multivariable H-function is represented here in the following manner (see [3, p. 251, eqn. (C. 1)]) :

$$(1.3)\quad H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P,Q:P_1,Q_1;\ldots;P_r,Q_r}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$$

$$\left.\begin{matrix}(a_j;\ \alpha'_j,\ldots,\alpha_j^{(r)})_{1,P} : (c'_j,\ \gamma'_j)_{1,P_1}\ ;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,P_r} \\ (b_j;\ \beta'_j,\ldots,\beta_j^{(r)})_{1,Q} : (d_j,\ \delta'_j)_{1,Q_1}\ ;\ldots;\ (d_j^{(r)},\ \delta_j^{(r)})_{1,Q_r}\end{matrix}\right],$$

For the definition of $H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ and the various conditions on the parameters as also the conditions of convergence, etc., we refer to [3, p. 252, eqns. (C 1) to (C. 8)]. These conditions are assumed to be satisfied throughout this paper. Also, the appearance of an asterisk (*) at a particular place indicates that parameters at that place are the same as on the right side of (1.3) at the corresponding place.

2. THE SINGLE INTEGRALS

For $i = 1, \ldots, r$ and with

$$X_i(x) = (x-\alpha)^{p_i}(\beta-x)^{q_i},\ p_i,\ q_i > 0$$

$$Y_i(x) = (x-\gamma)^{-p_i-q_i} X_i(x)$$

and

$$f(x) = (x-\alpha)^{u-1}(\beta-x)^{v-1}(x-\gamma)^{-u-v}$$

$g(x) = (x-\alpha)^{\mu}(\beta-x)^{\lambda}(x-\gamma)^{-\mu-\lambda},\ \mu,\ \lambda \geqslant 0$ (both not zero simultaneously), we have :

$$(2.1)\quad \int_{\alpha}^{\beta}(x-\alpha)^{u-1}(\beta-x)^{v-1} S_n^m[a(x-\alpha)^{\mu}(\beta-x)^{\lambda}]\, H\begin{bmatrix} z_1 X_1(x) \\ \vdots \\ z_r X_r(x)\end{bmatrix} dx$$

$$= (\beta-\alpha)^{u+v-1} \sum_{k=0}^{[n/m]} \left\{ \frac{(-n)_{mk}}{k!} A_{n,k}\ a^k (\beta-\alpha)^{k(\mu+\lambda)} \right.$$

$$. H^{0,\, N+2 : *\,;\ldots;\, *}_{P+2,\, Q+1 : *\,;\ldots;\, *} \left[\begin{array}{c|c} z_1\,(\beta-\alpha)^{p_1+q_1} & E : *\,,\ldots,\, * \\ \vdots & \\ z_r\,(\beta-\alpha)^{p_r+q_r} & F : *\,,\ldots,\, * \end{array} \right] \Bigg\}$$

$$(2.2) \quad \int_\alpha^\beta f(x)\, S_n^m [\, g(x)\,]\, H \begin{bmatrix} z_1\, Y_1(x) \\ \vdots \\ z_r\, Y_r(x) \end{bmatrix} dx$$

$$= R \sum_{k=0}^{[n/m]} \left\{ \frac{(-n)_{mk}}{k!} A_{n,k}\, [\alpha(\beta-\alpha)^{\mu+\lambda} (\beta-\gamma)^{-\mu} (\alpha-\gamma)^{-\lambda}]^k \right.$$

$$. H^{0,\, N+2 : *\,;\ldots;\, *}_{P+2,\, Q+1 : ;\ldots;\, *} \left[\begin{array}{c|c} Z_1 & E : *\,;\ldots;\, * \\ \vdots & \\ Z_r & F : *\,;\ldots;\, * \end{array} \right] \Bigg\}, \gamma < \alpha < \beta$$

with

$$R = (\beta-\alpha)^{u+v-1} (\beta-\gamma)^{-u} (\alpha-\gamma)^{-v}$$

and

$$Z_i = z_i\, (\beta-\alpha)^{p_i+q_i} (\beta-\gamma)^{-p_i} (\alpha-\gamma)^{-q_i}\,,\ i = 1\,,\ldots,\, r\,;$$

$$(2.3) \quad \int_\alpha^\beta f(x)\, S_n^m [\, (-1)^{\mu+\lambda}\, a\, g(x)\,]\, H \begin{bmatrix} (-1)^{p_1+q_1} z_1\, Y_1(x) \\ \vdots \\ (-1)^{p_r+q_r} z_r\, Y_r(x) \end{bmatrix} dx$$

$$= R \sum_{k=0}^{[n/m]} \left\{ \frac{(-n)_{mk}}{k!} A_{n,k}\, [\, a(\beta-\alpha)^{\mu+\lambda} (\gamma-\beta)^{-\mu} (\gamma-\alpha)^{-\lambda}]^k \right.$$

$$. H^{0,\, N+2 : *\,;\ldots;\, *}_{P+2,\, Q+1 : *\,;\ldots;\, *} \left[\begin{array}{c|c} (-1)^{p_1+q_1} Z_1 & E : *\,;\ldots;\, * \\ \vdots & \\ (-1)^{p_r+q_r} Z_r & F : *\,;\ldots;\, * \end{array} \right] \Bigg\},$$

$\alpha < \beta < \gamma$, where R and Z_i are defined with (2.2) above.

In equations (2.1) to (2.3),

$$E : (1-u-\mu k;\ p_1, \ldots, p_r), (1-v-\lambda k;\ q_1, \ldots, q_r), (a_j;\ \alpha'_j, \ldots, \alpha_j^{(r)})_{1,Q}$$

$$F : (1-u-v-(\mu+\lambda)\, k;\ p_1+q_1, \ldots, p_r+q_r), (b_j;\ \beta_j', \ldots, \beta_j^{(r)})_{1,Q}$$

and (2.1)—(2.3) are valid if

$$min\, \{Re\, (\, u + \sum_{i=1}^{r} p_i\, D_j^{(i)},\ v + \sum_{i=1}^{r} q_i\, D_j^{(i)})\, \} > 0$$

where

(2 4) $D_j^{(i)} = d_j^{(i)}/\delta_j^{(i)},\ j = 1, \ldots, M_i,\ i = 1, \ldots, r.$

3. THE MULTIPLE INTEGRALS

For $i = 1, \ldots, r$ and with

$$U_i\,(x_i) = (x_i - \alpha_i)^{p_i}\,(\beta_i - x_i)^{q_i},\ p_i, q_i > 0$$

$$W_i\,(x_i) = (x_i - \gamma_i)^{-p_i - q_i}\ U_i\,(x_i)$$

$V_i(x_i) = (x_i - \alpha_i)^{\mu_i}\ (\beta_i - x_i)^{\lambda_i},\ \mu_i, \lambda_i \geqslant 0$ (not all zero simultaneously),

$$T_i\,(x_i) = (x_i - \gamma_i)^{-\mu_i - \lambda_i}\ V_i(x_i),$$

and

$$f(x_i) = (x_i - \alpha_i)^{u_i - 1}\ (\beta_i - x_i)^{v_i - 1}\ (x_i - \gamma_i)^{-u_i - v_i},$$

we have :

$$(3.1)\quad \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} [\,(x_i-\alpha_i)^{u_i-1}\,(\beta_i-x_i)^{v_i-1}\,] \times$$

$$R_n^{m_1,\ldots,m_r}(a_1 V_1(x_i),\ldots,a_r V_r(x_r))\, H\begin{bmatrix} z_1\, U_1(x_1) \\ \vdots \\ z_r\, U_r(x_r)\end{bmatrix} dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} (B_i) \sum_{k_1,\ldots,k_r=0}^{J \leqslant n} \{\,(-n)_J\, A(n;\, k_1,\ldots,k_r)$$

$$\prod_{i=1}^{r} \left\{ \frac{a_i^{k_i}(\beta_i-\alpha_i)^{(\mu_i+\lambda_i)k_i}}{k_i\,!} \right\}$$

$$. H \begin{matrix} 0,\, N:\, M_1,\, N_1+2;\ldots;M_r,\, N_r+2 \\ P,\, Q:\, P_1+2,\, Q_1+1;\ldots;P_r+2,\, Q_r+1 \end{matrix} \left[\begin{matrix} z_1\,(\beta_1-\alpha_1)^{p_1+q_1} \\ \vdots \\ z_r\,(\beta_r-\alpha_r)^{p_r+q_r} \end{matrix} \middle| \begin{matrix} * : C \\ \\ * : D \end{matrix} \right] \Bigg\}$$

with

$$B_i = (\beta_i-\alpha_i)^{u_i+v_i-1}\,,\ i = 1,\ldots,r;$$

$$(3.2)\quad \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} [\, f(x_i)\,]\, R_n^{m_1,\ldots,m_r}(a_1 T_1(x_1),\ldots,a_r T_r(x_r))$$

$$. H \begin{bmatrix} z_1\, W_1(x_1) \\ z_r\, W_r(x_r) \end{bmatrix} dx_1 \ldots dx_r$$

$$= \sum_{i=1}^{r} (\eta_i) \prod_{k_1,\ldots,k_r=0}^{J \leqslant n} \{(-n)_J\, A(n;k_1,\ldots,k_r) \prod_{i=1}^{r} \left[\frac{a_i^{k_i}\, h_i^{k_i}}{k_i\,!} \right] \times$$

$$. H \begin{matrix} 0,\, N:\, M_1,\, N_1+2;\ldots;M_r,\, N_r+2 \\ P,\, Q:\, P_1+2,\, Q_1+1;\ldots;P_r+2,\, Q_r+1 \end{matrix} \left[\begin{matrix} Z_r' \\ \vdots \\ Z_r' \end{matrix} \middle| \begin{matrix} * : C \\ \\ * : D \end{matrix} \right] \Bigg\}$$

for $\gamma_i < \alpha_i < \beta_i$, $i = 1,\ldots,r$, *with*

$$\eta_i = (\beta_i - \alpha_i)^{u_i + v_i - 1} \; (\beta_i - \gamma_i)^{-u_i} \; (\alpha_i - \gamma_i)^{-v_i}$$

$$h_i = (\beta_i - \alpha_i)^{\mu_i + \lambda_i} \; (\beta_i - \gamma_i)^{-\mu_i} \; (\alpha_i - \gamma_i)^{-\lambda_i}$$

$$Z_i' = z_i \; (\beta_i - \alpha_i)^{p_i + q_i} \; (\beta_i - \gamma_i)^{-p_i} \; (\alpha_i - \gamma_i)^{-q_i}, \; i = 1, \ldots, r;$$

$$(3.3) \quad \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} [f(x_i)]$$

$$R_n^{m_1, \ldots, m_r} \; ((-1)^{\mu_1 + \lambda_1} a_1 T_1(x_1), \ldots, (-1)^{\mu_r + \lambda_r} a_r T_r(x_r))$$

$$H \begin{bmatrix} (-1)^{p_1 + q_1} z_1 \; W_1(x_1) \\ \vdots \\ (-1)^{p_r + q_r} z_r \; W_r(x_r) \end{bmatrix} dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} (\eta_i) \sum_{k_1, \ldots, k_r = 0}^{J \leqslant n} \left\{ (-n)_J \; A(n; k_1, \ldots, k_r) \prod_{i=1}^{r} \left[\frac{(-1)^{\mu_i + \lambda_i} \; a_i^{k_i} h_i^{k_i}}{k_i \, !} \right] \right.$$

$$\left. \cdot \; H \begin{matrix} 0, N: M_1, N_1 + 2; \ldots; M_r, N_r + 2 \\ P, Q: P_1 + 2, Q_1 + 1; \ldots; P_r + 2, Q_r + 1 \end{matrix} \left[\begin{matrix} (-1)^{p_1 + q_1} Z_1' \\ \vdots \\ (-1)^{p_r + q_r} Z_r' \end{matrix} \middle| \begin{matrix} * : C \\ * : D \end{matrix} \right] \right\}$$

for $\alpha_i < \beta_i < \gamma_i$, $i = 1, \ldots, r$,

where η_i, h_i and Z_i' are given above with (3.2).

In equations (3.1) to (3.3), we have

$$C : (1 - u_1 - \mu_1 k_1, p_1), (1 - v_1 - \lambda_1 k_1, q_1), (c_j', \gamma_j')_{1, P_1}; \ldots;$$

$$(1 - u_r - \mu_r k_r, p_r), (1 - v_r - \lambda_r k_r, q_r), (c_j^{(r)}, \gamma_j^{(r)})_{1, P_r}$$

$$D : (d_j', \delta_j)_{1, Q_1}, (1 - u_1 - v_1 - (\mu_1 + \lambda_1) k_1, p_1 + q_1); \ldots;$$

$(d_j^{(r)}, \delta_j^{(r)})_{1,Q_r}\ (1-u_r-v_r-(\mu_r+\lambda_r)k_r,\ p_r+q_r)$

and the conditions of validity of (3. 1) to (3.3) are :

$min\ \{Re\ (u_i+p_i\ D_j^{(i)},\ v_i+q_i\ D_j^{(i)})\ \} > 0,\ j = 2,\ldots,M_i;\ i=1,\ldots,r,$

where $D_j^{(i)}$ being given by (2.4);

$$(3.4)\quad \frac{1}{(2\pi\omega)^r}\int_{\eta_1-\omega\infty}^{\eta_1+\omega\infty}\cdots\int_{\eta_r-\omega\infty}^{\eta_r+\omega\infty} exp\left(\sum_{i=1}^{r} s_i\, t_i\right)\ [(t_i+p_i)^{-\alpha_i}\ ]$$

$$.\ R_n^{m_1,\ldots,m_r}\ (a_1(t_1+p_1)^{-\lambda_1}\ ,\ldots,\ a_r(t_r+p_r)^{-\lambda_r}\ )\ H\left[\begin{matrix} z_1(t_1+p_i)^{-\mu_1} \\ \vdots \\ z_r(t\ +p\ )^{-\mu_r}\end{matrix}\right| dt_1..dt_r$$

$$= \prod_{i=0}^{r}(s_i^{a_i-1})\ exp\ (-\sum_{i=1}^{r} s_i\, p_i)\ \sum_{k_1,\ldots,k_r=0}^{J\leqslant n}\{(-n)_J\ A(n;\ k_1,\ldots,k_r)$$

$$\prod_{i=1}^{r}\left[\frac{a_i^{k_i}\ s_i^{\lambda_i k_i}}{k_i\ !}\right]$$

$$.\ H\left[\begin{matrix} 0, \\ P,\ Q: P_1,\ Q_1+1;\ldots;P_r,\ Q_r+1\end{matrix}\ :\ \begin{matrix} M_1,\ N_1;\ldots;M_r,\ N_r \\ \end{matrix}\ \left[\begin{matrix} z_1\ s_1^{\mu_1} \\ \vdots \\ z_r\ s_r^{\mu_r}\end{matrix}\right|\ \begin{matrix} * : * ;\ldots; * \\ * : D_1 \end{matrix}\right]\Bigg\}$$

$(w = \sqrt{-1})$ with

$D_1 :\ (d_j',\ \delta_j')_{1,Q_1}\ ,\ (1-\alpha_1-\lambda_1 k_1,\ \mu_1);\ldots;(d_j^{(r)},\ \delta_j^{(r)},\ \delta_j^{(r)})_{1,Q_r}$

$(1-\alpha_r-\lambda_r\ k_r,\ \mu_r)$

provided that $\lambda_i,\ \mu_i,\ s_i > 0,\ \eta_i + Re\ (p_i) > 0,\ Re\ (\alpha_i) > 0,$

$Re\ (\alpha_i) + \eta_i \sum_{j=1}^{r} \mu_j > 0,\ i = 1,\ldots,r;$

$$(3.5)\ \frac{1}{(2\pi\omega)^r}\int_{\eta_1-\omega\infty}^{\eta_1+\omega\infty}\cdots\int_{\eta_r-\omega\infty}^{\eta_r+\omega\infty} exp\left(\sum_{i=1}^{r} s_i\, t_i\right)\prod_{i=1}^{r}\ [(t_i+p_i)^{-\alpha_i}\ ]$$

$$.\ R_n^{m_1,\ldots,m_r}\ (a_1(t_1+p_1)^{-\lambda_1}\ ,\ldots,\ a_r(t_r+p_r)^{-\lambda_r}\ )\ H\begin{bmatrix} z_1(t_1+p_1)^{\mu_1} \\ \vdots \\ z_r(t_r+p_r)^{\mu_r}\end{bmatrix} dt_1\ldots dt_r$$

$$= exp\ (-\sum_{i=1}^{r} s_i\, p_i)\ \prod_{i=1}^{r}\ (s_i^{\alpha_i-1}\ )\ \sum_{k_1,\ldots,k_r=0}^{J\leqslant n}\ \{(-n)_J\ A(n;\ k_1,\ldots,k_r)$$

$$.\ \prod_{i=1}^{r}\ \left\{\frac{a_i^{k_i}\ s_i^{\lambda_i k_i}}{k_i\ !}\right\} H\ \begin{matrix} 0,\ N:\ M_1,\ N_1;\ldots;M_r,\ N_r \\ P,\ Q:\ P_1+1,\ Q_1,\ldots,P_r+1,\ Q_r\end{matrix}$$

$$\left.\left[\begin{matrix} z_1\ s_1^{-\mu_1} \\ \vdots \\ z_r\ s_r^{-\mu}\end{matrix}\ \middle|\ \begin{matrix} *\ :\ C_1 \\ *\ :\ *\ ;\ldots;\ *\end{matrix}\right]\right\}$$

$(\omega = \sqrt{-1})$ with

$$C_1 : (c_j',\ \gamma_j')_{1,P_1},\ (\alpha_1+\lambda_1 k_1,\ \mu_1)\ ;\ldots;\ (c_j^{(r)},\ \gamma_j^{(r)})_{1,P_r},\ (\alpha_r+\lambda_r k_r,\ \mu_r)$$

provided that $\lambda_i,\ \mu_i > 0,\ \eta_i + Re\ (p_i) > 0,\ Re\ (\alpha_i) > 0,$

and $Re\ (\alpha_i) > \eta_i \sum_{j=1}^{r} \mu_j\ ,\ i = ,\ldots,\ r.$

Method of proof : To establish (2 1), we substitute for $S_n^m\ [x]$ occurring in the integrand on the left of (2.1), from (1.1), and then integrate term-by-term to get

$$\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!}\ A_{n,k}\ a^k \int_{\beta}^{\alpha} (x-\alpha)^{u+\mu k-1}\ \ (\beta-x)^{v+\lambda k-1}$$

$$H\begin{bmatrix} z_1 X_1(x) \\ \vdots \\ z_r X_r(x) \end{bmatrix} dx .$$

We next substitute for the multivariable H-function occurring in the integrand of the above integral interms of the multiple contour integral from [3, p. 251, eqn. (C. 1)], change the order of integrations, evaluate the x-integral by means of [1, p. 10, eqn. (13)] and then interpret the resulting integral with the help of [3, p. 251, eqn. (C. 1)] to get the desired result.

Formulas (2.2) and (2.3) are established similarly with the help of [1, p. 10, eqns. (14), (15)].

The multiple integrals (3.1) to (3.3) are established with the help of (1.2) in a similar manner, while as the multiple contour integrals (3.4) to (3.5) are established with the help of the following result :

$$\frac{1}{2\pi\omega}\int_{\eta-\omega\infty}^{\eta+\omega\infty} e^{st}(t+p)^{-\alpha}\, dt = \frac{s^{\alpha-1}\, e^{-sp}}{\Gamma(\alpha)}, \; s > 0,$$

($\omega = \sqrt{-1}$), where, for convergence $\eta + Re\,(p) > 0$ and $Re\,(\alpha) > 0$.

4. Particular Cases

(i) If, in (2.1), we take $\mu=0$ and $\lambda=1$, it reduces to the known result [4, p. 166, eqn. (2.2)].

(ii) If, in (3.1), we replace the general class of polynomials of r variables, viz.

$R_n^{m_1,\ldots,m_r}(x_1,\ldots,x_r)$ by the product of r polynomials of the type $S_{n_i}^{m_i}(x_i)$ it yields :

$$(4.1)\quad \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_r}^{\beta_r} \prod_{i=1}^{r} \left\{ (x_i-\alpha_i)^{u_i-1} (\beta_i-x_i)^{v_i-1} S_{n_i}^{m_i} [a_i V_i(x_i)] \right\} \times$$

$$. H \begin{bmatrix} z_1 U_1(x_1) \\ \vdots \\ z_r U_r(x_r) \end{bmatrix} dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} (B_i) \sum_{k_i=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left\{ \prod_{i=1}^{r} \left[\frac{(-n_i)_{m_i k_i}}{k_i\,!} A_{n_i, k_i} a_i^{k_i} (\beta_i-\alpha_i)^{(\mu_i+\lambda_i)k_i} \right] \right.$$

$$\left. .H^{\,0,\,N:\,M_1,\,N_1+2;\ldots;M_r,\,N_r+2}_{\,P,\,Q:\,P_1+2,\,Q_1+1;\ldots;P_r+2,\,Q_r+1} \begin{bmatrix} z_1 (\beta-\alpha)^{p_1+q_1} \\ \vdots \\ z_r (\beta-\alpha)^{p_r+q_r} \end{bmatrix} \begin{vmatrix} * : C \\ * : D \end{vmatrix} \right\}$$

where $U_i(x_i)$, $V_i(x_i)$ and B_i, C, D are given above. The conditions of validity of (4.1) are:

min $\{Re\,(u_i + p_i D_j^{(i)}, v_i + q_i D_j^{(i)})\} > 0$, $j=1, \ldots, M_i$; $i=1, \ldots, r$, where $D_j^{(i)}$ being given by (2.4).

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SOME RESULTS ON FIXED POINTS FOR THREE MAPS

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In this note we present some results on fixed point theorem for three maps which generalize the results of Park and Rhoades [4] and Park [3] respectively. First we give the following definitions and notations.

Throughout the paper w stands for the set of all nonnegative integers and R_+ for the set of all non-negative real numbers.

A point $x_0 \varepsilon X$ is called regular for f, g and h, or, simply regular, if there exists a sequence $\{ x_n \} \subset X$ satisfying, $hx_{2n+1} = fx_{2n}$ and $hx_{2n+2} = gx_{2n+1}$ for each $n \varepsilon w$, and $\sup \{ d(hx_i, hx_j) \mid i, j \varepsilon w \} < \infty$. It should be noted that for $f(X) \cup g(X) \subseteq h(X)$, such a $\{x_n\}$ always exists.

Let $O(x_0) = \{ hx_n \mid n \varepsilon w \}$, and $\delta [O(x_0)]$ denote the diameter of $O(x_0)$.

Let $\delta(x,y) =$ diameter $\{ O(x) \cup O(u) \}$ where x and y are regular We have the following results :

Theorem 1. *Let* f, g, h *be self-maps of a metric space* (X,d) *such that* fh = hf, gh = hg, $\phi : R_+ \longrightarrow R_+$, ϕ *non-decreasing, continuous from the right, and satisfying* $\phi(t) < t$ *for each* $t > 0$. *Suppose there*

exists a regular point $x_0 \in X$ *such that* $\{hx_n\}$ *has a cluster point* $a_0 \in X$, *which is regular.*

If

(1) $d(fx, gy) \leqslant \phi(\delta[O(x) \cup O(y)])$ *for each* $x, y \in \{x_n\} \cup \{a_n\} \cup ha_0$, *where* $\{a_n\}$ *is defined by* $ha_{2n+1} = fa_{2n}$ *and* $ha_{2n+2} = ga_{2n+1}$ *for* $n \in w$, *and*

(2) h *is continuous at* a_0,

then ha_0 *is a common fixed point for* f, g, h *and* $\{hx_n\}$ *converges to* a_0. *If* (1) *is satisfied for all regular points* $x, y \in X$, *then* ha_0 *is the unique common fixed point of* f, g and h.

The result can be easily proved by coupling the proof for two maps by Park and Rhoades [4] with the usual technique for the proof of fixed point for theree maps, given by Ganguly([1],[2]) and Singh[5].

Theorem 2. *Let* f, g, h *be self maps of a complete metric space* (X, d) *such that* $fh = hf$, $gh = hg$, h *is continuous, and* $\phi : R_+ \to R_+$, $\phi(t) < t$ *for each* $t > 0$. *If every point of* X *is a regular point and* (1) *is satisfied for all* $x, y \in X$, *then* f, g and h *have a unique common fixed point and* $\{hx_n\}$ *converges to the fixed point for each* $x \in X$.

The result follows from Theorem 1.

Remark 1. For $f = g$, we have Theorems 1 and 2 of Park and Rhoades [4].

Remark 2. From the logic of Park and Rhoades ([4], pp. 116–117) it follows that Theorem 1 of Canguly [1] is a special case of Theorems 2 and 1, respectively.

Now we state a different result which can also be easily proved.

Theorem 3. *Let* (X, d) *be a metric space,* f, g *and* h *be the same as above. Suppose there exists a regular point* $x_0 \in X$ *such that*

(1) *$O(x_0$ has a cluster point $a_0 \in X$ which is regular;*

(2) *for any $\epsilon > 0$, there exist $\epsilon_0 < \epsilon$ and $\delta_0 > 0$ such that for any $x, y \in \{x_n\} \cup \{a_n\} \cup \{ha_0\}$,*

$$\epsilon \leqslant \delta(x, y) < \epsilon + \delta_0 \text{ implies } d(fx, gy) \leqslant \epsilon_0,$$ *where $\{a_n\}$ is the same as in Theorem 1, and*

(3) *if ha_0 is continuous at a_0,*

then ha_0 is a common fixed point of f, g and h, and $\{hx_n\}$ converges to a_0. If (2) is satisfied for all regular points $x, y \in X$, then ha_0 is the unique common fixed point of f, g and h.

Remark 3. For $f=g$, we have Theorem 2 (C δ') of Park [3].

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A NOTE ON ENTIRE FUNCTIONS OF IRREGULAR (p, q)-GROWTH

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1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a nonconstant entire function. Set $M(r) \equiv M(r, f) = max_{|z|=r} |f(z)|$, $\mu(r) \equiv \mu(r, f) = max_{n \geqslant 0} \{|a_n| r^n\}$, $M(r)$ and $\mu(r)$ are called the maximum modulue and the maximum term of $f(z)$ respectively. The concept of (p, q)-order, lower (p, q)-order, (p, q)-type and lower (p, q)-type of an entire function $f(z)$ having index-pair $(p, q), p \geqslant q \geqslant 1$ has recently been introduced by Juneja *et al.* ([2], [3]). Thus $f(z)$ is said to be of (p, q)-order ρ and lower (p, q)-order λ if

$$(1.1) \quad \lim_{r\to\infty} \begin{matrix} sup \\ inf \end{matrix} \frac{log^{[p]} M(r)}{log^{[q]} r} = \begin{matrix} \rho(p, q) \equiv \rho \\ \lambda(p, q) \equiv \lambda \end{matrix}$$

and the function $f(z)$ having (p, q)-order $\rho (b < \rho < \infty)$ is said to be of (p, q)-type T and lower (p, q)-type t if

$$(1.2) \quad \lim_{r\to\infty} \begin{matrix} sup \\ inf \end{matrix} \frac{log^{[p-1]} M(r)}{(log^{[q-1]} r)^{\rho}} = \begin{matrix} T(p, q) \equiv T \\ t(p, q) \equiv t, \end{matrix}$$

where $b = 0$ if $p > q$ and $b = 1$ if $p = q$ and $log^{[p]} x$ stands for the

pth iterate of log x.

Definition. An entire function for which (p, q)-order and lower (p, q)-order are equal is said to be of regular (p, q)-growth. Functions which are not of regular (p, q)-growth are said to be of irregular (p, q)-growth.

It is easy to see that the lower (p, q)-type of an entire function of irregular (p, q)-growth is always zero. Hence the study of growth properties of entire functions with nonzero lower (p,q)-type is limited to functions of regular (p, q)-growth. Thus we need to define a new constant to study the growth of such functions.

Let $f(z)$ be an entire function of lower (p, q)-order $\lambda (b < \lambda < \infty)$. A real valued positive function $\lambda(r)$ defiend on $(0, \infty)$ is said to be a lower proximate order of an entire function $f(z)$ with index-pair (p, q) if

(i) $\lambda(r) \to \lambda$ as $r \to \infty$ and

(ii) $\Lambda_{[q]}(r)\, \lambda\ (r) \to 0$ as $r \to \infty$

where $\Lambda_{[q]}(r) = \log^{[q]}{}_r \log^{[q-1]}{}_r \ldots \log r.\, r.$

If such a function exists and

$$(1.3)\quad \liminf_{r\to\infty} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]}{}_r)\, \lambda(r)} = t_\lambda,\ 0 < t_\lambda < \infty,$$

then t_λ is termed as generalized $\lambda(p,q)$-type of $f(z)$ with respect to the comparison function $\lambda(r)$.

The object of this note is to prove the following theorem which generalizes a result due to Kumar [4].

2. Theorrem. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of lower (p, q) order $\lambda (b < \lambda < \infty)$. Then a necessary and sufficient condition for $f(z)$ to be of generalized $\lambda(p, q)$-type t_λ is that

$$(2.1) \quad \lim_{n\to\infty} \inf \left[\frac{\phi(\log^{[p-2]} n)}{\log^{[q-1]} (-1/n \log |a_n|)} \right]^{\lambda - A} = t_\lambda / M,$$

provided $\Psi(n) = |a_n/a_{n+1}|$ forms a nondecreasing function of n for $n \geqslant n_0$, where

$$M = \begin{cases} 1/\lambda \left(\frac{(\lambda-1}{\lambda}\right)^{\lambda-1} & if\ (p, q) = (2,2) \\ \frac{1}{e\lambda} & if\ (p, q) = (2,1) \\ 1 & \text{for all other index-pairs } (p, q), \end{cases}$$

$A = 1$ $if (p, q) = (2,2)$ and 0 otherwise and $\Phi(t)$ is a real valued function defined by

$$(2.2) \quad t = (\log^{[q-1]} r)^{\lambda(r)-A} \Leftrightarrow \Phi(t) \log^{[q-1]} r.$$

3. To prove this theorem we need the following lemma :

Lemma 1. [1]. $(\log^{[q-1]} r)^{\lambda(r)-A}$ is a monotone increasing function for sufficiently large values of r, where $0 \leqslant A \leqslant \lambda$.

Lemma 2. Let $\lambda(r)$ be a lower proximate order of an entire function with index-pair (p, q). Then

$$(3.1) \quad \lim_{t\to\infty} \frac{\Phi(ht)}{\Phi(t)} = h^{1/\lambda - A}, 0 < h < \infty.$$

Proof. In view of (2.2), we have

$$\frac{d[\log \Phi(t)]}{d[\log t]} = \frac{d[\log^{[q]} r]}{d[(\lambda(r) - A) \log^{[q]} r]}$$

$$= \frac{1}{(\lambda(r) - A) + \Lambda_{[q]}(r)\lambda'(r)}.$$

Hence,

$$\lim_{t \to \infty} \frac{d[\log \Phi(t)]}{d[\log t]} = \frac{1}{\lambda - A}.$$

It follows that for given $\varepsilon > 0$ and $t > t_0$,

$$\left(\frac{1}{\lambda - A} - \varepsilon\right) d[\log t] < d[\log \Phi(t)] < \left(\frac{1}{\lambda - A} + \varepsilon\right) d[\log t],$$

which gives on integrating between the limits t and ht

$$(3.2) \quad \left(\frac{1}{\lambda - A} - \varepsilon\right) \log h < \log \frac{\Phi(ht)}{\Phi(t)} < \left(\frac{1}{\lambda - A} + \varepsilon\right) \log h.$$

Since $\varepsilon > 0$ is arbitrary, on passing to limits in (3.2) we get the required result.

4. **Proof of the Theorem.** Let $0 < t_\lambda < \infty$. Then from (1.3), we have for any $\varepsilon > 0$ and $r > r_0(\varepsilon)$,

$$(4.1) \quad \log^{[p-1]} M(r) > (t_\lambda - \varepsilon) \left(\log^{[q-1]} r\right)^{\lambda(r)}$$

Since $f(z)$ is of finite (p,q)-order, following Valiron [5], it can easily be proved that

$$\log^{[p-1]} M(r) \simeq \log^{[p-1]} \mu(r) \text{ as } r \to \infty.$$

Thus, from (4.1), we have

$$\log \mu(r) > \exp^{[p-2]}\{(t_\lambda-\varepsilon)\ (\log^{[q-1]}r)^{\lambda(r)}\}.$$

Since $\Psi(n)$ is a nondecreasing function of n, we have for r satisfying $\Psi(n-1) \leqslant r < \Psi(n)$,

$$\log |a_n| + n \log r > \exp^{[p-2]}\{(t_\lambda-\varepsilon)\ (\log^{[q-1]}r)^{\lambda(r)}\}$$

or,

$$(4.2)\quad X \equiv \frac{[\Phi(\log^{[p-2]}n)]^{\lambda-A}}{\exp[(\lambda-A)\log^{[q-1]}\{-1/n \log |a_n|\}]} >$$

$$\frac{[\Phi(\log^{[p-2]}n)]^{\lambda-A}}{\exp[\lambda-A)\log^{[q-1]}\{\log r-1/n \exp^{[p-2]}\{(t_\lambda-\varepsilon)\ (\log^{[q-1]}r)^{\lambda(r)}\}\}]}.$$

With little calculation and the use of properties of $\Psi(t)$ and $\lambda(r)$, we can see that the minimum value of the expression on the right hand side of the above inequality is attained for a value of $r=r(n)$ given by

$$(4.3)\quad \frac{E_{[p-2]}\{(t_\lambda-\varepsilon)\ (\log^{[q-1]}r)^{\lambda(r)}\}}{\Lambda_{[q-1]}(r)} = \frac{n}{\lambda r},$$

where $E_{[q]}(x) = \prod_{i=0}^{q} \exp^{[i]}x$.

For the index-pair $(p,q) = (2,1), (4,3)$ reduces to

$$r^{\lambda(r)} = \frac{n}{\lambda(t_\lambda-\varepsilon)} \Leftrightarrow r = \Phi\left(\frac{n}{\lambda(t_\lambda-\varepsilon)}\right).$$

Therefore,

$$= \frac{[\Phi(n)]^{\lambda}}{\exp[\lambda\{\log r - r^{\lambda(r)}\ (t_\lambda-\varepsilon)/n\}]}$$

$$= e\left[\frac{\Phi(n)}{\Phi(n/\lambda(t_\lambda-\varepsilon))}\right]^\lambda,$$

or,

(4.4) $X > e\,\lambda\,(t_\lambda-\varepsilon)$.

For $(p, q) = (2,2), (4,3)$ reduces to

$$(\log r)^{\lambda(r)-1} = \frac{n}{\lambda(t_\lambda-\varepsilon)} \Leftrightarrow \log r = \Phi\left(\frac{n}{\lambda(t_\lambda-\varepsilon)}\right).$$

Therefore, we have in this case

$$X > \left[\frac{\Phi(n)}{\frac{(\lambda-1)}{\lambda}\,\Phi\left(\frac{n}{\lambda(t_\lambda-\varepsilon)}\right)}\right]^{\lambda-1}$$

i. e.,

(4.5) $X > \dfrac{\lambda^\lambda}{(\lambda-1)^{\lambda-1}}\,(t_\lambda-\varepsilon)$.

For $(p, q) \neq (2,1)$ and $(2,2)$, (4.3) gives

$$\log^{[q-1]} r)^{\lambda(r)} = \frac{1}{t_\lambda-\varepsilon}\,\log^{[p-2]} n/\lambda \Leftrightarrow \log^{[q-1]} r = \Phi\left(\frac{1}{t_\lambda-\varepsilon}\,\log^{[p-2]} n/\lambda\right).$$

Thus

$$X > \frac{[\Phi(\log^{[p-2]} n)]^\lambda}{\exp\{\lambda \log^{[q]} (re^{-1/\lambda})\}}$$

$$\geq \left[\frac{\Phi(\log^{[p-2]} n)}{\Phi\left(\frac{1}{t_\lambda-\varepsilon}\,\log^{[p-2]} n/\lambda\right)}\right]^\lambda$$

or, from Lemma 2,

(4.6) $X > (t_\lambda - .\epsilon)$

(4,2), (4.4), (4.5) and (4.6) combine into

(4.7) $\lim_{n \to \infty} \inf X \geqslant t_\lambda / M.$

The inequality in (4.7) obviously holds when $t_\lambda = 0$ and for $t_\lambda = \infty$, the same arguments with an arbitrary large number in place of $(t_\lambda - \varepsilon)$ lead to the result $\lim_{n \to \infty} \inf X = \infty$.

To obtain (2.1), we show that the strict inequality in (4.7) can not hold. Hence, let a number $t_\alpha > t_\lambda$ be given such that

$$\lim_{n \to \infty} \inf \left[\frac{\Phi(\log^{[p-2]} n)}{\log^{[q-2]} \{-1/n \log | a_n | \}} \right]^{\lambda - A} = \frac{t_\alpha}{M}.$$

Let us choose a number t_β such that $t_\lambda < t_\beta < t_\alpha$. Then we have for alll arge values of $n > n_0$,

$$\log | a_n | - n \exp^{[q-2]} \left[\frac{\Phi(\log^{[q-2]} n)}{(t_\beta / M)^{1/\lambda - A}} \right].$$

Hence, by Cauchy's inequlity $| a_n | r^n \leqslant M(r)$, we, again have for $r > r_0$ and $n > n_0$,

$$(4.8) \quad \log M(r) > -n \exp^{[q-2]} \left[\frac{\Phi(\log^{[p-2]} n)}{(t_\beta / M)^{[1/\lambda - A}} \right] + n \log r.$$

For $(p, q) = (2, 1)$, choose r so that $n = \lambda t_\beta r^{\lambda(r)}$. Then, in view of Lemma 2, we have

$$\log M(r) > -n \log \left[\frac{\Phi(n)}{(e \lambda t_\beta)^{1/\lambda}} + n \log \Phi\left(\frac{n}{\lambda t_\beta} \right) \right.$$

$$= n \log \left[(e\lambda \ t_\beta)^{1/\lambda} \ \frac{\Phi^{(n/\lambda \, t_\beta)}}{\Phi(n)} \right]$$

(4.5) $\simeq t_\beta \, r^{\lambda(r)}$.

In case $(p, q) = (2.2)$, choose

$$(\log \ r)^{\lambda(r)-1} = \frac{M \, n}{t_\beta \left(\frac{\lambda-1}{\lambda} \right)^{\lambda-1}} \Leftrightarrow \log r = \Phi \left(\frac{M \, n}{t_\beta \left(\frac{\lambda-1}{\lambda} \right)^{\lambda-1}} \right),$$

then (4 8) gives

$$\log M(r) > n \left[\log r - (n) \, (M/t_\beta)^{1/\lambda-1} \right]$$

$$\simeq n/\lambda \log r$$

(4.10) $$\simeq t_\beta \, (\log r)^{\lambda(r)}.$$

Lastly let us take $(p, q) \neq (2,1)$ and $(2,2)$. Then we choose r such that

$$t_\beta \, (\log^{[q-1]} r/e^{\varepsilon})^{\lambda(r/e^{\varepsilon})} = \log^{[p-2]} n/\lambda \Leftrightarrow \log^{[q-1]} (r/e^{\varepsilon}) =$$

$$= \Phi \left(\frac{\log^{[p-2]} n/\lambda)}{t_\beta} \right), \qquad \varepsilon > 0.$$

Using (3.1), (4.8) reduces to

$$\log M(r) > n \left[\log r - \exp^{[q-2]} \left\{ \frac{\Phi \left(\log^{[p-2]} n \right\}}{(t_\beta)^{1/\lambda}} \right\} \right]$$

$$\simeq n \left[\varepsilon + \exp^{[q-2]} \left\{ \frac{\Phi \left(\log^{[p-2]} n/\lambda \right.}{t_\beta^{1/\lambda}} \right\} - \exp^{[q-2]} \left\{ \frac{\Phi \left(\log^{[p-2]} n \right.}{t_\beta^{1/\lambda}} \right\} \right]$$

or,

(4.11) $\log^{[p-1]} M(r) > (\log^{[q-1]} r)^{\lambda(r)} \{ t_\beta + o(1) \}$.

Thus, (4.9), (4.10) and (4.11) lead to

which is a contradiction. Thus, strict inequality in (4.7) can not hold. This proves the result (2.1) and hence the theorem.

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APPLICATIONS OF SRIVASTAVA'S HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES IN HEAT CONDUCTION

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1. INTRODUCTION

Singh [6] evaluated some integrals involving Kampé de Fériet's function and employed one of them to obtain a solution of a problem in heat conduction considered by Bhonsle [1]. Some expansion formulae involving above functions have also been obtained. Appell's functions and the functions related to them have many applications in mathematical physics. Here we evaluate an integral involving Srivastava's hypergeometric function $F^{(3)}[x, y, z]$ of three variables ([7], P. 428) and discuss it's applications in solving a problem on heat conduction considered by Bhonsle [1] and in establishing an expansion formula involving Srivastava function.

2. AN INFINITE INTEGRAL

Multiplying both sides of the equation [4, *p*, 74, E_g (4. 16. 1)] by $e^{-z^2} H_{2\nu}(z)$ and using the orthogonality property of Hermite polyno-

mlals [5], we have

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z)dz = \frac{\sqrt{\pi}2^{2(\nu-\rho)}\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)}, \quad \rho = 0, 1, 2, \ldots$$

which, on application to Srivastava's triple series $F^{(3)}[uz^2, vz^2, w]$, further gives

$$(2.1)\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z) F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b'')::(c);(c');(c'');\\(e)::(g);(g');(g'')::(h);(h');(h'');\end{matrix}\; uz^2, vz^2, wz^2\right]dz$$

$$= \frac{\Gamma(2\rho+1)\sqrt{\pi}}{\Gamma(1+\rho-\nu)} 2^{2(\nu-\rho)} F^{(3)}\left[\begin{matrix}(a)\ \frac{2\rho+1}{2}, \rho+1::(b);(b');(b'')::(c);(c');(c'');\\(e), 1+\rho-\nu::(g);(g');(g'')::(h);(h');(h'');\end{matrix}\; u, v, w\right],$$

where $\rho = 0, 1, 2, \ldots$ and $A + B + B'' + C \leqslant E + G + G'' + H$, etc.

An application of the above integral will be shown in our present investigation.

3. APPLICATION TO HEAT CONDUCTION

Bhonsle [1] employed Hermite polynomials in solving the partial differential equation

$$(3.1) \quad \frac{\partial\Phi}{\partial t} - K\frac{\partial^2\Phi}{\partial z^2} - K\Phi z^2,$$

where $\Phi(z,t)$ tends to zero for large value of it; and when $|z| \to \infty$, this equation is related to the problem of heat conduction

$$(3.2)\quad \frac{\partial \Phi}{\partial t} = K \frac{\partial^2 \Phi}{\partial z^2} - h_1 (\Phi - \Phi_0),$$

provided that

$$\Phi_0 = 0 \text{ and } h_1 = Kz^2;$$

The solution of equation (3.1) given by Bhonsle [1] is

$$(3.3)\quad \Phi(z, t) = \sum_{r=0}^{\infty} \Phi_r \, e^{-(1+2r)Kt - z^2/2} \, H_r(z).$$

We shall consider the problem of determining a function $\Phi\ (z, t)$ such that

$$(3.4)\quad \Phi(z, 0) = z^2 \, e^{-z^2 \rho} \, F^{(3)} \left[\begin{matrix} (a) :: (b); (b'); (b''); (c); (c'); (c''); \\ (e) :: (g); (g'); (g''); (h); (h'); (h''); \end{matrix} \; uz^2, vz^2, wz^2 \right].$$

Now from (3.3) and (3.4), we have

$$(3.5)\quad z^{2\rho} \, e^{-z^2} \, F^{(3)} \left[\begin{matrix} (a) :: (b); (b'); (b''); (c); (c'); (c''); \\ (e) :: (g); (g'); (g''); (h); (h'); (h''); \end{matrix} \; uz^2, vz^2, wz^2 \right].$$

$$= \sum_{r=0}^{\infty} Q_r \, e^{-z^2/2} \, H_r(z),$$

and, by (2.1) and the orthogonality property of Hermite polynomials [2, p.289], we find that

$$(3.6)\quad Q_\mu = \frac{\Gamma(2\rho+1) \; 2^{\mu-2\rho-1/2}}{\Gamma(1+\rho-\mu/2) \;\; \mu!} \, F^{(3)} \left[\begin{matrix} (a), \frac{2\rho+1}{2}, \rho+1 :: (b); \\ (e), 1+\rho-\mu/2 :: (g); \end{matrix} \right.$$

$$(b'); (b''): (c); (c'); (c''); \quad u, v, w \Big].$$
$$(g'); (g''): (h); (h'); (h'');$$

Thus the solution of (3.3) of the problem reduces to

$$(3.7) \quad \Phi(x, t) = \sum_{r=0}^{\infty} \frac{2^{r-2\rho-1/2}\,\Gamma(2\rho+1)}{\Gamma(1+\rho-r/2)\;\;r!}\; e^{-(1+2r)Kt-z^2/2}$$

$$H_r(z)\, F^{(3)}\left[\begin{matrix}(a), \frac{2\rho+1}{2}, \rho+1 :: (b); (b'); (b''): (c); (c'); (c''); \\ (e), 1-\rho-r/2 \;\; :: (g); (g'); (g''): (h); (h'); (h'');\end{matrix}\; u, v, w\right].$$

where $\rho = 0, 1, 2, \ldots$, and $A+B+B''+C \leqslant E+G+G''+H$, etc.

4. EXPANSION FORMULA

By (3.5) and (3.6) we establish the following expansion formula

$$(4.1) \quad z^{2\rho}\, e^{-z^2/2}\; F^{(3)}\left[\begin{matrix}(a) :: (b); (b'); (b''): (c); (c'); (c''); \\ (e) :: (g); (g'); (g''): (h); (h'); (h'');\end{matrix}\; uz^2, vz^2, wz^2\right]$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(2\rho+1)}{\Gamma(1+\rho-r/2)}\,\frac{2^{r-2\rho-1/2}}{r!}\, H_r(z).$$

$$F^{(3)}\left[\begin{matrix}(a), \frac{2\rho+1}{2}, \rho+1 :: (b); (b'); (b''): (c); (c'); (c''); \\ (e), 1-\rho-r/2 \;\; :: (g); (g'); (g''): (h); (h'); (h'');\end{matrix}\; u, v, w\right].$$

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INTEGRALS INVOLVING THE H-FUNCTION

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ABSTRACT

The aim of this paper is to evaluate two new integrals involving Fox's H-function. Besides the H-function, our integrands also involve Bessel functions and Legendre functions. Since the H-function is sufficiently general in nature, a number of other integrals involving various special functions can also be obtained from these integrals simply on specializing the parameters of the H-functions involved in the main integrals. For the sake of illustration we also evaluate here two such integrals; these latter integrals are of interest in themselves and are also believed to be new.

1. INTRODUCTION

The following formulae will be required for evaluating the main integrals of this paper [1, *p*. 91, *Eq*. (3.1); 4, *p*. 3, *Eq*. (5)]

$$\text{(i)} \int_0^1 x^{\sigma-\lambda-\delta} (1-x^2)^{-\frac{1}{2}\mu} p_\nu^\mu (x) J_\delta(xt) J_\lambda(xt)\, dx$$

$$= \frac{2^{\mu-\lambda-\delta-1} \Gamma(\frac{1}{2}+\frac{1}{2}\sigma) \Gamma(1+\frac{1}{2}\sigma) t^{\lambda+\delta}}{\Gamma(\delta+1) \Gamma(\lambda+1) \Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)}$$

$$\cdot\ {}_4F_5\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}\sigma,\ 1+\frac{1}{2}\sigma,\ \frac{1}{2}(\lambda+\delta+1),\ \frac{1}{2}(\lambda+\delta+2)\\ 1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu,\ 3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu,\ \delta+1,\lambda+1,\lambda+\delta+1\end{matrix};-t^2\right], \tag{1.1}$$

where $\mathrm{Re}\,(\mu) < 1$, $\mathrm{Re}\,(\sigma) > -1$;

(ii) $$\int_0^\infty x^{\nu+1}\,(\alpha^2+x^2)^{-\lambda/2-5/4}\,exp\left(-\frac{p^2\,\alpha}{\alpha^2+x^2}\right)J_\nu\left(\frac{p^2x}{\alpha^2+x^2}\right)dx$$

$$= 2^{-\lambda-\nu-\frac{1}{2}}\,p^{2\nu}\sqrt{\pi}\,\alpha^{-\lambda-\frac{1}{2}}\,\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda/2+3/4)\,\Gamma(\lambda/2+\nu+5/4)}$$

$$\cdot\ {}_1F_1\,[\lambda+\tfrac{1}{2};\ \nu+\lambda/2+5/4;\ -\frac{p^2}{2\alpha}], \tag{1.2}$$

where

$$R(\alpha) > 0,\ p > 0,\ R\,(\lambda+\tfrac{1}{2}) > 0\ \ R(\nu+1) > 0.$$

MAIN INTEGRALS

(i) $$\int_0^1 x^{\sigma-\lambda-\delta}\,(1-x)^{-\frac{1}{2}\mu}\ P_\nu^\mu\,(x)\,J_\delta(xt)\,J_\lambda(xt)$$

$$\cdot\ H^{m,\,n}_{p,\,q}\left[z\,x^\rho\,\middle|\,\begin{matrix}(a_j,\,\alpha_j)_{1,p}\\(b_j,\,\beta_j)_{1,q}\end{matrix}\right]dx.$$

$$= \frac{2^{\mu-\lambda-\delta-\sigma-1}\ t^{\lambda+\delta}\sqrt{\pi}}{\Gamma(\delta+1)\,\Gamma(\lambda+1)}$$

$$\cdot\ \sum_{r=0}^{\infty}\ \frac{2^{-2r}\left(\frac{\lambda+\delta+1}{2}\right)_r\left(\frac{\lambda+\delta+2}{2}\right)_r\left(-t^2\right)^r}{(\lambda+1)_r\,(\delta+1)_r\,(\lambda+\delta+1)_r\ \ r!}$$

$$\cdot H^{m,\, n+1}_{p+1,\, q+2}\left[2^{-\rho}z \,\middle|\, \begin{matrix} A \\ B \end{matrix}\right], \tag{2.1}$$

where $A = (-\sigma - 2r, \rho), (a_j, \alpha_j)_{1,p}$,

$$B = (b_j, \beta_j)_{1,q}, (-\tfrac{1}{2}\sigma + \tfrac{1}{2}\nu + \tfrac{1}{2}\mu - r, \tfrac{1}{2}\rho),$$

$$(-\tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{2}\nu + \tfrac{1}{2}\mu - r, \tfrac{1}{2}\rho),$$

and $Re(\mu) < 1$, $Re(\sigma) > -1$

$$Re(\sigma + 1) + \rho \min_{1 \leqslant j \leqslant n} Re\left(\frac{b_j}{\beta_j}\right) > 0$$

(ii) $$\int_0^\infty x^{\nu+1} (\alpha^2 + x^2)^{-\lambda/2-5/4} \exp\left(-\frac{p^2 \alpha}{\alpha^2+x^2}\right) J_\nu\left(\frac{p^2 x}{\alpha^2+x^2}\right)$$

$$\cdot H^{m,\, n}_{p,\, q}\left[z(\alpha^2 + x^2)^{\alpha} \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] dx$$

$$= 2^{-\lambda-\nu-\frac{1}{2}}\, p^{2\nu}\, \pi^{\frac{1}{2}}\, \alpha^{-\lambda-\frac{1}{2}}$$

$$\cdot \sum_{r=0}^{\infty} \frac{\left(-\frac{p^2}{2\alpha}\right)^r}{r!} H^{m+1,\, n}_{p+2,\, q+1}\left[(2\alpha)^{2\mu} z \,\middle|\, \begin{matrix} E \\ F \end{matrix}\right], \tag{2.2}$$

where $E = (a_j, \alpha_j)_{1,p}, (\lambda/2+3/4, \mu), (\nu+\lambda/2+5/4+r, \mu)$,

$$F = (\lambda+\tfrac{1}{2}+r, 2\mu), (b_j, \beta_j)_{1,q},$$

and

$$Re(\alpha) > 0, p > 0, Re(\lambda+1/2) > 0$$

$$Re(\nu+1) > 0, 2\mu \max_{1 \leqslant j \leqslant n} Re\left(\frac{a_j-1}{\alpha_j}\right) < Re(\lambda+\tfrac{1}{2}),$$

The H-functions occurring in (2.1) and (2.2) stand for the well -

$$\cdot H^{m,\,n+1}_{p+1,\,q+2}\left[2^{-\rho}z \,\middle|\, \begin{matrix} A \\ B \end{matrix}\right], \tag{2.1}$$

where $A = (-\sigma - 2r, \rho), (a_j, \alpha_j)_{1,p}$,

$$B = (b_j, \beta_j)_{1,q}, (-\tfrac{1}{2}\sigma + \tfrac{1}{2}\nu + \tfrac{1}{2}\mu - r, \tfrac{1}{2}\rho),$$

$$(-\tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{2}\nu + \tfrac{1}{2}\mu - r, \tfrac{1}{2}\rho),$$

and $Re(\mu) < 1, Re(\sigma) > -1$

$$Re(\sigma + 1) + \rho \min_{1 \leqslant j \leqslant n} Re\left(\frac{b_j}{\beta_j}\right) > 0$$

(ii) $$\int_0^\infty x^{\nu+1}(\alpha^2 + x^2)^{-\lambda/2-5/4} \exp\left(-\frac{p^2\alpha}{\alpha^2+x^2}\right) J_\nu\left(\frac{p^2 x}{\alpha^2+x^2}\right)$$

$$\cdot H^{m,\,n}_{p,\,q}\left[z(\alpha^2 + x^2)^{\alpha} \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] dx$$

$$= 2^{-\lambda-\nu-\frac{1}{2}}\, p^{2\nu}\, \pi^{\frac{1}{2}}\, \alpha^{-\lambda-\frac{1}{2}}$$

$$\cdot \sum_{r=0}^{\infty} \frac{\left(-\frac{p^2}{2\alpha}\right)^r}{r!} H^{m+1,\,n}_{p+2,\,q+1}\left[(2\alpha)^{2\mu} z \,\middle|\, \begin{matrix} E \\ F \end{matrix}\right], \tag{2.2}$$

where $E = (a_j, \alpha_j)_{1,p}, (\lambda/2+3/4, \mu), (\nu+\lambda/2+5/4+r, \mu)$,

$$F = (\lambda+\tfrac{1}{2}+r, 2\mu), (b_j, \beta_j)_{1,q},$$

and

$$Re(\alpha) > 0, p > 0, Re(\lambda+1/2) > 0$$

$$Re(\nu+1) > 0, 2\mu \max_{1 \leqslant j \leqslant n} Re\left(\frac{a_j-1}{\alpha_j}\right) < Re(\lambda+\tfrac{1}{2}),$$

The H-functions occurring in (2.1) and (2.2) stand for the well -

known (Fox's) H-function defined and represented as follows:

$$H^{m,\,n}_{p,\,q}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1) \ldots (a_p, \alpha_p)\\ (b_1, \beta_1) \ldots (b_q, \beta_q)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds, \quad (2.3)$$

where the nature of contour L in (2.3), the conditions of its convergence and some of the properties of the H-function can be found in Chapter 2 of a recent book on the subject [5]. We have restricted ourselves in the present study to the case where the parameters of the H-function occurring throughout the present paper satisfy conditions corresponding appropriately to the following conditions for (2.3)

(i) $A = \sum_{j=1}^{n} \alpha_j - \sum_{n+1}^{p} \alpha_j + \sum_{1}^{m} \beta_j - \sum_{m+1}^{q} \beta_j > 0$

(ii) $|\arg x| < \frac{1}{2} A\pi$

Proof. To derive (2.1), we first express the H-function on the left-hand side of it with the help of (2.3), and obtain

$$\int_0^1 x^{\sigma-\lambda-\delta}\,(1-x^2)^{-\frac{1}{2}\mu}\, P_\gamma^\mu\, J_\delta(xt)\, J_\lambda(\lambda t)\, dx$$

$$\cdot\ \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}\, z^\xi\, x^{\rho\xi}\, d\xi. \quad (A)$$

On changing the order of integration in (A) [which is easily permissible under the conditions stated in (2.1)], putting the value of x —integral in (A) from (1.1), we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} z^{\xi}\,d\xi$$

$$\cdot\frac{2^{\mu-\lambda-\delta-1}\,\Gamma[\frac{1}{2}+\frac{1}{2}(\sigma+\rho\xi)]\,\Gamma[1+\frac{1}{2}(\sigma+\rho\xi)]\,t^{\lambda+\delta}}{\Gamma(\delta+1)\,\Gamma(\lambda+1)\,\Gamma(1+\frac{1}{2}(\sigma+\rho\xi)-\frac{1}{2}\nu-\frac{1}{2}\mu)\,\Gamma(3/2+\frac{1}{2}(\sigma+\rho\xi)+\frac{1}{2}\nu-\frac{1}{2}\mu)}$$

$$\cdot\,{}_4F_5\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}(\sigma+\rho\xi),\ 1+\frac{1}{2}(\sigma+\rho\xi),\ \frac{1}{2}(\lambda+\delta+1),\ \frac{1}{2}(\lambda+\delta+2)\\ 1+\frac{1}{2}(\sigma+\rho\xi)-\frac{1}{2}\nu-\frac{1}{2}\mu,\ 3/2+\frac{1}{2}(\sigma+\rho\xi)+\frac{1}{2}\nu-\frac{1}{2}\mu,\delta+1,\lambda+1,\lambda+\delta+1\end{matrix};-t^2\right] \tag{B}$$

Now representing the function ${}_4F_5$ involved in the above expression in a series form, and using the duplication formula for the gamma function, we easily get after a little simplification:

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} z^{\xi}\,d\xi$$

$$\cdot\frac{2^{\mu-\lambda-\delta-\sigma-1}\,t^{\lambda+\delta}\,\sqrt{\pi}}{\Gamma(\delta+1)\,\Gamma(\lambda+1)}$$

$$\sum_{r=0}^{\infty}\frac{2^{-2r}\,\Gamma(1+\sigma+2r+\rho\xi)\left(\frac{\lambda+\delta+1}{2}\right)_r\left(\frac{\lambda+\delta+2}{2}\right)_r}{(\delta+1)_r\,(\lambda+1)_r\,(\lambda+\delta+1)_r}$$

$$\cdot\frac{1}{\Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu+r+\frac{1}{2}\rho\xi)\,\Gamma(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+r+\frac{1}{2}\rho\xi)}\,\frac{(-t^2)^r}{r!}$$

Changing the order of integration and summation in (C) and interpreting the result thus obtained with the help of (2.3), we get the required result (2.1). Proof of (2.2) follows on lines similar to that of (2.1) on using the formula (1.2).

3. SPECIAL CASES

(i) On specializing the parameters of the H-functions involved in (2.1) suitably, we get by means of a known formula [2, p. 600] the following interesting integral after a little simplification:

$$\int_0^1 x^{\sigma-\lambda-\delta}\,(1-x^2)^{-\frac{1}{2}\mu}\; P_\nu^\mu(x)\, J_\delta(xt)\, J_\lambda(xt)$$

$$\cdot\; {}_2F_1\,[a,\, b:\, c;\, zx^2]\, dx$$

$$= \frac{2^{\mu-\lambda-\delta-\sigma-1}\; t^{\lambda+\delta}\,\sqrt{\pi}}{\Gamma(\lambda+1)\,\Gamma(\delta+1)\,\Gamma(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu)\,\Gamma(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)}$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(1+\sigma+2r)\left(\frac{\lambda+\delta+1}{2}\right)_r \left(\frac{\lambda+\delta+2}{2}\right)_r}{(\lambda+1)_r\,(\delta+1)_r\,(3/2+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)_r\,(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu)_r}$$

$${}_3F_3\left[\begin{matrix} a,\, b,\, 1+\sigma+2r \\ c,\; 1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu+r,\;\; \frac{1}{2}+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+r \end{matrix}\; ;\, z2^{-\rho}\right]$$

where Re $(\mu) < 1$, Re $(\sigma) > -1$ and $|\arg(1-z)| < \pi$.

(ii) On specializing the parameter of the H-function involved in (2.2) suitably we get by means of a known formula [3, p, 11 (1. 7. 5)]:

$$\int_0^\infty x^{\nu+1}\,(\alpha^2+x^2)^{-\lambda/2-5/4}\; exp\left(-\frac{p^2\,\alpha}{\alpha^2+x^2}\right) J_\nu\left(\frac{p^2 x}{\alpha^2+x^2}\right)$$

$$\cdot\; K_\sigma\;[2z^{\frac{1}{2}}\,(\alpha^2+x^2)^{\mu/2}\;]dx$$

$$= 2^{-\lambda-\nu+a-3/2}\, p^{2\nu}\, \pi^{\frac{1}{2}}\, \alpha^{-\lambda-\frac{1}{2}}$$

$$\cdot \sum_{r=0}^{\infty} \frac{\left(-\frac{p^2}{2\alpha}\right)^r}{r!} H^{3,0}_{2,3}\left[(2\alpha)^{2\mu} z \,\middle|\, \begin{matrix} I \\ J \end{matrix}\right]$$

where $I = (\lambda/2+3/4, \mu), (\nu+\lambda/2+5/4+r, \mu),$

$$J = \left(\frac{a-\sigma}{2}, 1\right), \left(\frac{a+\sigma}{2}, 1\right), (\lambda+\tfrac{1}{2}+r, 2\mu),$$

and

$Re(\alpha) > 0, p \geqslant 0, R(\lambda+\frac{1}{2}) > 0, R(\nu+1) > 0.$

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ON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS IN COMPACT METRIC SPACES

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ABSTRACT

In this paper, our main fixed point theorem, given in a compact metric space, improves a recent result of the second author using a condition of weak commutativity indebted to the first author. Further, we prove that other related results, established for commuting mappings, are not extendable to the case of weakly commuting mappings.

1. Main theorem

Let (X, d) be a metric space and let S, I be two selfmaps of X. Sessa [6], generalizing a result of Das and Naik [1], defined the pair (S, I) to be weakly commuting if

$$d(SIx, ISx) \leqslant d(Ix, Sx) \tag{1}$$

for all x in X.

Clearly two commuting mappings weakly commute but the converse is not generally true as is shown in example 1 below.

In the following we study common fixed points of mappings dfiened on compact metric spaces. In this direction, we recall the following theorem of Fisher [2].

THEOREM 1. Let S, T, I and J be continuous selfmaps of a compact metric space (X, d) satisfying the inequality

$$d(Sx, Ty) < \max\{ d(Ix,Jy),\ d(Ix,Ty),\ d(Sx,Jy),$$

$$\tfrac{1}{2}\, d(Ix,Sx),\ \tfrac{1}{2}\, d(Jy,Ty) \} \qquad (2)$$

for all x,y in X for which the right-hand side of the inequality (2) is positive. If

$$S(X) \subseteq I(X), \qquad T(X) \subseteq J(X) \qquad (3)$$

and S and T commute with I and J respectively, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Slightly modifying the proofs of [2], we can show that

THEOREM 2. Theorem 1 also holds if S and T weakly commuts with I and J respectively.

PROOF. We give it for the convenience of the reader. First of all, we observe that (2) implies that

$$d(Sx, Ty) < \max\{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy),$$

$$\tfrac{1}{2}[\, d(Ix, Jy) + d(Sx, Jy)\,],\ \tfrac{1}{2}[\, d(Ix, Jy) + d(Ix, Ty)\,] \}$$

and therefore

$$d(Sx, Ty) < \max\{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy) \} \qquad (4)$$

for all x, y in X for which the rigth-hand side of inequality (4) is positive.

We distinguish two cases.

(i) If the right-hand side of (4) is positive for all x, y in X, then the function

$$g(x, y) = \frac{d(Sx, Ty)}{\max \{ d(Ix, Jy), d(Ix, Ty), d(Sx, Jy) \}}$$

is continuous on the compact metric space $X \times X$ and so achieves its maximum value c. In virtue of (4), $c < 1$. This means that

$$d(Sx, Ty) < c. \max \{ d(Ix, Jy), d(Ix, Ty) \ d(Sx, Jy) \} \tag{5}$$

for all x, y in X. Let x_0 (resp. y_0) be an arbitrary point in X and define a sequence $\{x_n\}$ (resp. $\{y_n\}$) inductively by choosing a point x_n (resp. y_n) such that

$$Sx_{n-1} = Ix_n \text{ (resp. } Ty_{n-1} = Jy_n)$$

for $n=1,2,\ldots$ This can be done because of (3). It follows similarly to the proof of Theorem 1 of [2] that the sequences $\{Sx_n\} = \{Ix_{n+1}\}$ and $\{Ty_n\} = \{Jy_{n+1}\}$ have the same limit z in X. Since S and I are continuous, the sequences $\{ISx_n\}$ and $\{SIx_n\}$ converge to Iz and Sz respectively. Further using (1) we have

$$d(SIx_n, Iz) \leqslant d(SIz_n, ISx_n) + d(ISx_n, Iz)$$

$$\leqslant d(Ix_n, Sx_n) + d(ISx_n, Iz).$$

Letting n tend to infinity, we deduce that $\{SIx_n\}$ converges to Iz so $Sz=Iz$. Using (5), it follows that

$$d(Sz, Ty_n) \leqslant c. \max \{ d(Iz, Jy_n), d(Iz, Ty_n), d(Sz, Jy_n) \}$$

and on letting n tends to infinity we aveh

$$d(Sz, z) \leqslant c. \max \{ d(Iz, z), d(Iz, z), d(Sz, z) \}$$

$$c. d(Sz, z).$$

Since $c < 1$, we see that z is a fixed point of S and therefore also a fixed point of I. Similarly, we can prove that z is a fixed point of T and J.

(ii) Suppose that the right-hand side of (4) is zero for some x, y in X. Then, as in the proof of Theorem 5 of [2], we have

$$Ix = Jy = Sx = Ty$$

Since S weakly commutes with I, we have

$$d(SIx, ISx) \leqslant d(Ix, Sx) = 0$$

and so

$$ISx = SIx = S^2x = STy.$$

If $S^2x \neq Ty$, then using (4) we have

$$d(S^2x, Ty) < \max \{ d(ISx, Jy), d(ISx, Ty), d(S^2x, Jy) \}$$

$$= d(S^2x, Ty),$$

a contradiction. Hence $S^2x = Ty = STy$ and so $Ty = z$ is a fixed point of S. We now have

$$Iz = ITy = ISx = SIx = STy = Sz = z$$

and so z is also a fixed point of I.

Similarly, using the weak commutativity of T and J, we can prove

that T and J have a common fixed point w. We now claim that $z = w$. Otherwise (4) implies

$$d(z, w) = d(Sz, Tw)$$

$$< \max \{ d(Iz, Jw), d(Iz, Tw), d(Sz, Jw) \}$$

$$= d(z, w),$$

a contradiction.

Thus in both cases we have shown that S, T, I and J have a common fixed point z It follows easily that z is the unique common fixed point of S and I and of T and J. This completes the proof of the theorem.

The corollary follows easily.

COROLLARY. Let S and I be two continuous selfmaps of a compact metric space (X, d) satisfying the inequality

$$d(Sx, Sy) < d(Ix, Iy)$$

for all distinct x, y in X. If the range of I contains the range of S and if S weakly commutes with I, then S and I have a unique common fixed point.

EXAMPLE 1. Let $x = [0, 1]$ with the euclidean metric d and let S and I be two selfmaps of X defined by

$$Sx = \tfrac{1}{4}x - \tfrac{1}{8}x^2, \; Ix = \tfrac{1}{4}x$$

for all x in X. Note that $S(x) = [0, 1/8] \subset [0, 1/4] = I(X)$ and

$$d(SIx, ISx) = x/16 - x^2/128 - x/16 + x^2/32 = 3x^2/128$$

$$\leqslant x^2/8 = d(Ix, Sx)$$

for all x in X. Thus S and I weakly commute. Further

$$d(Sx, Sy) = \tfrac{1}{4} \mid x - y \mid [1 - \tfrac{1}{2}(x + y)]$$

$$< \tfrac{1}{4} \mid x - y \mid = d(Ix, Iy)$$

for all distinct x, y in X. All the assumptions of the corollary are therefore satisfied but Theorem 1 is not applicable since

$$SIx = \frac{1}{16} x - \frac{1}{128} x^2 \neq \frac{1}{16} x - \frac{1}{32} x^2 = ISx$$

for any $x \neq 0$ and so S and I do not commute.

Results related to Theorem 2 can be found in [4] and [7].

2. Another fixed point theorem.

In this section we present a second generalization of Theorem 1 showing that condition (3) and the continuity of the mappings I and J are unnecessary.

We first recall the following result of Leader [5].

LEMMA. For a continuous selfmap S on a compact metric space (X, d), the core

$$Y = \bigcap_{n=1}^{\infty} S^n(X)$$

is compact and $SY = Y$.

PROOF. See proposition 2 of [5].

THEOREM 3. Let S and I be commuting selfmaps and let T and

J be commuting selfmaps of a compact metric space (X,d) satisfying inequality (2) for all x, y in X for which the right-hand side of inequality (2) is positive. If S and T are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

PROOF. By the lemma, the core

$$Y = \bigcap_{n=1}^{\infty} S^n(X) = SY$$

is a compact subset of X of course, Y is non-empty because the family $\{S^n(X) : n = 1,2, \ldots\}$ has the finite intersection property. Now let x be an arbitrary point in Y. Then x is in $S^n(X)$ for $n = 1,2, \ldots$ and so Ix is in $IS^n(X) = S^nI(X)$ for $n=1,2, \ldots$ since I commutes with S. Thus

$$Ix \in \bigcap_{n=1}^{\infty} S^nI(X) \subseteq \bigcap_{n=1}^{\infty} S^n(X) = Y$$

and so I maps Y into Y.

Similarly, the set

$$W = \bigcap_{n=1}^{\infty} T^n(X) = TW$$

is a non-empty compact subset of X and J maps W into W.

Since d is a continuous mapping of the compact set $Y \times W$ into the reals, there exist points z' in Y and w' in W such that

$$d(z', w') = \sup\{d(x, y) : x \in X, y \in W\} = L.$$

Since $SY = Y$, there exists a point z in Y such that $Sz = z'$.

Similarly, there exists a point w in W such that $Tw = w'$.

We then have $d(Sz, Tw) = L$. Suppose that

$$\max \{ d(Iz, Jw), d(Iz, Tw), d(Sz, Jw) \} > 0.$$

Then we have from (2):

$$\begin{aligned} L &= d(Sz, Tw) \\ &< \max \{ d(Iz, Jw), d(Iz, Tw), d(Sz, Jw), \\ &\qquad \tfrac{1}{2} d(Iz, Sz), \tfrac{1}{2} d(Jw, Tw) \} \\ &\leqslant \max \{ d(Iz, Jw), d(Iz, Tw), d(Sz, Jw), \\ &\qquad \tfrac{1}{2} [d(Iz, Jw) + d(Jw, Sz)], \\ &\qquad \tfrac{1}{2} [d(Iz, Jw) + d(Iz, Tw)] \} \\ &\leqslant L \end{aligned}$$

since Sz and Iz are in T and Yw and Tw are in W. The foregoing inequality gives a contradiction and therefore

$$d(Iz, Jw) = d(Iz, Tw) = d(Sz, Jw) = 0$$

or equivalently

$$Iz = Jw = Sz = Tw. \tag{6}$$

It now follows that $d(Sz, Tw) = 0 = L$. The set $Y = SY$ therefore consists of the single point $z = Sz$ and the set $W = TW$ consists of the single point $w = Tw$. Since I maps Y into Y, $Iz = z$ and since J maps W into W, $Jw = w$. Further (6) implies $z = w$, showing that z is a common fixed point of S, T, I and J. The uniqueness of z is easily proved.

This concludes the proof.

To see the condition that X is compact in Theorem 3 is necessary, consider the following example:

EXAMPLE 2. Let $X = [1, \infty)$ with the euclidean metric d and define S, T, I and J by

$$Sx = 3x,\ Tx = 6x,\ Ix = 4x,\ Jx = 8x$$

for all x in X. Since

$$d(Sx, Ty) = \begin{cases} 3.\ |\ x - 2y\ | < 4\ |\ x - 2y\ | = d(Ix, Jy) & \text{if } x \neq 2y \\ 0 < x/2 = (4x - 3x)/2 = d(Ix, Sx)/2 & \text{if } x = 2y \end{cases}$$

for all x, y in X, it is easily seen that all the conditions of Theorem 3 are satisfied except the compactness of X, but S, T, I and J have no fixed points.

The next example shows that the condition that S and T commute with I and J respectively is necessary in Theorem 3.

EXAMPLE 3. Let $X = [0, \frac{1}{4}] \cup \{1/3\}$ with the cuclidean metric d and define S, T, I and J by

$$Sx = 1/4,\ Tx = x,\ Ix = 1/3,\ Jx = x/2$$

for all x in X. Since

$$d(Sx, Ty) = \begin{cases} 1/4 - y < 1/3 - y/2 = d(Sx, Jy) & \text{if } y \neq 1/3 \\ 1/3 - \frac{1}{4} < 1/3 - 1/6 = d(Ix, Jy) & \text{if } y = 1/3 \end{cases}$$

for all x, y in X, then all the conditions of Theorem 3 are satisfied except the commutativity of S and I, but S, T, I and J have no common fixed points

The following example shows that the continuity of S and T is a

necessary condition in Theorem 3.

EXAMPLE 4. Let $X=[0,1]$ with the euclidean metric d and define

$$Sx = \begin{cases} 1/2 & if\ x = 0, \\ x/4 & if\ x = 0, \end{cases} \qquad Ix = \begin{cases} 1 & if\ x = 0, \\ x/2 & if\ x \neq 0 \end{cases}$$

$$Tx = 0, \qquad Jx = x$$

for all x in X. Since

$$d(Sx, Ty) = \begin{cases} \frac{1}{2} < 1 = d(I0, Ty) & if\ x = 0,\ y \in Y \\ x/4 < x/2 = d(Ix, Ty) & if\ x \neq 0,\ y \in Y \end{cases}$$

it is easily seen that all the conditions of Theorem 3 are satisfied except the continuity of S, but S, T, I and J have no common fixed points.

The next example shows that theorem 3 is not extendable to the case for weakly commuting mappings.

EXAMPLE 5 Let $X = \{x, y, z, w\}$ be a finite set with metric d defined by

$$d(x, x) = d(y, y) = d(z, z) = d(w, w) = 0,$$

$$d(x, z) = d(x, w) = d(y, z) = d(y, w) = 1.$$

$$d(x, y) = d(z, w) = 2$$

and define $S = T$, I and J by

$$Sx = Sy = Sz = y,\ Sw = z,$$

$$Ix = Iy = Iz = x,\ Iw = w,$$

$$Jw = Jy = Jz = Jw = y.$$

Obviously X is compact and S is continuous on X.

With a routine calculation, it is easily proved that S and I weakly commute whereas S commutes with J.

Further, since

$$d(Sx, Sy) = d(Sx, Sz) = d(Sy, Sz) = d(y, y) = O,$$

$$d(Sx, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Ix, Jw),$$

$$d(Sy, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Iy, Jw),$$

$$d(Sz, Sw) = d(y, z) = 1 < 2 = d(x, y) = d(Iz, Jw),$$

we see that condition (2) is satisfied but S, I and J do not have a common fixed point.

We also note that since

$$S(X) = \{y, z\} \not\subset \{x, w\} = I(X),$$

$$T(X) = \{y, z\} \supset \{y\} = J(X),$$

this example shows that condition(3) cannot be dropped in Theorem 2.

We conclude this section pointing out that Theorems 2 and 3 are two distinct generalizations of Theorem 1. Of course, in virtue of example 5, Theorems 2 and 3 are not comparable.

3. A related result.

Using different assumptions on the mappings under discussion and satisfying a more general condition than (2), Fisher proved the following result in [3].

THEOREM 4. Let S, T, I and J be selfmaps of a compact metric space (X, d) satisfying the inequality

$$d(Sx, Ty) < \max \{ d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx) \} \tag{7}$$

for all x, y in X for which the right-hand side of the inequality (7) is positive. If S and T commute, if I and J commute with ST and if ST is continuous. then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Note that in this theorem the assumption (3) and the continuity of S, T, I and J are unnecessary whereas the commutativity of the pairs (S,T), (I, ST) is assumed.

We now show in the next example that Theorem 4 fails if only the weak commutativity of the pair (I, ST) is assumed, even if $S=T$.

EXAMPLE 6. Let (X,d), $S=T$ and J be as in example 5. Define I by $Ix = Iy = Iz = Iw = x$.

It is easily seen that J commutes with S^2 and S^2 is continuous. Since

$$IS^2x = IS^2y = IS^2z = IS^2w = Iy = x,$$

$$S^2Ix = S^2Iy = S^2Iz = S^2Iw = S^2x = y,$$

S^2 weakly commutes (but does not commute) with I.

Further, as already seen in example 5, condition (7) holds but S, I and J do not have a common fixed point.

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REMARKS ON SOME FIXED POINT THEOREMS AND THEIR EXTENSIONS

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INTRODUCTION

The main aim of the present paper is to show that some results of Ghosh [4], Khan [6] and Ray [9] are particular cases of well known results. We also present a theorem, which extends of Ghosh and Chatterjee [5]. Also we remark that the results of Chatterjee [1] hold for a more general class of mappings.

1. Preliminaries and Basic Definitions

Definition 1.1. Let X be a metric space. A mapping $T: X \to X$ is said to be a *Banach operator* if there exists k, $0 \leqslant k < 1$, such that $d(T^2x, Tx) \leqslant k\, d(Tx, x)$ for all $x \in X$. T is called a *contraction* if there exists k, $0 \leqslant k < 1$, such that $d(Tx, Ty) \leqslant kd(x, y)$ for all $x, y \in X$.

Remark 1. 1 Clearly any contraction is a Banach operator, but not conversely. Let $X = R$. Define $T: R \to R$ by $T(x) = x^2$. Then T is a contraction on any closed interval $[a, b] \subset (-\frac{1}{2}, +\frac{1}{2})$, but a Banach operator on $[a, b] \subset (-1, 1)$. A Banach operator may not be continuous and may have more than one fixed point. In fact, let $X = [0, 1]$. Define $T: X \to X$ by $T(x) = 0, 0 \leqslant x < \frac{1}{2}$, $T(x) = 1, \frac{1}{2} \leqslant x \leqslant 1$.

Then T is a discontinuous Banach operator with fixed points 0 and 1.

Definition 1. 2 Let B be a bounded set in a normed linear space X and let $\delta(B)$ be its diameter. A point $x \in B$ is said to be a *diametral point* of B if $\sup_{y \in B} \| x-y \| = \delta(B)$.

A convex set S of X is said to have *normal structure* if every bounded convex subset S_1 of S which contains more than one point, has a point that is not a diametral point of S_1.

Lemma 1. 1. [8, Lemma 1. 5] A Banach operator $T: X \to X$, where X is a complete metric space, has a fixed point.

Lemma 1. 2. [10, Theorem 1. 3] Let C, a non-empty weakly compact convex subset of a normed linear space, possess normal structure.
If $T: C \to C$ satisfies $\| Tx - Ty \| \leqslant \frac{1}{2} [\| x - Tx \| + \| y - Ty \|]$ for all $x, y \in C$, then it has a fixed point.

2. Results.

Ray [9] proved the following theorem.

Theorem [Ray]. Let K be a nonempty, bounded closed and convex subset of a reflexive Banach space X and let K have normal structure.

Let T be a mapping of K into itself, such that

$$\text{(A)} \quad \| Tx - Ty \| \leqslant \alpha \| x - y \| + \beta \| [\| x - Tx \| + \| y - Ty \|] + \gamma [\| x - Ty \| + \| y - Tx \|]$$

for all $x, y \in K$ and for some $\alpha, \beta, \gamma \in R^+$ (nonnegative reals) with $3\alpha + 2\beta + 4\gamma \leqslant 1$, then T has a unique fixed point.

Lemma 2.1 If $\beta \geqslant 3$, $\gamma > 0$ and T satisfies A, then there exists a constant k, $0 < k < 1$, such that $\| Tx - T^2x \| \leqslant k \| x - Tx \|$.

Proof. Setting $y = Tx$ in (A) we have

$$\| Tx - T^2x \| \leqslant \alpha \| x - Tx \| + \beta [\| x - Tx \| + \| Tx - T^2x \|]$$

$$+ \gamma [\| x - T^2x \| + \| Tx - Tx \|]$$

$$\leqslant (\alpha + \beta + \gamma) \| x - Tx \| + (\beta + \gamma) \| Tx - T^2x \|.$$

Thus $[1 - (\beta + \gamma)] \| Tx - T^2x \| \leqslant [\alpha + \beta + \gamma] \| x - Tx \|$

or $$\| Tx - T^2x \| \leqslant \frac{[\alpha + \beta + \gamma]}{1 - (\beta + \gamma)} \| x - Tx \|.$$

Let $\dfrac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} = k$. If $k \geqslant 1$, then $\alpha + \beta + \gamma \geqslant 1 - \beta - \gamma$,

or $\alpha + 2\beta + 2\gamma \geqslant 1$, and hence $3\alpha + 2\beta + 4\gamma \geqslant 1 + 2\alpha + 2\gamma > 1$, contradiction.

Remark 2.1 If $\beta > 0$, $\gamma > 0$, it follows from Lemma 2. 1 that T is a Banach operator and has a fixed point.

In view of Lemma 2.1 it is enough to prove the theorem for the cases $\beta = 0$ and $\gamma = 0$ only.

Case 1. If $\beta = 0$ then (A) reduces to

$$\| Tx - Ty \| \leqslant \alpha \| x - y \| + \gamma [\| x - Ty \| + \| y - Tx \|].$$

Lemma 2. 2 If $\gamma > 0$, there exists a constant k, $0 < k < 1$, such that $\| Tx - T^2x \| \leqslant k \| x - Tx \|$.

Proof. As in Lemma 2. 1 we have $\| Tx - T^2x \| \leqslant \frac{\alpha + \gamma}{1 - \gamma} \| x - Tx \|$. Let $\frac{\alpha + \gamma}{1 - \gamma} = k$, $k < 1$. Suppose not, $k \geqslant 1$ will imply $\alpha + 2\gamma \geqslant 1$, which in turn implies $3\alpha + 4\gamma \geqslant 1 + 2\alpha + 2\gamma > 1$, a contradiction.

If $\gamma > 0$, the result follows from Lemma 1. 1. If $\gamma = 0$, $\alpha = 1/3$, the result is a particular case of Banach's contraction principle.

Case 2. $\gamma = 0$. The case $\beta = \gamma = 0$ need not be considered, due to Banach's contraction principle, and $\gamma = \alpha = 0$ follows from Lemma 1. 2. Thus we may assume $\gamma = 0$ and $\alpha > 0$, $\beta > 0$.

Lemma 2. 3 If $\alpha > 0$, $\beta > 0$, then there exists a k, $0 < k < 1$, such that $\| Tx - T^2x \| \leqslant k \| x - Tx \|$.

Proof $\| Tx - T^2x \| \leqslant \alpha \| x - Tx \| + \beta [\| x - Tx \| + \| Tx - T^2x \|]$. Thus $\| Tx - T^2x \| \leqslant \frac{\alpha + \beta}{1 + \beta} \| x - Tx \|$. Let $k = \frac{\alpha + \beta}{1 - \beta}$. Clearly $k < 1$. Suppose not, that is $k \geqslant 1$, then $\alpha + 2\beta \geqslant 1$ implies $3\alpha + 2\beta > 1 + 2\alpha > 1$, a contradiction. Thus in this case T is a Banach operator and has a fixed point.

Remark 2. 2. Result of Kirk is not a special case of [9] as remarked on page 904. In fact $3\alpha = 1$ implies $\alpha = 1/3$, the result of Ray becomes a particular case of Banach's contraction principle for $\beta = \gamma = 0$.

The following is proved in [4].

Theorem B [4, Theorem 3]. Let E be a rotund Banach space,

M be a compact convex subset of E and let T be a self-mapping of M. If T is continuous and satisfies

$$\| TT_\lambda x - TT_\lambda y \| \leqslant a_1 \| x - y \| + a_2 \| x - TT_\lambda x \| + a_3 \| y - TT_\lambda y \|$$

$$+ a_4 \| x - TT_\lambda y \| + a_5 \| y - TT_\lambda x \| \qquad (C),$$

where a_i $(i = 1, 2, \ldots, 5)$ are non-negative real numbers such that $\sum_{i=1}^{5} a_i = 5$ and $T_\lambda x = \lambda x + (1 - \lambda) Tx$, $0 < \lambda \leqslant 1$, for all $x, y \in E$. If $TT_\lambda x = T_\lambda Tx$ for any $x \in M$, then T has a fixed point.

Remark 2.3. The condition (C) and rotundity of the space E is unnecessary in Theorem B. Since T is a continuous self-map of a compact convex subset of a Banach space, T has a fixed point by Schauder's fixed point theorem. In fact $F(T)$ and $F(T_\lambda)$ are the same, where $F(T) = \{x \in M : T(x) = x\}$. Indeed let $y \in F(T)$, then $Ty = y$. Now $T_\lambda(y) = \lambda y + (1-\lambda) y = y$, thus $y \in F(T_\lambda)$, hence $F(T) \subset F(T_\lambda)$. Conversely suppose $z \in F(T_\lambda)$, then $T_\lambda z = z$, which in turn implies $z = \lambda z + (1 - \lambda) Tz$ which implies $z = Tz$. Thus $F(T_\lambda) \subset F(T)$.

Remark 2.4. Theorem 1 [1, p. 91] can be extended to the following type of mapping : There exists a constant k, $0 \leqslant k < 1$, such that

$$d(TT_m x, TT_m y) \leqslant k \max \{ d(x,y), d(x, TT_m x), d(y, TT_m y), d(x, TT_m y), d(y, TT_m x)\}$$

for all $x, y \in X$, where X and $T_m x$, $T_m y$ are as in [1].

In fact the result is true for any contractive type mapping [11] for which the fixed point is unique.

Remark 2.5. The claim of lines B_1, B_2 [1, p. 89] is incorrect.

Let $X = \{a, b\}$ with the usual norm.

Define $T: X \to X$ by $Ta = b$ and $Tb = a$, X is compact T is continuous, but T is without fixed point.

Remark 2.6. The statement line T_1 [1, p. 90] is incorrect. Indeed, let X be any normed linear space. Let $B = \{x \in X: \| x \| < 1\}$. Clearly B is convex. Define $T: B \to B$ by $T(x) = x/2 + a/2$, where $a \neq 0$ and $\| a \| = 1$. Clearly T is nonexpansive and satisfies the conditions of T_1 [1, p. 90]. However, T does not have a fixed point.

Definition 2.1. A metric space X is said to be *metrically convex* if for any $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$ such that $d(x, z) + d(z, y) = d(x, y)$.

Definition 2.2. Let K be a nonempty closed subset of a metric space X and let S, T be mappings of K into $CB(X)$ (the set of nonempty closed bounded subsets of X). Then (S, T) is said to be a *generalized contraction pair* of K into $CB(X)$ if there exist nonnegative reals α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that for any $x, y \in K$

$$H(Sx, Ty) \leqslant \alpha d(x, y) + \beta [D(x, Sx) + D(y, Ty)] + \gamma [D(x, Ty) + D(y, Sx)].$$

Note

For any nonempty subsets A, B of X we define

$$D(A,B) = \inf d\{a, b) : a \in A \text{ and } b \in B\}$$

and

$$H(A, B) = \max \{ \sup \{D(a, B) : a \in A\}, \sup \{D(A, b) : b \in B\} \}.$$

Theorem D. [6, Theorem 3. 1] Let X be a complete and metri-

cally convex metric space, K a nonempty closed subset of X. Let (S, T) be a generalized contraction pair of K into $CB(X)$.

If for any

$x \in \partial K$, $S(x) \subset K$, $T(x) \subset K$ and $(a+\beta+\gamma)\ (1+\beta+\gamma)/(1-\beta-\gamma)^2 < 1$, then there exists $z \in K$ snch that $z \in S(z)$ and $z \in T(z)$.

Theorem E [12, Theorem 3. 2]. Let X be a complete metrically convex metric space, K be a closed subset of X, and S, T be two mappings of K into $CB(X)$ satisfying the boundary condition: $Sx \subset K$, $Tx \subset K$, $(V\, x \in \partial K)$. Suppose there exist nonnegative numbers $a_1, \ldots, a_5$ $(\Sigma a_i < 1)$ with

$$a_i + a_j < \frac{1 - a_5}{3 + a_5}, (i = 1, 2;\ j = 3, 4) \text{ such that}$$

$$D(Sx, Ty) \leqslant a_1\, d(x, Sx) + a_2\, d(y, Ty) + a_3\, d(x, Ty) + a_4\, d(y, Sx)$$

$$+ a_5\, d(x, y) \text{ for all } x, y \in K.$$

Then the fixed point set of each S, T is nonempty and these two sets coincide.

Lemma 2.4. The condition $\dfrac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1$

implies

$$a_i + a_j < \frac{1 - a_5}{3 + a_5}, (i = 1, 2;\ j = 3, 4).$$

Proof. $\dfrac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1$ implies

$$\alpha+\beta+\gamma+\alpha\beta+\beta^2+\gamma\beta+\alpha\gamma+\beta\gamma+\gamma^2 < 1-2\beta-2\gamma+\beta^2+\gamma^2+2\beta\gamma.$$

Thus, $\alpha+\alpha\beta+\alpha\gamma+3\beta+3\gamma < 1$. Let $\alpha=a_5$, $\beta=a_1+a_2$, $\gamma=a_3+a_4$.

Then we have,

$$a_5+a_5\,[a_1+a_2]+a_5\,[a_3+a_4] + 3a_1+3a_2+3a_3+3a_4 < 1.$$

$$a_i + a_j < \frac{1-a_5}{3+a_5} \quad (i = 1, 2;\ j = 3, 4)$$

can be written as $3(a_i + a_j) + a_5(a_i + a_j) < 1 - a_5$

or $a_5 + 3(a_i + a_j) + a_5(a_i + a_j) < 1$

(i) If $i = 1$, $j = 3$, then $a_5 + 3(a_1 + a_3) + a_5(a_1 + a_3) < 1$.

(ii) If $i = 1$, $j = 4$, then $a_5 + 3(a_1 + a_4) + a_5(a_1 + a_4) < 1$.

(iii) If $i = 2$, $j = 3$, then $a_5 + 3(a_2 + a_3) + a_5(a_2 + a_3) < 1$.

(iv) If $i = 2$, $j = 4$, then $a_5 + 3(a_2 + a_4) + a_5(a_2 + a_4) < 1$.

Thus the Lemma.

Using symmetric properties of a metric, it can be easily seen that Definition 2.2 is equivalent to the condition of Theorem E.

Remark 2.7. It follows from Lemma 2.4 that Theorem 3.1 [6] is a special case of Theorem 3.2 [12].

The following result extends Theorem 5, 6 and 7 of [5].

Theorem 1. Let S be a compact metric space and T be a continuous mapping of S into itself. Suppose there exists a family of functions $F = \{f_\alpha\}_{\alpha \in S}$ from [0, 1] into S such that for each $\alpha \in S$, $f_\alpha(1) = \alpha$ and for a function Φ: $(0, 1) \to (0, 1)$, $d(f_{Tx}(t), f_{Ty}(t)) \leqslant$

$\phi(t) \max\{d(x, y), d(x, f_{Tx}(t)), d(y, f_{Ty}(t)), d(x, f_{Ty}(t)), d(y, f_{Tx}(t))\}$ for all $x, y \in S$ and for all $t \in [0,1]$. Further, assume that for $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in S, $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in S, then T has a fixed point.

Proof. For each $n=1,2,3,\ldots$, let $k_n = \dfrac{n}{n+1}$, and let $T_n: S \to S$ be defined by $T_n(x) = f_{Tx}(k_n)$ for all $x \in S$.

Since $T(S) \subset S$ and $0 < k_n < 1$, each T_n is well defined and maps S into S. Now for each $x, y \in S$ we have $d(T_n x, T_n y) = d(f_{Tx}(k_n), f_{Ty}(k_n)) \leqslant$

$\phi(k_n) \max\{d(x, y), d(x, f_{Tx}(k_n)), d(y, f_{Ty}(k_n)), d(x, f_{Ty}(k_n)), d(y, f_{Tx}(k_n))\}$

$= \phi(k_n) \max\{d(x, y), d(x, T_n(x)), d(y, T_n(y)), d(x, T_n y)), d(y, T_n(x))\}$.

Thus each T_n satisfies Ciric s condition. Since S is compact (hence complete), it follows from Theorem 1 [2] each T_n has a unique fixed point $x_n \in S$. Since S is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x \in S$. Now $T_{n_j} x_{n_j} = x_{n_j} \to x \in S$.

Using the continuity of T, we conclude that $Tx_{n_j} \to Tx$.

Finally $T_{n_j} x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx}(1) = Tx$. It follows that $Tx = x$,

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ON $|V, \lambda|_k$ SUMMABILITY OF ULTRASPHERICAL SERIES

By

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In the present paper, a new theorem on $|V, \lambda|_k$ factors of ultraspherical series has been proved. The work is motivated by the recent papers [1], [2], [5] and [6].

1. Let Σu_n be a given series with the sequence of partial sums $\{S_n\}$ and let $\lambda=\{\lambda_n\}$ be a monotonic non decreasing sequence of natural numbers with $\lambda_{n+1}-\lambda_n \leqslant 1$ and $\lambda_1=1$. The sequence to sequence transformation

$$V_n(\lambda) = 1/\lambda_n \sum_{v=n-\lambda_n+1}^{n} S_v$$

defines the generalized de la Vallée Poussin means of the sequence $\{S_n\}$ generated by λ.

The series Σu_n is said to be summable $|V, \lambda|$, if the sequence $\{V_n(\lambda)\}$ is of bounded variation, *i.e.*,

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda) - V_n(\lambda)| < \infty.$$

We say that the series Σu_n is summable $|V, \lambda|_k$, $k \geqslant 1$, if

$$\Sigma \lambda_n^{k-1} |V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty.$$

On taking $\lambda_n = n$, this summability reduces to $|C, 1|_k$ summability and for $k = 1$ this is the same as summability $|V, \lambda|$.

Let $f(\theta, \phi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. The ultraspherical series corresponding to $f(\theta, \phi)$ on the sphere S is

$$f(\theta, \phi) \sim 1/2\pi \sum_{n=0}^{\infty} (n+\lambda) \int\int_S \frac{f(\theta', \phi) P_n^{(\lambda)} (\cos w) \sin \theta' \, d\theta' \, d\phi'}{[\sin^2 \theta' \sin^2 (\phi - \phi')^{(1-2\lambda)/2}}$$

$$\equiv \sum_{n=0}^{\infty} u_n$$

where

$$\cos w = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

A generalized mean value of $f(\theta, \phi)$ on the sphere S has been defined by Kogbetliantz [3] as follows:

$$f(w) = \frac{1}{2\pi (\sin w)^{2\lambda}} \int_{C_w} \frac{f(\theta', \phi) \, ds'}{[\sin^2 \theta' \sin^2 (\phi - \phi')]^{(1-2\lambda)/2}}$$

where the integral is taken around the small circle C_w whose centre is (θ, ϕ) on the sphere S and whose curvilinear radius is w.

We write

$$\phi(w) = f(w) (\sin w)^{2\lambda - 1}$$

$$\phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x - t)^{(p-1)} \phi(t) \, dt, \qquad p > 0$$

$$\Phi_{k,\, p}(x) = \frac{1}{\Gamma(kp - k + 1)} \int_0^x (x - t)^{k(p-1)} \phi(t)^k \, dt, \; p > 1 - (1/k)$$

$$\Phi_0(x) = \phi(x),\ \phi_p(x) = \Gamma(p+1)x^{-p}\,\Phi_p(x).\ p \geqslant 0$$

$$\Phi_p(x) = d/dx\ \Phi_{p+1}(x)$$

$$\log^k x = \log\log^{k-1} x, \ldots, \log^2 x = \log\log x.$$

In this paper we prove the following results:

Theorem 1. *If $\phi(w)$ is of bounded variation in (η, π), where*

$$\eta = \frac{\mu'}{n^{\Delta}},\ \frac{1-\lambda}{\lambda} = \Delta,\ 0 < \lambda \leqslant \tfrac{1}{2},$$

μ' is a large constant, and if

$$\int_0^t |\phi(u)|^k\, du = 0\{t^{1+(\alpha-1)k}(\log^{\mu} 1/t)^{\beta}\},\ \text{as } t \to 0, \tag{1.2}$$

$$\alpha = \frac{2\lambda + 1 - \Delta}{\Delta},\ \beta \geqslant 0,\ k \text{ and } \mu \text{ are integers such that}$$

$1 \leqslant k,\ \mu \leqslant \infty$, then the series

$$\Sigma\ \frac{\mu_n\, u_n(t)}{[\log^{\mu}(n+1)]^{\beta+\delta-\frac{1}{2}}}$$

at $t=x$ is summable $|V,\lambda|_k$, provided $\{\mu_n\}$ is a convex sequence such that

$$\Sigma\ \frac{n\,\mu_n(\log^{\mu} n)^{1/2-\delta}}{\lambda_n^2} < \infty \qquad (|\delta| \leqslant \tfrac{1}{2})$$

and

$$\Sigma\ \frac{n\,(\log^{\mu} n)^{1/2-\delta}\,\Delta\mu_n}{\lambda_n} < \infty.$$

Theorem 2. *If $\phi(w)$ is of bounded variation in (η,π) where $\eta=\mu/n^{\Delta}$, Δ is a positive real number less than unity satisfying*

$$\frac{1-\lambda}{\lambda} > \Delta > \frac{1+2\lambda}{3+\beta-\alpha}, (0 \leqslant \alpha \leqslant \beta < 1)$$

μ' is a large constant, and if

$$\Phi_{k,\alpha}(t) = O\{t^{1+k\beta}(\log^{\mu} 1/t)^{\gamma}\}, \text{ as } t \to 0, \gamma \geqslant 0, k \text{ integer}, \quad 1 \leqslant k < \infty \tag{1.5}$$

then the series

$$\Sigma \frac{\mu_n u_n(t)}{[\log^{\mu}(n+1)]^{\gamma+\delta-1/2}}$$

at $t = x$ is summable $|V, \lambda|_k$, provided $\{\mu_n\}$ is a convex sequence such that (1. 3) *and* (1. 4) *hold.*

2. We require the following lemmas:

Lemma 1. *The conditions* (1. 2) *and* (1. 5) *imply, respectively, that*

$$\int_0^t |\phi(u)|\, du = O\{t^{\alpha}(\log^{\mu} 1/t)^{\beta/k}\} \tag{2.1}$$

and

$$\Phi_{1,\alpha}(t) = \Phi_{\alpha}(t) = O\{t^{1+\beta}(\log^{\mu} 1/t)^{\gamma/k}\}. \tag{2.2}$$

Proof. For $k = 1$ the lemma follows directly from (1. 2) and (1. 5). Using Hölder's inequality we have, for $1 < k < \infty$,

$$\int_0^t |\phi(u)|\, du \leqslant \left\{\int_0^t |\phi(u)|^k du\right\}^{1/k} \cdot \left\{\int_0^t du\right\}^{1-(1/k)}$$

$$= O\{t^{\alpha}(\log^{\mu} 1/t)^{\beta/k}\}.$$

By a similar proof we obtain (2.2).

Lemma 2. *If $\{\mu_n\}$ is a convex sequence such that*

$$\Sigma^{n} \frac{\mu_n \log n}{\lambda_n^2} < \infty,$$

then

$$\sum_{n=1}^{m} \log^{\mu} (n+1) . \triangle \mu_n = O(1), \qquad m \to \infty \tag{2.3}$$

$$\sum_{n=1}^{m} n \log^{\mu} (n+1) . \triangle^2 \mu_n = O(1), \qquad m \to \infty. \tag{2.4}$$

Proof. The convergence of $\Sigma \frac{n \mu_n \log n}{\lambda_n^2}$ implies the convergence of $\Sigma \frac{\mu_n \log n}{n}$, and the latter implies $\mu_n \log n \to 0$ and $n \log n \triangle n \to 0$.

Now using Abel's transformation, we have

$$\sum_{n=1}^{m} \frac{\mu_n}{n} = \sum_{n=1}^{m-1} \left[\sum_{r=1}^{n} 1/r \right] (\triangle \mu_n) + \left[\sum_{r=1}^{m} 1/r \right] u_m$$

which implies

$$\sum_{n=1}^{m} \log n \, \triangle \mu_n = O(1).$$

Again, by Abel's transformation, we have

$$\sum_{n=1}^{m} \log n \, \triangle \mu_n = \sum_{n=1}^{m-1} \left[\sum_{r=1}^{n} \log r \right] \triangle^2 \mu_n + \left[\sum_{r=1}^{m} \log r \right] \triangle \mu_m$$

which implies

$$\sum_{n=1}^{m} n \log n \, \triangle^2 \mu_n = O(1).$$

3. **Proof of Theorem 1.** Let S_n be the nth partial sum of the series (1.1). Then we have (Szegö [4], p. 84)

$$S_n = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} f(w) \sum_{k=0}^{n} (k+\lambda) P_k^{(\lambda)} (\cos w) (\sin w)^{2\lambda}\, dw$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} f(w) \left[\frac{d}{dx}\left\{P_{n+1}^{(\lambda)}(x) + P_n^{(\lambda)}(x)\right\}\right] (\sin w)^{2\lambda}\, dw$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} \phi(w) \frac{d}{dw}\left\{P_{n+1}^{(\lambda)}(\cos w) + P_n^{(\lambda)}(\cos w)\right\} dw$$

$$= S_n^1 + S_n^2, \text{ say.}$$

$$S_n^1 = -\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \left\{\int_0^{\eta} + \int_{\eta}^{\pi}\right\} = J_1 + J_2, \text{ say.}$$

$$J_1 = O(n^{2\lambda+1}) \int_0^{\eta} \phi(w)\, dw = O\{n^{2\lambda+1-\Delta(\alpha+1)} (\log^{\mu} n)^{\beta/k}\}$$

$$= O\{(\log^{\mu} n)^{\beta/k}\}.$$

$$J_2 = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_{\eta}^{\pi} \phi(w) \frac{d}{dw}\{P_{n+1}^{(\lambda)}(\cos w)\}\, dw$$

$$= A\left\{\int_{\eta}^{\pi/2} + \int_{\pi/2}^{\pi-\eta} + \int_{\pi-\eta}^{\pi}\right\} \phi(w) \frac{d}{dw}\{P_{n+1}^{(\lambda)}(\cos w)\}\, dw$$

$$= J_{2,1} + J_{2,2} + J_{2,3}, \text{ say.}$$

$$J_{2,1} = A\left[\phi(w) P_{n+1}^{(\lambda)}(\cos w)\right]_{\eta}^{\pi/2} - A\int_{\eta}^{\pi/2} P_{n+1}^{(\lambda)}(\cos w)\, d\phi(w)$$

$$= O(n^{\lambda-1+\Delta\lambda}) = O(1).$$

$$J_{2,\,2} = A\left[\phi(w)\,P_{n+1}^{(\lambda)}(\cos w)\right]_{\pi/2}^{\pi-\eta} - A\int_{\pi/2}^{\pi-\eta} P_{n+1}^{(\lambda)}(\cos w)\,d\phi(w)$$

$$= A\left[\phi(\pi-w)\,(-1)^{n+1}\,P_{n+1}^{(\lambda)}(\cos w)\right]_{\pi/2}^{\eta} +$$

$$(-1)^n A\int_{\pi/2}^{n} P_{n+1}^{(\lambda)}(\cos w)\,d\phi(\pi-w) = O(n^{\lambda-1+\Delta\lambda}) = O(1).$$

$$J_{2,\,3} = A\left[\phi(w)\,P_{n+1}^{(\lambda)}(\cos w)\right]_{\pi-\eta}^{\pi} - A\int_{\pi-\eta}^{\pi} P_{n+1}^{(\lambda)}(\cos w)\,d\phi(w)$$

$$= O(n^{2\lambda-1}) = O(1).$$

Therefore

$$J_2 = O(1)$$

and

$$S_n^1 = O\{(\log^{\mu} n)^{\beta/k}\}.$$

Other parts of the sum are treated similarly, and consequently we have

$$S_n = O\{(\log^{\mu} n)^{\beta/k}\}$$

and

$$\sum_{\nu=0}^{n} |S_\nu(x)|^k = O\{n(\log^{\mu} n)^{\beta}\}.$$

Now let

$$T_n(x) = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu u_\nu(x).$$

By Abel's transformation, we have

$$T_n(x) = S_n(x) - \frac{1}{n+1} \sum_{v=0}^{n} S_v(x) = O\{(\log^\mu n)^{\beta/k}\},$$

which implies

$$\sum_{v=1}^{n} |T_v(x)|^k = O\{n(\log^\mu n)^\beta\}.$$

Let $C_n = V_{n+1}(\lambda; x) - V_n(\lambda; x)$, where $V_n(\lambda; x)$ is the nth de la Vallée Poussin mean of the series

$$\Sigma \frac{\mu_n u_n(x)}{[\log^\mu (n+1)]^{\beta+\delta-1/2}}.$$

By an easy computation, we have

$$C_n = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{v=n-\lambda_n+2}^{n+1} \{(\lambda_{n+1} - \lambda_n)(v - n - 1) + \lambda_n\}$$

$$\frac{\mu_v u_v(x)}{[\log^\mu (v+1)]^{\beta+\delta-\frac{1}{2}}}.$$

Therefore, in order to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |C_n|^k < \infty.$$

Let $\overset{(i)}{\sum_n}$ be the summation over all n satisfying $\lambda_{n+1} = \lambda_n$ and $\overset{(ii)}{\sum_n}$ be the summation over all n, where $\lambda_{n+1} > \lambda_n$.

When $\lambda_{n+1} = \lambda_n$, Abel's transformation gives that

$$C_n = \frac{1}{\lambda_{n+1}}\left[\sum_{v=n-\lambda_n+2}^{n}\left\{\sum_{r=1}^{v} ru_r(x)\right\}\Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}-\right.$$

$$-\frac{\mu_{n-\lambda_n+2}}{(n-\lambda_n+2)\,[\log^{\mu}(n-\lambda_n+3)]^{\beta+\delta-1/2}}\left\{\sum_{r=1}^{n-\lambda_n+1} ru_r(x)\right\}$$

$$\left.+\frac{\mu_{n+1}}{(n+1)\,[\log^{\mu}(n+2)]^{\beta+\delta-1/2}}\left\{\sum_{r=1}^{n+1} ru_r(x)\right\}\right]$$

$$= L_n^1 + L_n^2 + L_n^3, \text{ say.}$$

By Minkowski's inequality it is therefore sufficient to prove that

$$\sum_{n}^{(i)} \lambda_n^{k-1} \mid L_n^r \mid^k < \infty \text{ for } r=1, 2, 3.$$

Now

$$\sum_{n}^{(i)} \lambda_n^{k-1} \mid L_n^1 \mid^k = O(1) \sum_{n}^{(i)} \frac{1}{\lambda_n}\left[\sum_{v=n-\lambda_n+2}^{n} v \mid T_v(x) \mid \right.$$

$$\left.\Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}\right]^k$$

$$= O(1) \sum_{n}^{(i)} \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+2}^{n} v \mid T_v(x) \mid^k \Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}$$

$$= O(1) \sum_{v=1}^{\infty} v \mid T_v(x) \mid^k \Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\} \cdot \sum_{n=v}^{n+\lambda_v-1} \frac{1}{\lambda_n}$$

$$= O(1) \sum_{v=1}^{\infty} v \mid T_v(x) \mid^k \Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}.$$

Using Abel's transformation again, by (3. I), we easily have

$$\sum_n^{(i)} \lambda_n^{k-1} \, |L_n^1|^k = O(1) \sum_{n=1}^{\infty} n^2 (\log^\mu n)^\beta \triangle^2 \left\{ \frac{\mu_n}{n[\log^\mu(n+1)]^{\beta+\delta-1/2}} \right\}$$

$$= O(1) \sum_{n=1}^{\infty} n (\log^\mu n)^{(1-2\delta)/2} \triangle^2 \mu_n + O(1) \sum_{n=1}^{\infty} (\log^\mu n)^{1/2-\delta} \triangle \mu_n$$

$$+ O(1) \sum_{n=1}^{\infty} \frac{1}{n} \mu_n (\log^\mu n)^{1/2-\delta}$$

$$= O(1), \tag{3.2}$$

by (2.3) and (2.4); and hypothesis (1.3).

Further, applying Abel's transformation and (3.1), it is easy to see that

$$\sum_n^{(i)} \lambda_n^{k-1} \, |L_n^2|^k + \sum_n^{(i)} \lambda_n^{k-1} \, |L_n^3|^k$$

$$= O(1) \sum_{n=1}^{\infty} |T_n(x)|^k \frac{\mu_n}{\lambda_n [\log^\mu(n+1)]^{\beta+\delta-1/2}}$$

$$= O(1) \sum_{n=1}^{\infty} \frac{n \triangle \mu_n (\lg^\mu n)^{1/2-\delta}}{\lambda_n}$$

$$+ O(1) \sum_{n=1}^{\infty} \frac{n \mu_n (\log^\mu n)^{1/2-\delta}}{\lambda_n^2}$$

$$+ O(1) \sum_{n=1}^{\infty} \frac{\mu_n (\log^\mu n)^{1/2-\delta}}{\lambda_n}$$

$$= O(1), \tag{3.3}$$

by hypothesis (1.3) and (1.4).

Now, in order to estimate $\sum_{n}^{(ii)}$ we have, with the aid of Abel's transformation,

$$|C_n| \leqslant \frac{1}{\lambda_n \lambda_{n+1}} \Bigg[\sum_{v=n-\lambda_n+2}^{n} v \,|T_v(x)| \,.$$

$$\left| \triangle \left\{ (\lambda_n + v - n - 1) \frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}} \right\} \right|$$

$$+ (n-\lambda_n+1) \,|T_{n-\lambda_n+1}(x)| \frac{\mu_{n-\lambda_n+2}}{(n-\lambda_n+2)[\log^{\mu}(n-\lambda_n+3)]^{\beta+\delta-1/2}}$$

$$+ (n+1) \,|T_{n+1}(x)| \frac{\lambda_n \mu_{n+1}}{(n+1)[\log^{\mu}(n+2)]^{\beta+\delta-1/2}} \Bigg]$$

$$= M_n^1 + M_n^2 + M_n^3 \text{, say.}$$

By Minkowski's inequality, it is therefore sufficient to prove that

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \,|M_n^r|^k < \infty \quad \text{for} \quad r = 1, 2, 3.$$

Now

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \,|M_n^1|^k \leqslant \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \Bigg[\sum_{v=n-\lambda_n+1}^{n} v \,|T_v(x)| \,$$

$$\left\{ \lambda_v \triangle \left(\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}} \right) + \frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}} \right\} \Bigg]^k$$

$$\leqslant \Bigg(\Bigg[\sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \left\{ \sum_{v=n-\lambda_n+2}^{n} v \,|T_v(x)| \,\lambda_v \triangle \right.$$

$$\left(\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right)\Bigg\}^k\Bigg]^{1/k}$$

$$+\left[\sum_{n}^{(ii)}\frac{1}{\lambda_n^{k+1}}\left\{\sum_{v=n-\lambda_n+2}^{n}\lambda_v\,|\,T_v(x)\,|\right.\right.$$

$$\left.\left.\frac{\mu_v}{\lambda_v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}^k\right]^{1/k}\Bigg)^k=\left(N_1^{1/k}+N_2^{1/k}\right)^k, \text{ say}.$$

We observe that

$$N_1 = O(1)\sum_{n}^{(ii)}\frac{1}{\lambda_n^{k+1}}\sum_{v=n-\lambda_n+2}^{n} v|\,T_v(x)\,|^k\lambda_v^k\,\Delta$$

$$\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}$$

$$= O(1)\sum_{v=1}^{\infty} v\,|\,T_v(x)\,|^k\lambda_v^k\,\Delta$$

$$\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}\cdot\sum_{n\geqslant v}^{(ii)}\frac{1}{\lambda_n^{k+1}}$$

$$= O(1)\sum_{v=1}^{\infty} v\,|\,T_v(x)\,|^k\,\Delta\left\{\frac{\mu_v}{v[\log^{\mu}(v+1)]^{\beta+\delta-1/2}}\right\}$$

$$= O(1), \text{ by } (3.2).$$

And similarly

$$N_2 = O(1)\sum_{n}^{(ii)}\frac{1}{\lambda_n^{k+1}}\sum_{v=n-\lambda_n+2}^{n}\frac{|\,T_v(x)\,|^k\lambda_v^{k-1}\mu_v}{[\log^{\mu}(v+1)^{\beta+\delta-1/2}}$$

$$= O(1) \sum_{v=1}^{\infty} \frac{|T_v(x)|^k \mu_v}{\lambda_v [\log^{\mu}(v+1)]^{\beta+\delta-1/2}} = O(1), \text{ by (3. 3).}$$

Therefore

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \; |M_n^1|^k = O(1).$$

Finally,

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \; |M_n^2|^k + \sum_{n}^{(ii)} \lambda_n^{k-1} \; |M_n^3|^k$$

$$= O \sum_{n=1}^{\infty} \frac{|T_n(x)|^k \mu_n}{\lambda_n [\log^{\mu}(n+1)]^{\beta+\delta-1/2}} = O(1), \quad \text{by (3, 3).}$$

4. Proof of Theorem 2. To prove this theorem it is sufficient to calculate the order of the integral

$$I = \int_0^{\eta} f(w) \left\{ \frac{d}{dw} P_{n+1}^{(\lambda)} (\cos w) \right\} (\sin w)^{2\lambda-1} \, dw$$

under the hypothesis of the theorem.

We have

$$I = \int_0^{\eta} \phi(w) \left\{ \frac{d}{dw} P_{n+1}^{(\lambda)} (\cos w) \right\} dw$$

$$= \Big[2\lambda \sin w \, P_n^{(\lambda+1)} (\cos w) \, \Phi_1(w) \Big]_0^{\eta}$$

$$- 2\lambda \int_0^{\eta} \Phi_1(w) \frac{d}{dw} \left\{ \sin w \, P_n^{(\lambda+1)} (\cos w) \right\} dw$$

$$= I_1 - I_2, \text{ say.}$$

But,

$$I_1 = O\{ n^{2\lambda+1}\, \eta\, \eta^{2+\beta-\alpha} (\log^{\mu} 1/\eta)^{\gamma/k} \} = O\{ (\log^{\mu} n)^{\gamma/k} \}.$$

$$I_2 = 2\lambda \int_0^{\eta} \frac{d}{dw} \left\{ \sin w\, P_n^{(\lambda+1)} (\cos w) \right\}$$

$$\left\{ \frac{1}{\Gamma(1-\beta)} \int_0^{w} (w-u)^{-\alpha}\, \Phi_\alpha (u)\, du \right\} dw$$

$$= \frac{2\lambda}{\Gamma(1-\beta)} \int_0^{\eta} \Phi_\alpha (u) \left[\int_u^{\eta} \frac{d}{dw} \{ \sin w\, P_n^{(\lambda+1)} (\cos w) \} \right.$$

$$\left. (w-u)^{-\alpha}\, dw \right] du = \left\{ \int_0^{1/n} + \int_{1/n}^{\eta} \right\} \Phi_\alpha (u)\, F(\eta, u)\, du$$

$$= I_{2,1} + I_{2,2}, \text{ say.}$$

We approximate $F(\eta, u)$ for the first integral $I_{2,1}$ i e , when u is ranging from 0 to $1/n$

$$F(\eta, u) = \int_0^{\eta} (w-u)^{-\alpha} \frac{d}{dw} \left\{ \sin w\, P_n^{(\lambda+1)} (\cos w) \right\} dw$$

$$= \left\{ \int_u^{2u} + \int_{2u}^{\eta} \right\} (w-u)^{-\alpha} \frac{d}{dw} \left\{ \sin w\, P_n^{(\lambda+1)} (\cos w) \right\} dw$$

$$= K \int_u^{2u} (w-u)^{-\alpha} \left\{ \cos w\, P_n^{(\lambda+1)} (\cos w) \right.$$

$$\left. + \sin w \frac{d}{dw} \left[P_n^{(\lambda+1)} (\cos w) \right] \right\} dw +$$

$$+ O\, u^{-\alpha} \int_{2u}^{\xi} \frac{d}{dw} [\sin w\, P_n^{(\lambda+1)} (\cos w)] dw \qquad 2u \leqslant \xi \leqslant \eta$$

$$= O\,(n^{2\lambda+1})\,u^{1-\alpha} + O\,(n^{2\lambda+3})\,u^{3-\alpha} + O\,u^{-\alpha}\left[\sin w\,P_n^{(\lambda+1)}(\cos w)\right]_{2u}^{\xi}$$

$$= O\,(n^{2\lambda+1})\,u^{1-\alpha} + O\,(n^{2\lambda+3})\,u^{3-\alpha} + O\,(n^{2\lambda+1-\Delta})\,u^{-\alpha}.$$

Again we have the following order estimate for $F(\eta, u)$ when u ranges from $1/n$ to η.

$$F(\eta,u) = \left\{ \int_u^{u+(1/n)} + \int_{u+(1/n)}^{\eta} \right\} (w-u)^{-\alpha}$$

$$\frac{d}{dw}\left\{ \sin w\,P_n^{(\lambda+1)}(\cos w) \right\} dw = O(n^{\lambda})n^{\alpha-1}\,u^{-\lambda-1} + u^{2-\lambda-2}\,n^{\lambda+1}\,O(n^{\alpha-1}) +$$

$$+ O(n^{\alpha}) \max \left[\sin w\,P_n^{(\lambda+1)}(\cos w) \right]_{u+(1/n)}^{\xi} \quad ; \; u+(1/n) \leqslant \xi \leqslant \eta$$

$$= O\,(n^{\lambda+\alpha-1})\,u^{-\lambda-1} + O\,(n^{\lambda+\alpha})\,u^{-\lambda} + O(n^{\alpha+\lambda})\,u^{-\lambda-1}.$$

If $u + (1/n)$ is not less than η we do not need to break the integral into two parts.

Now

$$I_{2,\,1} = \int_0^{1/n} O\,(u^{2+\beta-\alpha})\,(\log^{\mu} 1/u)^{\gamma/k}\,n^{2\lambda+1}\,du +$$

$$+ \int_0^{1/n} O\,(u^{4+\beta-\alpha})\,(\log^{\mu} 1/u)^{\gamma/\beta}\,n^{2\lambda+3}\,du$$

$$+ \int_0^{1/n} O\,(u^{1+\beta-\alpha})\,(\log^{\mu} 1/u)^{\gamma/k}\,n^{2\lambda+1-\Delta}\,du$$

$$= O\left\{n^{-(3+\beta-\alpha)+2\lambda+1}(\log^{\mu} n)^{\gamma/k}\right\} + O\left\{n^{-(5+\beta-\alpha)+3+2\lambda}(\log^{\mu} n)^{\gamma/k}\right\}$$

$$+ O\{n^{-(2+\beta-\alpha)+2\lambda+1-\Delta}(\log^{\mu} n)^{\lambda/k}\}$$

$$= O\{n^{-(2+\beta-\alpha)+2\lambda+1}(\log^{\mu} n)^{\gamma/k}\} +$$

$$+ O\{n^{2\lambda+1-\Delta(3+\beta-\alpha)+\Delta(2+\beta-\alpha)-(2+\beta-\alpha)}(\log^{\mu} n)^{\gamma/k}\}$$

$$= O\{(\log^{\mu} n)^{\gamma/k}\},$$

and

$$I_{2,2} = = O[n^{\lambda+\alpha-1}]\int_{1/n}^{\eta} u^{\beta-\lambda}(\log^{\mu} 1/u)^{\lambda/k}\,du +$$

$$+ O[n^{\lambda+\alpha}]\int_{1/n}^{\eta} u^{1+\beta-\lambda}[\log^{\mu} 1/u)^{\gamma/k}\,du$$

$$+ O[n^{\alpha+\lambda-\Delta}\int_{1/n}^{\eta} u^{1+\beta-\lambda-1}[\log^{\mu} 1/u]^{\gamma/k}$$

$$= O\{n^{\lambda+\alpha-1-\Delta(1+\beta-\lambda)}(\log^{\mu} n)^{\gamma/k}\} +$$

$$+ O\{n^{2\lambda+1-\Delta(3+\beta-\alpha)-(1+\lambda-\alpha)+\Delta(1+\lambda-\alpha)}(\log^{\mu} n)^{\gamma/k}\}$$

$$+ O\{n^{2\lambda+1-\Delta(3+\beta-\alpha)-1-\lambda+\alpha-\Delta(-1+\alpha-\lambda)}(\log^{\mu} n)^{\gamma/k}\}$$

$$= O\{(\log^{\mu} n)^{\gamma/k}\},$$

and therefore

$$I = O\{(\log^{\mu} n)^{\gamma/k}\}$$

Now, using steps parallel to those used in the proof of Theorem 1, we see that the proof is complete.

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ON STRONG APPROXIMATION TO A FUNCTION BY ITS JACOBI SERIES

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1. DEFINITIONS

An infinite series Σu_n is said to be strongly summable with index $k > 0$ to the sum S, if

$$\sum_{r=0}^{n} | S_r - S |^k = O(n), \; n \to \infty$$

where S_r denotes the rth partial sum of the series Σu_n.

Let $f(x)$ be a function defined on the interval $-1 \leqslant x \leqslant 1$, such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)\, dx$$

exists in the sense of Lebesgue. The Fourier-Jacobi expansion, generally known as Jacobi series corresponding to the function $f(x)$ given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{1.1}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α, β) and degree n in x,

$$a_n = \frac{1}{g_n} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)$$

and

$$g_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(n+1)\ \Gamma(n+\alpha+\beta+1)}.$$

Ultraspherical and Legendre series are particular cases of the series (1.1) when $\alpha = \beta = \lambda - \frac{1}{2}$ and $\alpha = \beta = 0$, respectively.

We write

$$\phi(w) = f(\cos w) - f(1),$$

and establish the following result.

Theorem. *If* $\alpha > -1$, $\beta > -1$, $\alpha + \beta > 0$, *and if* $\Psi(t)$ *is a positive increasing function such that*

$$\int_{0}^{t} |d\phi(u)| \leqslant A\Psi(t), \text{ A } \textit{being a constant},$$
$$t \to 0$$

$$\int_{t}^{\delta} w^{\alpha-1/2}\, \Psi(w)\, dw = O\{t^{\alpha+1/2}\, \Psi(t)\}, \tag{1.2}$$

and

$$\Phi(t) = \int_{t}^{a} w^{\beta+1/2}\, \phi(\pi - w)\, dw = O\{t^{\alpha+1/2}\Psi(t)\}, \tag{1.3}$$
$$t \to 0$$

then

$$\frac{1}{n+1} \sum_{r=0}^{n} |S_r(1) - f(1)| = O\{\Psi(1/n)\}.$$

2. LEMMAS

We require the following lemmas for the proof of the theorem.

Lemma 1 (Szegö [2]). *Let α, β be arbitrary real numbers and* C *a fixed positive constant Then, for $n \to \infty$.*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}.\, O(n^{-1/2}), & C/n \leqslant \theta \leqslant \pi/2 \\ O(n^{\alpha}) & , \quad 0 \leqslant \theta \leqslant C/n \end{cases}$$

Lemma 2 (Szegö [2]). *If $\alpha > -1$, $\beta > -1$, then*

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}K(\theta)\{\cos(N\theta + \gamma) + (n\sin\theta)^{-1}.\, O(1)\}$$
$$C/n \leqslant \theta \leqslant -C/n$$

where

$$K(\theta) = \pi^{-1/2}(\sin\theta/2)^{-\alpha-1/2}(\cos\theta/2)^{-\beta-1/2}$$

$$N = n + \tfrac{1}{2}(\alpha + \beta + 1),\ \gamma = -(\alpha + \tfrac{1}{2})\pi/2.$$

Lemma 3. *The condition* (1.3) *implies*

$$\int_0^t w^{2\beta} \,|\, \phi(\pi - w) \,|\, dw = O\{t^{\alpha+\beta}\Psi(t)\}. \tag{2.1}$$

Proof of Lemma 3 is fairly straightforward, and we omit the details.

3. PROOF OF THE THEOREM

Let S_u be the nth partial sum of the series (1. 1). Hence we have

$$S_n(1) = \int_0^{\pi} (1 - \cos w)^{\alpha+1/2}(1 + \cos w)^{\beta+1/2} f(\cos w).$$

$$\sum_{m=0}^{n} \frac{1}{g_m} P_m^{(\alpha,\beta)} (\cos w) P_m^{(\alpha,\beta)} (1)\, dw$$

$$S_n(1) - f(1) = \int_0^{\pi} (\sin w/2)^{2\alpha+1} (\cos w/2)^{2\beta+1} \phi(w),$$

$$\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)} (\cos w)\, dw$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \cdot$$

$$\cdot \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \right\}$$

$$= J_1 + J_2 + J_3 + J_4 \text{, say.}$$

$$J_1 = O(n^{2\alpha+2}) \int_0^{1/n} w^{2\alpha+1} \mid \phi(w) \mid dw$$

As $\phi(0) = 0$, we see that

$$\mid \phi(w) \mid \leqslant \int_0^{w} d \mid \phi(u) \mid \leqslant A\Psi(w).$$

Therefore

$$J_1 = O(n^{2\alpha+2}) \int_0^{1/n} w^{2\alpha+1} \Psi(w)\, dw = O\{\Psi(1/n)\}.$$

$$J_2 = O(n^{\alpha+1}) \int_{1/n}^{\delta} (\sin w/2)^{2\alpha+1} (\cos w/2)^{2\beta+1} \mid \phi(w) \mid [O(n^{-1/2}).$$

$$\sum_{m=0}^{n} \frac{1}{g_m} P_m^{(\alpha,\beta)} (\cos w) P_m^{(\alpha,\beta)} (1)\, dw$$

$$S_n(1) - f(1) = \int_0^{\pi} (\sin w/2)^{2\alpha+1} (\cos w/2)^{2\beta+1} \phi(w),$$

$$\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)} (\cos w)\, dw$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \cdot$$

$$\cdot \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \right\}$$

$$= J_1 + J_2 + J_3 + J_4 \text{, say.}$$

$$J_1 = O(n^{2\alpha+2}) \int_0^{1/n} w^{2\alpha+1} \mid \phi(w) \mid dw$$

As $\phi(0) = 0$, we see that

$$\mid \phi(w) \mid \leqslant \int_0^{w} d \mid \phi(u) \mid \leqslant A\Psi(w).$$

Therefore

$$J_1 = O(n^{2\alpha+2}) \int_0^{1/n} w^{2\alpha+1} \Psi(w)\, dw = O\{\Psi(1/n)\}.$$

$$J_2 = O(n^{\alpha+1}) \int_{1/n}^{\delta} (\sin w/2)^{2\alpha+1} (\cos w/2)^{2\beta+1} \mid \phi(w) \mid [O(n^{-1/2}).$$

$$\cdot \{\sin w/2)^{-\alpha-3/2} \,.\, (\cos w/2)^{-\beta-1/2} (O(1) + n(\sin w)^{-1})\}]$$

$$= O(n^{\alpha+1/2}) \int_{1/n}^{\delta} w^{\alpha-1/2} \mid \phi(w) \mid dw + O(n^{\alpha-1/2}) \int_{1/n}^{\delta} w^{\alpha-3/2} \mid \phi(w) \mid dw$$

$$= O(n^{\alpha+1/2}) \int_{1/n}^{\delta} w^{\alpha-1/2} \Psi(w) dw$$

$$= O\{\Psi(1/n)\} \quad \text{by (1.2)} \tag{3.2}$$

$$J_3 = O(n^{\alpha+1/2}) \int_{\delta}^{\pi-1/n} (\sin w/2)^{\alpha-1/2} (\cos w/2)^{\beta+1/2} \mid \phi(w) \mid dw$$

$$= O(n^{\alpha+1/2}) \int_{1/n}^{n-\delta} (\cos w/2)^{\alpha-1/2} (\sin w/2)^{\beta+1/2} \mid \phi(\pi - w) \mid dw$$

$$= O(n^{\alpha+1/2}) \,.\, O(1/n)^{\alpha+1/2} \Psi(1/n), \text{ by (1.3)}$$

$$= O\{\Psi(1/n)\}. \tag{3.3}$$

$$J_4 = \lambda_n \int_{\pi-1/n}^{\pi} (\sin w/2)^{2\alpha+1} (\cos w/2)^{2\beta+1} \mid \phi(w) \mid P_n^{(\alpha+1,\beta)} (\cos w) dw$$

where

$$\lambda_n = \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\, \Gamma(n + \beta + 1)}.$$

Therefore

$$J_6 = (-1)^n \lambda_n \int_0^{1/n} (\cos w/2)^{2\alpha+1} (\sin w/2)^{2\beta+1} \phi(\pi - w).$$

$$\cdot P_n^{(\beta,\ \alpha+1)} \cos w)\, dw.$$

Since $P_n^{(\alpha,\ \beta)}(-x) = (-1)^n P_n^{(\beta,\ \alpha)}(x)$, it follows ther

$$J_4 = O(n^{\alpha+\beta+1}) \int_0^{1/n} w^{2\beta+1} \mid \phi(\pi - w) \mid dw$$

$$= O(n^{\alpha+\beta})\, O(1/n)^{\alpha+\beta}\, \Psi(1/n), \quad \text{by (2.1)} \tag{3.4}$$

$$= O\{\Psi(1/n)\}.$$

In view of (3.1), (3.2), (3.3) and (3.4), we get

$$\mid S_n(1) - f(1) \mid = O\{\Psi(1/n)\}.$$

Therefore

$$\frac{1}{n+1} \sum_{r=0}^{n} \mid S_r(1) - f(1) \mid = O\{\Psi(1/n)\}$$

This completes the proof of the theorem.

For several analogous results, see the works of Singh [1] and Yadav [3].

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A NOTE ON A CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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ABSTRACT

It is the purpose of the present paper to prove some distortion inequalities for the derivative of functions belonging to a certain class of analytic functions considered earlier by K. S. Padmanabhan [3], and by S. Chandra and P. Singh [1].

1. INTRODUCTION

Let A (α) denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and satisfy the following condition

$$(2) \qquad \left| \frac{f(z)}{z} - 1 \right| < \alpha \qquad (z \in U)$$

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* This research was completed at the University of Victoria (Victoria, British Columbia, Canada) while the author was on study leave from Kinki University.

for some α $(0 < \alpha \leqq 1)$.

This class $A(\alpha)$ was studied by Padmanabhan [3], and by Chandra and Singh [1].

Now, we need the following lemma which is immediately obtained from Nehari's lemma [2].

Lemma, *Let $\phi(z)$ be analytic in the unit disk U and satisfy $|\phi(z)| \leqq \alpha$ for $z \in U$. Then*

$$(3) \qquad \left|\frac{\phi'(z)}{\alpha}\right| \leqq \frac{1-|\phi(z)|^2/\alpha^2}{1-|z|^2} \qquad (z \in U),$$

where $0 < \alpha \leqq 1$.

2. MAIN RESULTS

We now state and prove our results which provide upper and lower bounds for the derivative of functions belonging to the class $A(\alpha)$

Theorem 1. *Let the function $f(z)$ defined by* (1) *belong to the class $A(\alpha)$. Then*

$$(4) \qquad |f'(z)| \leqq 1 + 2\alpha|z| + \frac{\alpha|z|^2}{1-|z|^2}$$

and

$$(5) \qquad |f'(z)| \geqq 1 - 2\alpha|z| - \frac{\alpha|z|^2}{1-|z|^2}$$

for $0 < \alpha \leqq 1$ *and* $z \in U$.

PROOF. Let the function $g(z)$ be defined by

$$(6) \qquad g(z) = \frac{f(z)}{z} - 1.$$

Then $g(z)$ is analytic in the unit disk U and has simple zero at the origin. Hence we can write that

$$(7) \quad g(z) = \frac{f(z)}{z} - 1 = z\phi(z),$$

where $\phi(z)$ is analytic in the unit disk U and satisfies $|\phi(z)| < \alpha$ for $z \in U$. Thus we have

$$(8) \qquad f(z) = z + z^2\phi(z);$$

hence further,

$$(9) \qquad f(z) = 1 + 2z\phi(z) + z^2\phi(z).$$

Consequently, by using the lemma, we obtain two inequalities (4) and (5) for $0 < \alpha \leqslant 1$ and $z \in U$.

Next we shall consider functions with initial zero coefficients.

THEOREM 2. *Let the function defined by*

$$(10) \qquad f(z) = z + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N = \{1, 2, 3, \ldots\})$$

belong to the class $A(\alpha)$. *Then*

$$(11) \qquad |f(z)| \leqslant 1 + (p+1)\alpha |z|^p + \frac{\alpha |z|^{p+1}}{1 - |z|^2}$$

and

$$(12) \qquad |f'(z)| \geqslant 1 - (p+1)\alpha |z|^p - \frac{\alpha |z|^{p+1}}{1 - |z|^2}$$

for $0 < \alpha \leqslant 1$ *and* $z \in U$.

PROOF. Let the function h (z) be defined by

$$(13) \quad h(z) = \frac{f(z)}{z} - 1.$$

Then $h(z)$ is analytic in the unit disk U and has p zeros at the origin. Therefore, we can write

$$(14) \quad h(z) = \frac{f(z)}{z} - 1 = z^p \phi(z),$$

where ϕ (z) is analytic in the unit disk U and satisfies $|\phi(z)| < \alpha$ for $z \in U$. This gives two inequalities of the theorem with the aid of Lemma.

3. REMARK

We have not been able to obtain sharp estimates for $|f'(z)|$ in our theorems.

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A NOTE ON THE RADII OF STARLIKENESS AND CONVEXITY

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1. INTRODUCTION AND DEFINITIONS

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{ z: |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ of A and by S^* and K the subclasses of S whose members are starlike with respect to the origin and convex in the unit disk U, respectively. A function $f(z)$ in A is said to be starlike of order α $(0 \leqq \alpha < 1)$ in the unit disk U if and only if

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \tag{1.2}$$

for $z \in U$. Further, a function $f(z)$ in A is said to be convex of order α $(0 \leqq \alpha < 1)$ in the unit disk U if and only if

*This research was completed at the University of Victoria (Victoria, British Columbia, Canada) while the author was on study leave from Kinkl University.

$$(1.3) \qquad Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for $z \in U$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of A whose members satisfy (1.2) and (1.3), respectively. Then it is well known that $S^*(\alpha) \subset S^*$, $K(\alpha) \subset K$ for $0 < \alpha < 1$ and that $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\alpha = 1$.

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [9], and were latter studied by Merkes, Robertson and Scott [5], Schild [12], Bogowski, Jablonski and Stankiewicz [1] and Jack [3]. In particular, the class $S^*(1/2)$ was studied by Schild [13] and MacGregor [4].

Now, Several essentially equivalent definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals, have been given in the literature (*cf.*, *e.g.*, [2, Chapter 13], [6], [7], [10] [11], and [14, p. 28 *et seq.*]). We find it convenient to restrict ourselves to the following definitions used recently by Owa [8].

DEFINITION 1. The fractional integral of order λ is defined by

$$(1.4) \qquad D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta,$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$(1.5) \qquad D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\,d\zeta,$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$(1.6) \quad D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),$$

where $0 \leqq \lambda < 1$, $n \in N \cup \{0\}$ and $N = \{1, 2, 3, \ldots\}$.

Recently, by using the fractional derivative $D_z^{\lambda} f(z)$ of $f(z)$ of order λ, Srivastava and Owa [15] introduced the following class.

Definition 4. We say that $f(z)$ is in the class $P_{\lambda}^*(\alpha, \beta)$ if $f(z)$ of S defined by

$$(1.7) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

satisfies the following condition

$$(1.8) \quad \left| \frac{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) - 1}{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) + (1-2\alpha)} \right| < \beta \qquad (z \in U)$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$.

2. Radii of starlikeness and convexity

In this section, we determine the radii of starlikeness and convexity of functions $f(z)$ belonging to the class $P_{\lambda}^*(\alpha, \beta)$. We first need the following lemma given by Srivastava and Owa [15].

Lemma. *A function $f(z)$ defined by* (1. 7) *is in the class $P_\lambda^*(\alpha, \beta)$ if and only if*

$$(2.1) \quad \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (1+\beta) a_n \leqq 2\beta(1-\alpha).$$

The result (2. 1) *is sharp.*

Now we state and prove

Theorem 1. *Let the function $f(z)$ defined by* (1.7) *be in the class $P_\lambda^*(\alpha, \beta)$ for $0 \leqq \lambda < 1, 0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$. Then $f(z)$ is starlike of order γ $(0 \leqq \gamma < 1)$ in the disk $|z| < r_0$, where*

$$(2.2) \quad r_0 = \inf_{n \in N-\{1\}} \left\{ \frac{(1+\beta)(1-\gamma)\,\Gamma(n+1)\,\Gamma(2-\lambda)}{2\beta(n-\gamma)(1-\alpha)\,\Gamma(n+1-\lambda)} \right\}^{1/(n-1)}$$

with equality for the function $f(z)$ given by

$$(2.3) \quad f(z) = z - \frac{2\beta(1-\alpha)\,\Gamma(n+1-\lambda)}{(1+\beta)\,\Gamma(n+1)\,\Gamma(2-\lambda)} z^n.$$

Proof. It suffices to show that

$$(2.4) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \gamma$$

for $|z| < r_0$. Now, we can observe that

$$(2.5) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leqq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}$$

$$\leqq 1 - \gamma$$

if and only if

$$(2.6) \quad \sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1-\gamma}\right) a_n \, |z|^{n-1} \leqq 1 .$$

Consequently, in view of the lemma, we need only find values of $|z|$ for which

$$(2.7) \quad \left(\frac{n-\gamma}{1-\gamma}\right) |z|^{n-1} \leqq \frac{(1+\beta)\,\Gamma(n+1)\,\Gamma(2-\lambda)}{2\beta(1-\alpha)\,\Gamma(n+1-\lambda)} \qquad (n \geqq 2)$$

which will be true when $|z| \leqq r_0$ This completes the proof of the theorem.

Theorem 2. *Let the function $f(z)$ defined by (1.7) be in the class $P_\lambda^*(\alpha, \beta)$ for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$. Then $f(z)$ is convex of order γ $(0 \leqq \gamma < 1)$ in the disk $|z| < r_1$, where*

$$(2.8) \quad r_1 = \inf_{n \in N-\{1\}} \left\{ \frac{(1+\beta)(1-\gamma)\,\Gamma(n)\,\Gamma(2-\lambda)}{2\beta(n-\gamma)(1-\alpha)\,\Gamma(n+1-\lambda)} \right\}^{1/(n-1)}$$

with equality for the function $f(z)$ given by

$$(2.9) \quad f(z) = z - \frac{2\beta(1-\alpha)\,\Gamma(n+1-\lambda)}{(1+\beta)\,n\Gamma(n+1)\,\Gamma(2-\lambda)} z^n .$$

Proof. Since $f(z) \in K(\gamma)$ if and only if $zf'(z) \in S^*(\gamma)$, we can see that the theorem follows that of Theorem 1. with a_n replaced by na_n.

Corollary. Let the function $f(z)$ defined by (1.7) be in the class $P_\lambda^*(\alpha, \beta)$ for $0 \leqq < \lambda\ 1$, $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$. Then $f(z)$ in univalent and starlike for $|z| < r_2$, where

$$(2.10) \quad r_2 = \inf_{n \in N-\{1\}} \left\{ \frac{(1+\beta)\,\Gamma(n)\,\Gamma(2-\lambda)}{2\beta(1-\alpha)\,\Gamma(n+1-\lambda)} \right\}^{1/(n-1)}$$

Proof. By taking $\gamma = 0$ in Theorem 1, we have the above corollary.

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SOME REMARKS ON COMMON FIXED POINTS OF FOUR MAPPINGS

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ABTTRACT

In this paper, inspired by a recent result of Fisher [4], we point out a common fixed point theorem for four selfmappings of a complete metric space, using a well known contractive condition of Meade and Singh [13] and the concept of weak commutativity of the second author [16]. Our theorem generalizes results of Chang [1], Imdad and Khan [8], Sessa and Fisher [17] and Singh and Singh [19].

1. INTRODUCTION

Let R^+ be the set of nonnegative reals and let (X, d) be a complete metric space. Meade and Singh [13], improving a result of Husain and Sehgal [6], proved a common fixed point theorem for two selfmappings of (X, d) considering a real function $f : (R^+)^5 \to R^+$ satisfying the following properties :

(i) f is upper semi-continuous

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(ii) f is non-decreasing in each coordinate variable,

(iii) $f(t, t, at, bt, t) < t$ for any $t > 0$ where $a \geqslant 0, b \geqslant 0$ and $a + b = 3$.

Let F be the family of such functions f. Many authors studied contractive conditions with functions $f \in F$ or using functions with similar properties : for instance, see C. Chang [1], S. Chang [2], Danes [3], Guay, Singh Whitfield [5], Husain and Sehgal [7]. Imdad and Khan [8], Imdad, Khan and Sessa [9], Kasahara and Singh [10], Matkowski [12], Park and Rhoades [14], Rhoades [15], Sessa and Fisher [17], Sharma [18], and Yeh [20, 21].

Inspired by a recent paper of Fisher [4], we prove a common fixed point theorem for four selfmappings of (X, d) extending the results of [1], [8], [13], [17], and [18].

We also use the following notion of weakly commuting selfmappings of (X, d) given in [16].

Definition. Two selfmappings S and I of (X, d) weakly commute if

$$d(SIx, ISx) \leqslant d(Ix, Sx)$$

for any $x \in X$.

Obviously, if S commutes with I, then S also weakly commutes with I but when S weakly commutes with I, then S does not necessarily commute with I as is shown in Example 1 below.

2. A fixed point theorem.

As in [1], [5] we put

$\gamma(t) = \max\{ f(t, t, t, t, t), f(t, t, 2t, 0, t), f(t, t, 0, 2t, t) \}$
for any $t > 0$ and further, we assume a slight modified version of the property (iii), *i. e.*

(iii′) $\gamma(t) < t$ for any $t > 0$.

Now let S, T, I and J be four selfmappings of (X, d) such that

(1) $T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$

(2) $d(Sx, Ty) \leqslant f(\, d(Ix, Jy), d(Ix, Sx), d(Ix, Ty),$
$d(Jy, Sx), d(Jy, Ty)\,)$

for all x, y in X, where f satisfies (i), (ii) (iii′).

Let x_0 be an arbitrary point of X and x_1, x_2 in X such that $Sx_0 = Jx_1$, $Tx_1 = Ix_2$. This can be done since (1) holds. In according to Fisher [4], we can inductively define a sequence

(3) $Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, Sx_{2n+2}, \ldots$

such that $Sx_{2n} = Jx_{2n+1}$, $Tx_{2n+1} = Ix_{2n+2}$ for each integer $n \in N = \{ 0, 1, 2, \ldots \}$. Employing the method of proof of [13], it is proved that

Lemma. The sequence (3) is a Cauchy sequence.

See also an analogous result in [17].

Meade and Singh [13] established the following result :

THEOREM 1. Let S and T two selfmappings of (X, d) satisfying

$$d(Sx, Ty) \leqslant f(d(x, y), d(x, Sx), d(x, Ty)$$
$$d(y, Sx), d(y, Ty))$$

for all x, y in X, where $f \in F$. Then S and T have a unique common fixed point.

Bearing in mind the proofs of the results of [1], [5], [18], it is not hard to verify that Theorem 1 holds also under the assumption (iii′) instead of (iii). Analogous consideration holds for the main Theorem of [8]. Drawing inspiration from Fisher [4], we generalize Theorem 1 with the following

Theorem 2. Let S, T, I and J are four selfmappings of (X,d) satisfying conditions (1) and (2), where f satifies properties (i), (ii), (iii′). If one of S, T, I and J is continuous and if S and T weakly commute with I and J respectively, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. It is similar to that of Fisher and Sessa [17].

However we outline the essential steps in order to show where the weak commutativity plays the key role

By lemma, the sequence (3) converges to a point z.
Suppose that I is continuous. Since the sequences

$$\{Sx_{2n}\} = \{Jx_{2n+1}\} \text{ and } \{Tx_{2n-1}\} = \{Ix_{2n}\}$$

converge also to z, we have that the sequence $\{ISx_{2n}\}$ converges to Iz. S being weakly commuting with I, we deduce

$$d(SIx_{2n}, Iz) \leqslant d(SIx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz)$$

$$\leqslant d(Ix_{2n}, Sx_{2n}) + d(ISx_{2n}, Iz),$$

which implies, as $n \to \infty$, that $\{SIx_{2n}\}$ converges to Iz. As in [17],using

twice (2) and properties (i), (ii), (iii′) and the fact that $\{I^2x_{2n}\}$ converges also to Iz, we ascertain $Iz = Sz = z$. Since the range of J contains the range of S let z be a point in X such that $Jz' = z$. Then using (2) we have

$$d(z, Tz') = d(Sz, Tz) \leqslant$$

$$f(0, 0, d(z, Tz'), 0, d(z, Tz')) \leqslant \gamma\,(d(z, Tz')),$$

which implies $z = Tz'$ by property (iii′). Since T is weakly commuting with J, we have

$$d(TJz', JTz') \leqslant d(Jz', Tz') = d(z, z) = 0$$

and then

$$Jz = JTz' = TJz' = Tz.$$

Using again (2) and (iii), one deduces $Tz = Jz = z$. Therefore z is a common fixed point of S, T, I and J.

Analogous proof can be given if one supposes the continuity of J instead of I,

Now we suppose the continuity of S. Then the sequence $\{SIx_{2n}\}$ converges to Sz. Since S weakly commutes with I, we have

$$d(ISx_{2n}, Sz) \leqslant d(ISx_{2n}, SIx_{2n}) + d(SIx_{2n}, Sz)$$

$$\leqslant d(Sx_{2n}, Ix_{2n}) + d(SIx_{2n}, Sz)$$

which implies, as $n \to \infty$, that the sequence $\{ISx_n\}$ converges to Sz. By (2) and properties (i), (ii), (iii), and observing that $\{S^2x_{2n}\}$ converges also to Sz, one proves that $Sz = z$. As above, one shows that $Jz = Tz = z$. Since the range of I contains the range of T, let z'' be a

point in X such that $Iz'' = z$. Using again (2), we have

$$d(Sz'', z) = d(Sz'', Tz) \leqslant$$

$$f(d(Iz'', Jz), d(Iz'', Sz''), d(Iz'', Tz), d(Jz, Sz''), d(Jz, Tz)) \leqslant$$

$$f(0, d(z, Sz''), 0, d(z, Sz'', 0) \leqslant \gamma\,(d(z, Sz'')),$$

which implies $Sz'' = z$ by property (iii'). Since S weakly commutes with I, we have

$$d(SIz'', ISz'') \leqslant d(Iz'', Sz'') = d(z, z) = 0$$

and therefore

$$Iz = ISz'' = SIz'' = Sz = z.$$

Thus z is a common fixed point of S, T, I and J and making a similar proof, the same conclusion is achieved supposing the continuity of T insetead of S.

The uniqueness of z is easily proved.

3. Some remarks.

Remark 1. Assuming $I = J =$ identity of X, Theorem 2 becomes Theorem 1.

Remark 2. Recently S. L. Singh and S. P. Singh [19] proved a common fixed point theorem for three selfmappings S, T, I of (X, d) satisfying the following condition :

$$(4) \quad d(Sx, Ty) \leqslant h \max \{ d(Ix, Iy), d(Ix, Sx), 1/2 [d(Ix, Ty) + d(Iy, Sx)], d(Iy, Ty) \}$$

for all x, y in X, where $0 \leqslant h < 1$.

C. C. Chang [1] studied the following condition

$$(5)\quad d(Sx, Ty) \leqslant f(\, d(Ix, Iy),\ d(Ix, Sx),\ d(Ix, Ty),\ d(Iy, Sx),\ d(Iy, Ty))$$

for all x, y in X, where f verifies properties (i), (ii), (iii ').

Further, the cited authors assume I continuous and commuting with S and T, $S(X) \subseteq I(X)$, $T(X) \subseteq J(X)$.

Of course (4) is a consequence of (5) assuming

$$f(t_1, t_2, t_3, t_4, t_5) = h.\ max\ \{t_1, t_2, (t_3 + t_4)/2, t_5\}$$

for any $t_1, t_2, t_3, t_4, t_5 \geqslant 0$. However, (2) becomes (5) for $I = J$ and moreover our assumptions of Theorem 2 are more general than those cited.

Remark 3. Imdad and Khan [8] proved a common fixed point theorem for three selfmappings S, I, J of (X,d) satisfying the following condition

$$(6)\quad d(Sx, Sy) \leqslant f(\, d(Ix, Iy),\ d(Ix, Sx),\ d(Ix, Sy),\ d(Jy, Sx),\ d(Jy, Ty)\,)$$

for all x, y in X, where $f \in F$. These authors assume I and J continuous, S commuting with I and J and $S(X) \subseteq I(X) \cap J(X)$. Clearly (2) becomes (6) for $S = T$ and therefore our Theorem 2 is a stronger result.

Example 1. Let $X = [0, 1]$ with euclidean metric d and let S, T, I and J defined by

$$Sx = x/(x + 2),\ Tx = x/(x + 3),\ Ix = x/2,\ Jx = x/3$$

for any x in X. As shown in [16], S weakly commutes with I. Since

$$d(TJx, JTx) = x/(x+9) - x/(3x+9) = 2x^2/(x+9).(2x+9)$$

$$\leqslant x^2/(3x+9) = x/3 - x/(x+3) = d(Jx, Tx)$$

for any x in X, then T weakly commutes with J but T does not commute with J being $TJx \neq JTx$ for any non-zero x in X. Let

$f(t_1, t_2, t_3, t_4, t_5) = t_1/(6+t_1) = g(t_1)$ for all $t_1, t_2, t_3, t_4, t_5 \geqslant 0$. It is immediately seen that f enjoys properties (i), (ii) and (iii').
Further we have

$$T(X) = [0, 1/4] \subseteq [0, 1/2] = I(X), S(X) = [0, 1/3] = J(X)$$

and

$$d(Sx, Ty) = |3x - 2y| / (x+2).(y+3) \leqslant$$

$$\leqslant |3x-2y| /(6 + |3x - 2y|)$$

$$= g(d(Ix, Jy))$$

for all x, y in X. Being one of S, T, I and J continuous, then all the assumptions of Theorem 2 are verified resulting 0 the unique common fixed point of S, T, I and J

The idea of this example appears in [17].

Remark 4. Fisher and Sessa [17], generalizing the results of [4], established a common fixed point theorem for four selfmappings S, T, I and J of (X, d) satisfying the following condition

(7) $d(Sx, Ty) \leqslant \phi(\max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\})$

for all x, y in X, where $\phi : R^+ \to R^+$ satisfies properties (i), (ii) and

(iii″) $\phi(t) < t$ for any $t > 0$.

Obviously, by putting $f(t_1, t_2, t_3, t_4, t_5) = \phi$ (max $\{t_1, t_2, t_5\}$) for any $t_1, t_2, t_3, t_4, t_5 \geqslant 0$, the condition (2) becomes (7). Now we give an examle showing that our Theorem 2 is a more general than result of [17], even if one supposes $I = J =$ identity of X.

Example 2. Let $X = \{A, B, C, D, E\}$ the subset of R^2, where $A \equiv (-1, 0)$, $B \equiv (0, 0)$, $C \equiv (0, 1/2)$, $D \equiv (0, 1)$, $E \equiv (-1, 1)$, with euclidean metric d. Let S and T two selfmappings of X defined as $SA = SB = SC = SD = C$, $SE = D$ and $TA = B$,

$TB = TC = TD = C$, $TE = D$.

Then it is not hard to verify that condition (2) is satisfied if we choose

$$f(t_1, t_2, t_3, t_4, t_5) = (2t_1/5 + t_2/6\,(t_3 + t_4)/5$$

for any $t_1, t_2, t_3, t_4, t_5 \geqslant 0$. Condition (7) does not hold otherwise if there exists a function satisying properties (i), (ii), (iii′), we should obtain for $x = E$ and $y = A$:

$$d(SE, TA) = d(D, B) = 1 \leqslant \phi\,(\max\{\,d(A, E), d(E, D), d(A, B)\}) =$$

$$\phi\,(1, 1, 1),$$

a contradiction to the required condition (iii″).

The idea of this example appears in [11].

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SOME FIXED POINT THEOREMS FOR PAIRS OF MAPPINGS

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In a recent paper [1] fixed point theorems were established for certain expansion mappings. An examination of the inequalities used in both [1] and this paper discloses the fact that further generalizations are doubtful. This paper establishes some fixed polnt theorems for pairs of mappings.

Theorem 1· *Let f, g be surjective selfmaps of a complete metric space (X, d). Suppose there exists a constant $a > 1$ such that*

$$(1) \qquad d(fx, gy) \geqslant ad(x, y)$$

for each x, y in X. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since f is surjective there exists a point $x_1 \in f^{-1}x_0$. Since g is surjective there exists a point $x_2 \in g^{-1}x_1$. Continuing in this manner one obtains a sequence$\{x_n\}$with $x_{2n+1} \in f^{-1}x_{2n}$, $x_{2n+2}=g^{-1}x_{2n+1}$.

Suppose $x_{2n+1}=x_{2n}$ for some n. Since $d(x_{2n+1}, x_{2n})=d(gx_{2n+2}, fx_{2n+1})$, from (1), $d(x_{2n+2}, x_{2n+1}) = 0$. The condition $x_{2n+1} = x_{2n}$ implies that x_{2n} is a fixed ponit of f. Since also $x_{2n+2} = x_{2n+1}$, x_{2n} is a fixed point of g. Similarly, $x_{2n+2} = x_{2n+1}$ leads to x_{2n+1} being a common fixed point of f and g.

Assume $x_n \neq x_{n+1}$ for each n. From (1), $d(x_{2n}, x_{2n+1}) =$

$d(fx_{2n+1}, gx_{2n+2}) \geqslant ad(x_{2n+1}, x_{2n+2})$ and $d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) \geqslant ad(x_{2n+2}, x_{2n+3})$. Therefore $d(x_n, x_{n+1}) \geqslant ad(x_{n+1}, x_{2n+2})$, which implies $\{x_n\}$ converges to a point $x \in X$. Let $y \in f^{-1}x$. The assumption $x_n \neq x_{n+1}$ for each n implies that $x_n \neq x$ for almost all n, From (1),

$$d(x_{2n+1}, x) = d(gx_{n+2}, fy) \geqslant ad(x_{2n+2}, y).$$

Taking the limit as $n \to \infty$ yields $x = fx$. Let $z \in g^{-1}x$. Then

$$d(x_{2n}, x) = d(fx_{2n+1}, gz) \geqslant ad(x_{2n+1}, z),$$

and we obtain $z = x$, and x is a common fixed point of f and g.

Condition (1) forces uniqueness of the fixed point.

Theorem 2. *Let f, g be surjective selfmaps of a complete metric space (X, d). Suppose there exist nonnegative functions p, q, r, s, t, satisfying*

$$\inf_{x, y \in X} (p(x, y) + q(x, y) + t(x, y)) > 1, \tag{2}$$

$$\inf_{x, y \in X} \{(1 - q(x, y) + r(x, y)), (1 - p(x, y) + s(x, y))\} > 0, \tag{3}$$

$$\sup_{x, y \in X} \{p(x, y), q(x, y))\} < 1, \tag{4}$$

and

$$d(fx, gy) \geqslant p(x, y)\, d(x, fx) + q(x, y)\, d(y, gy) + r(x, y)\, d(x, gy) + s(x, y)\, d(y, fx) + t(x, y)\, d(x, y) \tag{5}$$

for all $x, y \in X$, $x \neq y$. Then f and g have common fixed points.

Proof. Define $\{x_n\}$ as in Theorem 1. Suppose $x_{2n} = x_{2n+1}$ for some n.

If $x_{2n+1} \neq x_{2n+2}$, then from (5),

$$d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})$$

$$\geqslant pd(x_{2n+1}, x_{2n}) + qd(x_{2n+2}, x_{2n+1})$$

$$+ sd(x_{2n+2}, x_{2n}) + td(x_{2n+1}, x_{2n+2}),$$

where p, q, r, s, and t are evaluated at (x_{2n+1}, x_{2n+2}).

Thus

$$0 \geqslant (q + s + t)\, d(x_{2n+1}, x_{2n+2}).$$

If $q + s + t = 0$ then $q + t = 0$ which, since $p < 1$, contradicts (2). Therefore $x_{2n+1} = x_{2n+2}$ and x_{2n} is a common fixed point of f and g. Similarly, $x_{2n+1} = x_{2n+2}$ for some n leads to x_{2n+1} being a common fixed point of f and g.

Assume $x_n \neq x_{n+1}$ for each n. From (5),

$$d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})$$

$$\geqslant pd(x_{2n+1}, x_{2n}) + qd(x_{2n+2}, x_{2n+1})$$

$$+ sd(x_{2n+2}, x_{2n}) + td(x_{2n+1}, x_{2n+2}),$$

or

$$(6) \quad (1-p+s)\, d(x_{2n}, x_{2n+1}) \geqslant (q + s + t)\, d(x_{2n+1}, x_{2n+2}),$$

where p, q, r, s, and t are evaluated at (x_{2n+1}, x_{2n+2}).

Again from (5),

$$d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3})$$

$$\geqslant p'd(x_{2n+3}, x_{2n+2}) + q'd(x_{2n+2}, x_{2n+1})$$

$$+ r'd(x_{2n+3}, x_{2n+1}) + t'd(x_{2n+2}, x_{2n+3}),$$

or

(7) $(1 - q' + r')\, d(x_{2n+1}, x_{2n+2}) \geqslant (p' + r' + t')\, d(x_{2n+2}, x_{2n+3})$,

where p', q', r', s and t' are evaluated at (x_{2n+3}, x_{2n+2}).

Inequalities (6) and (7), along with conditions (2) and (3), imply that $\{x_n\}$ is Cauchy, hence convergent to some x in X..

Without loss of generality we may assume that $x_n \neq X$ for infinitely many n since, otherwise, f and g have a common fixed point. If there exists an infinite number of integers n such that $x_{2n} \neq x$, define $y \in g^{-1}x$. Then, from (5),

$$d(x_{2n}, x) = d(fx_{2n+1}, gy)$$

$$\geqslant pd(x_{2n+1}, x_{2n}) + qd(y, gy) + rd(x_{2n+1}, gy)$$

$$+ sd(y, x_{2n}) + td(x_{2n+1}, y),$$

where p, q, r, s and t are evaluated at (x_{2n+1}, y). The above inequality implies that

$$d(x_{2n}, x) \geqslant (q + s + t)\, min\, \{ d(y, x), d(y, x_{2n}), d(x_{2n+1}, y) \}$$

$$\geqslant \inf_{x, y \in X} (q+s+t)\, min\, \{d(y,x), d(y, x_n), d(x_{2n+1},y)\}.$$

Taking the limit as $n \to \infty$ yields

$$0 \geqslant \inf_{x, y \in X} (q + s + t)\, d(x, y),$$

which implies that either $x = y$ or $\inf_{x, y \in X} (q + s + t) = 0$. However, the latter condition, along with (4), contradicts (2). Therefore $x = y$.

If $x_{2n+1} \neq x$ for all n sufficently large, then $x_{2n} = fx_{2n+1} = x$. Taking the limit as $n \to \infty$ yield x as a fixed point of f.

In $x_{2n+1} \neq x$ infinitely many n, define $f^{-1}x$.

Then, from (5), with p, q, r, s, and t evaluated (z, x_{2n+2})

$$d(x_{2n+1}, x) = d(x_{2n+2}, fz)$$

$$\geqslant pd(z, fz) + qd(x_{2n+2}, x_{2n+1}) + rd(z, x_{2n+1})$$

$$+ sd(x_{2n+2}, x) + td(z, x_{2n+2})$$

$$\geqslant \inf_{x,y \in X} (p+r+t) \min\{d(z,x), d(z,x_{2n+1}), d(z,x_{2n+2})\}.$$

Taking the limit as $n \to \infty$ yields $0 \geqslant \inf_{x, y \in X} (p+r+t)\, d(x, z)$, which, in light of (4) and (2), implies $x = z$.

Theorem 3· *Let f, g be surjective continuous selfmaps of a complete metric space X. If there exists a real number $a > 1$ such that*

(1) $\qquad d(fx, gy) \geqslant a \min \{ d(x, fx), d(y, gy), d(x,y) \}$

for each $x,y \in X$, then f or g has a fixed point or f and g have a common fixed point.

Proof. Define $\{x_n\}$ as in Theorem 1. If $x_n = x_{n+1}$ for any n, then f or g has a fixed polnt.

Assume $x_n \neq x_{n+1}$ for each n. From (8),

$$d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2}) \geqslant$$

$$\geqslant a \min \{ d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}) \}.$$

and

$$d(x_{2n+1}, x_{2n+2}) = d(gx_{2n+2}, fx_{2n+3}) \geqslant$$

$$\geqslant a \min \{ d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1}) \} .$$

Thus, for each n, $d(x_n, x_{n+1}) \geqslant a \min \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \}$,

which, since $a > 1$, implies that $\{x_n\}$ is Cauchy, hence convergent, to some x in X. The condition $x_{2n} = fx_{2n+1}$, $x_{2n+1} = gx_{2n+2}$ and the continuity of f and g imply that s is a common fixed point of f and g.

REMARK 1. Setting $f = g$ in Theorem 1 and 3 yields Theorems 1 and 3 of [1]

REMARK 2, Setting $f = g$, $s = t = 0$, p, q, r constants in Theorem 2 yields Theorem 2 of [1].

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EFFECT OF VISCOSITY ON RAYLEIGH–TAYLOR INSTABILITY IN THE PRESENCE OF A VERTICAL MAGNETIC FIELD

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ABSTRACT

The character of equilibrium of a heavy, viscous, incompressible, finitely condducting and rotating fluid of variable density in the presence of a vertical magnetic field is investigated, when the lower bounding surface is rigid and the upper is free. The nature of the boundaries chosen alter the order of the dispersion relation compared to the cases where the boundaries are both free or both rigid. It is found that the stability criterion is independent of the effects of viscosity, finite resistivity and rotation. It is further investigated that the growth rates both increase or decrease with the increase in viscosity.

1. INTRODUCTION

The character of an incompressible fluid of variable density stratified in the vertical direction was first investigated by Rayleigh [3] and he found that the configuration is stable or unstable according as $\frac{d\rho}{dz}$ is everywhere negative or anywhere positive. Chandrasekher [1] introduced the viscosity in the Rayleigh–Taylor instablity and observed that in the stable case, the fluid oscillates about the mean position with an amplitude which decays exponentially at a rate which increases with increasing viscosity.

The hydromagnetic version of the Rayleigh–Taylor instability was further studied by Hide [2] for a fluid having exponentially varying density in the vertical direction. He included the effects of viscosity and finite resistivity in his problem. Sharma and Ariel [4] investigated the effect of finite resistivity of the medium on the equilibrium of a heavy, viscous, incompressible, rotating fluid of variable density in the presence of a vertical magnetic field. The present note investigates theoretically the same problem as considered by Sharma and Ariel [4], when the fluid is confined between two boundaries, the lower bounding surface being rigid and the upper free. Based on the existence of a variational principle we obtained the dispersion relation for a fluid having exponentially varying density. The object of this problem, under these boundary conditions, is to find out to what extent the instability of the configuration is affected by changing the viscosity of the medium.

2. PERTURBATIONS EQUATIONS

The linearised perturbation equations for the problem under consideration are (Sharma and Ariel [4], *p.* 104, *eqns.* (31), (32), (34) and (35))

$$n[k^2 \rho_0 w - D(\rho_0 Dw) + \frac{KH_0}{4\pi}(D^2 - k^2)Dh_z - \frac{gk^2}{n}(D\rho_0)w -$$

$$- D(2\rho_0 \Omega \zeta) + \mu_0(D^2 - k^2)^2 w + 2D\mu_0 (D^2 - k^2)Dw +^2$$

$$+ D^2\mu_0(D^2 + k^2)w = 0, \tag{1}$$

$$[n\rho_0 - \mu_0(D^2 - k^2) - D \mu_0 D] \zeta - \frac{KH_0}{4\pi} D\xi \, 2\rho_0\Omega \, Dw, \tag{2}$$

$$[n - \eta (D^2 - k^2)]h_z = H_0 \, Dw, \tag{3}$$

and $$[n - \eta (D^2 - k^2)] \xi = H_0 D\zeta, \tag{4}$$

where

ρ_0 denotes the density before the system is disturbed,

w denotes the z-component of the velocity $\vec{u}$ at a fixed point,

$\vec{g}$ is the acceleration due to gravity, components $(0, 0, -g)$,

μ_0 is the coefficient of viscosity, assumed to be variable in the undisturbed state,

K is the coefficient of magnetic permeability assumed to be constant,

$\vec{H}_0$ is the magnetic field directed towards the z-axis in the undisturbed state,

n is the rate at which the system departs from equilibrium, h is the z-component of the magnetic field in the perturbed state,

k is the total wavenumber of the initial disturbance,

ζ is the z-component of the vector curl $\vec{u}$,

ξ is the z-component of the vector curl $\vec{h}$,

$D = \dfrac{d}{dz}$ and $\eta = [4\pi K\rho]^{-1}$.

3. BOUNDARY CONDITIONS

The fluid is assumed to be confined to the planes $z = 0$ which is rigid and perfectly conducting and $z = d$. which is free boundary.

The appropriate boundary conditions for the present problem are

$$\text{(i)} \quad w(0) = Dw(0) = h(0) = \zeta(0) = D\xi(0) = 0, \tag{5}$$

$$\text{(ii)} \quad w(d) = D^2w(d) = Dh(d) = D\zeta(0) = \xi(d) = 0. \tag{6}$$

Multiplying the equation (1) for the characteristic value n_i by w_j and integrating across the vertical extent of the fluid we obtain on combining equations (2),(3)and(4),after a series of integrations by parts

$$n(I_1 + I_4 + I_5 + I_7 + I_9) + I_3 + I_{10} -$$

$$- \frac{gk^2}{n} I_2 + \eta k^2(I_4 + 2I_5 + I_6 + I_9 + I_{10}) = 0, \tag{7}$$

the integrated parts vanish on account of boundary conditions,

where

$$I_1 = \int_0^d \rho_0 \left[k^2 w^2 + (Dw)^2\right] dz, \tag{8}$$

$$I_2 = \int_0^d D\rho_0 w^2 \, dz, \tag{9}$$

$$I_3 = \int_0^d \mu_0 [k^2 w^2 + 2k^2 (Dw)^2 + (D^2 w)^2] + k^2 \int_0^d D^2 \mu_0 \, dz, \tag{10}$$

$$I_4 = \frac{Kk^2}{4\pi} \int_0^d h^2 \, dz, \tag{11}$$

$$I_5 = \frac{K}{4\pi} \int_0^d (Dh)^2 \, dz, \tag{12}$$

$$I_6 = \frac{K}{4\pi k^2} \int_0^d (D^2 h)^2 \, dz, \tag{13}$$

$$I_7 = \int_0^d \rho_0 \zeta^2 \, dz, \tag{14}$$

$$I_8 = \int_0^d \mu_0 \left[k^2 \zeta^2 + (D\zeta)^2\right] dz, \tag{15}$$

$$I_9 = \frac{K}{4\pi k^2} \int_0^d \xi^2 \, dz \tag{16}$$

$$I_{10} = \frac{K}{4\pi k^2}\int_0^d (D\xi)^2\, dz, \tag{17}$$

4. THE CASE OF EXPONENTIALLY VARYING DENSITY

A case for which a simple analytical solution can be found is one in which the undisturbed density distribution is given by

$$\rho_0(z) = \rho exp\ \beta z, \tag{18}$$

where ρ and β are consants.

Assuming ν, the coefficient of kinematic viscosity to be constant, we shall take

$$\mu_0(z) = \nu\rho \exp \beta z. \tag{19}$$

In order to ensure that the density variation within the fluid is small compared to the average density, we make an assumption that

$$|\ \beta d\ | << 1. \tag{20}$$

Let us assume the following trial functions which satisfy the boundary conditions (5) and (6) :

$$\left.\begin{aligned} w(z) &= W(\cos lz - \cos 3lz), \\ h(z) &= X_1 \sin lz + X_3 \sin 3lz, \\ \zeta(z) &= Z_1 \sin lz + Z_3 \sin 3lz, \\ \xi(z) &= K_1 \cos lz = K_3 \cos 3lz, \end{aligned}\right\} \tag{21}$$

where $l = (\pi s/2d)$, s being an odd integer.

Substituting the values of $w(z)$, $h(x)$, $\zeta(z)$ and $\xi(z)$ equations (2), (3) and (4), we get

$$[n + \eta\,(l^2 + k^2)\,]Z_1 + \frac{kH_0 l}{4\pi\rho}\, K_1 = -\,2\Omega l W, \tag{22}$$

$$[n + \eta (9l^2 + k^2)]Z_3 + \frac{3H_0 l}{4\pi\rho} K_3 = 6\Omega l W, \tag{23}$$

$$[n + \eta (l^2 - k^2)]X_1 = - lH_0 W, \tag{24}$$

$$[n + \eta (9l^2 + k^2)]K_3 = - 3lH_0 W, \tag{25}$$

$$[n + \eta (l^2 + k^2)]K_1 = lH_0 Z_1, \tag{26}$$

$$[n + \eta (9l^2 + k^2)]K_3 = 3lH_0 Z_3. \tag{27}$$

Evaluating the integrals in equation (7) with the assumed form of w, etc., and eliminating the constants from (22)–(27) we obtain the following dispersion relation between n and k:

$$n^2 (k^2 + 5l^2) - g\beta k^2 + nv (k^4 + 10l^2k^2 + 41l^4) +$$

$$\frac{nl^2 V^2}{2} \left[\frac{l^2 + k^2}{n+\eta (l^2 + k^2)} + \frac{9(l^2 + k^2)}{n + \eta (9l^2 + k^2)} \right]$$

$$+ 2\Omega^2 l^2 n \left[\frac{n + \eta (l^2 + k^2)}{\{n + \eta (l^2 + k^2)\} \{n + \eta(l^2 + k^2)\} + l^2V^2} + \right.$$

$$\left. + \frac{9n + 9\eta (9l^2 + k^2)}{\{n + \eta (l^2 + k^2)\} \{n + (9l^2 + k^2)\} + 9l^2V^2} \right] = 0 \tag{28}$$

where $V(= \{KH_0 / 4\pi\rho\})$ denotes the Alfven velocity.

It is convenient to discuss equation (28) in non-dimensional from, so that the important physical parameters of the problem may be brought out clearly. Let us choose a dimensionless growth rate y and a dimensionless wavenumber x by measuiing n and k in suitable units. We define :

$$x = \frac{kd}{\pi s}, \tag{29}$$

and $y = \frac{nd}{\pi s V}$. (30)

From eqns. (28), (29) and (30), we have

$256y^6 + 128y^7 [4Ra + S(a^2 + 2a + 16)] + 64y^6 [2R^2 (a^2 - 16) +$

$+ S^2(a^2 - 16) + 2RS(a^3 + 2a^2 + 16a + 64) + a + 36 + 4 (5A - 4Bx^2)]$

$+ 32y^5 [4R^3 (a^3 - 16a) + S^3 (a^4 - 256) + R^2S(5a^4 + 18a^3 + 64a^2 +$

$224a - 256) + 2RS^2 (2a^4 + 9a^3 + 48a^2 + 112a) + 2R (a^2 + 46a) +$

$2S (5a^2 + 36a + 80) + 20A \{4Ra + S (a - 3.2)\} - 16Bx^2 (2Ra +$

$2Sa)] + 16y^4 [R^4 (a^4 - 256) 2R^3S(2a^5 + 8a^4 - 64a^2 - 512a - 1000) +$

$2R^2S^2 (6a^5 + 19a^4 + 64a^3 - 512a + 256) + 2S^3R (2a^5 + 4a^4 -$

$512a - 1000) + R^2 (11a^3 + 66a^2 - 240a + 416) + S^2 (11a^3 + 4a^2 +$

$144a - 140) + | 2RS (22a^3 + 108a^2 + 96a + 460) + 10a + 369 +$

$2A \{10R^2 (6a^2 - 32) + 2RS (15a^2 - 54a + 18) + 18\} - 16Bx^2$

$\{R^2 (6a^2 - 32) + 8RSa^2 + S^2 (a^2 - 16)] + 8y^3 [R^4S (a^6 + 2a^5 -$

$16a^4 - 64a^3 - 256a^2 + 512a + 4096) + 4R^3 S^2(2a^6 + 5a^5 - 32a^3 - 512a^2$

$- 744a) + R^2 S^3(a^4 - 256) (6a^2 + 16a - 32) + RS^2 (36a^4 - 12a^3 +$

$+ 208a^2 - 1344a - 256) + 2R^2 S (26a^4 + 59a^3 - 224a^2 - 368a -$

$3072) + 5R^3 (5a^4 + 10a^3 - 192a^2 - 160a + 1692) + R (60a^2 +$

$346a - 1952) + S (19a^2 + 328a - 10008) + 2A \{R^2S (40a^2 - 472a$

$+ 584) + R^3(40a^3 - 64a) + 54Ra\} - 16Bx^2 \{4R^3 (a^3 - 16a) + 2R^2S$

$(6a^3 - 32a) + 4RS^2(a^3 - 16a) + R(30a - 32) + S(10a - 32)\}]$

$+ 4y^2[R^4S^2(a^2 - 16)^2(a^3 + a^2 + 16a - 16) + 4R^3S^3(a^4 - 256)$

$(a^2 - 16)(a + 2) + R^2S^2(31a^5 - 12a^4 - 512a^3 + 160a^2 - 1338a +$

$29696) + 2R^3S(a^2 - 16)(10a^3 - 16a^2 - 256) + 2RS(39a^3 + 104a^2 +$

$1228a + 4730) + R^2(a^2 - 16)(50a - 151) + 9(a + 36) +$

$2A\{10R^4(a^2 - 16)^2 + 2R^3S(5a^4 - 12a^3 + 6a^2 + 192a - 1376) +$

$18R^2(30a^2 - 16)\} - 16Bx^2\{2R^2S^2(a^2 - 16)(3a^2 - 16) + 2RS(15a^2 -$

$64a + 80) + 8R^3S(a^4 - 16a^2) + R^4(a^2 - 16)^2 + R^2(30a^2 - 64a -$

$160) + 9\}] + 2y[2R^4S^3(a^2 + 16)(a^2 - 16)^3 + R^3S^2(a^2 - 16)$

$(5a^4 - 64a^3 + 160a^2 - 1024a + 3840) + R^2S(a^2 - 16)(59a^2 - 160a +$

$224) + 45R(a^2 - 16) + 2A\{R^4S(a^2 - 16)^2(10a - 32) + 18R^3(a^3 -$

$16a)\} - 16Bx^2\{2R^3S^2a(a^2 - 16)^2 + 2R^2S(11a^3 - 80a^2 + 80a +$

$256) + 2SR^4a(a^2 - 16)^2 + 2S^2R^2a(a^2 - 16)^2 + R^3(a^2 - 16)(10a -$

$- 32) + 18Ra\}] - 16Bx^2R^2(a^2 - 16)\{R^2S^2(a^2 - 16)^2 + 2RS(5a^2 -$

$32a + 80) + 9\} = 0,$ (31)

where

$$R = (\eta\pi s/2dV), \tag{32}$$

$$S = (\nu\pi s/2dV), \tag{33}$$

$$B = (g\beta d^2/\pi^2 s^2 V^2), \tag{34}$$

$$A = (4\Omega^2 d^2/\pi^2 s^2 V^2), \tag{35}$$

and $a = 4x^2 + 5.$ (36)

From the above equation we find that there are four parameters required to specify y for any given x. These numbers R, S, B and A respectively represent measures of resistivity of the medium, coefficient of viscosity, bouyancy forces and coriolis forces in terms of magnetic field. The case of viscous, finitely conducting and non-rotating configuration has been discussed in detail by Sharma [5], where the nature of the boundaries chosen gives rise to an additional monotonically decreasing mode. Another particular case of equation (31), when the viscosity is absent, has been treated by Sharma [6]. The dispersion relation (31) is an algebrical equation of degree eight in y, hence it will have eight roots. To obtain the explicit expression for each value of y for general values of parameter is a task which is mathematically too involved. If we consider equation (31) in its general form, we observe that the stratification is stable or unstable thoroughly according as B is negative or positive. However, if we consider the fluid to be viscous but infinitely conducting ($R = 0$), the configuration is rendered unstable for the wavenumber range

$$x < \frac{1}{2}\left[\frac{41}{4B-5}\right]^{1/2}. \tag{37}$$

Thus the effect of resistivity is to cancel the stabilizing role of magnetic field altogether and render the system unstable for the whole wavenumber range.

We now examine the behaviour of growth rate with respect to viscosity analytically To investigate this, we require to discuss the nature of the positive root of y in equation (31) in detail. We observe if $B > 0$, the absolute term in equation (31) is negative, therefore, the

equation (31) has at least one real positive root. Hence, the equilibrium will be always unstable. The asymptotic behaviour of this root for $X \to 0$ and $X \to \infty$ are

$$y \to \frac{(9R^2S^2 + 10RS + 1)}{(9R^2S^2 + 10RS + 1)(41RS + S) + AR^2(9RS + 5)} \tag{38}$$

$$y \to \frac{B}{2Sx^2}. \tag{39}$$

Now we study the behaviour of y (growth rate) on varying the value of S from equation (31). To find the role of S, we examine the nature of dy/dS form (38), while keeping other nondimensional numbers R, A and B fixed. To bring out the peculiar features of the role of viscosity into focus, in the context of Rayleigh-Taylor instability, we divide our subsequent analysis of instability criterion into two cases by different range of R and A.

Case 1. $AR^2 > 1$. In this case we find that with increasing S, the value of y decreases for all wavenumbers. Thus, more the viscosity of the medium, more it tends to stabilizes the configuration. This type of behaviour can also be observed even in the absence of rotation.

Case II. $AR^2 < 1$. In this case a peculiar tendency is exhibited by S, when (a) $S < \bar{S}$, (b) $S > \bar{S}$, where

$$\bar{S} = \frac{1}{18R^2}\{-10R + [100R^2 - \frac{1}{82}(82 - 9AR^3 - (82A^2R^4 + 5248AR^2)^{1/2})]^{1/2}\}. \tag{40}$$

So long as $S < \bar{S}$, an increase in the value of S leads to the decrease in the value of y for all wavenumbers. Thus, we can say that viscosity has a stabilizing influence on the configuration. When S becomes smaller than $\bar{S}$, this behaviour is reversed for small values of

x. Now an increase in the value of S leads to a increase in the value of y. This particular behaviour is however, marked only for small x. After reasonably large values of x, once again the uniform pattern can be observed, namely that with increasing S, more stability is imparted to the system. Thus, we conclude that viscosity has a destabilizing influence for small wavenumbers of disturbanca but it has a stabilizing influence for large wavenumbers.

Lastly, when $B<0$ we find that all roots of y are either real and negative or there are complex roots with negative real parts. The system is therefore stable in each case. Hence, the potentially stable configuration remains stable whether the effects of viscosity, resistivity and rotation are included or not.

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SOME INTEGRALS AND SERIES OF THE PRODUCT OE TWO MULTIVARIABLE H-FUNCTIONS

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ABSTRACT

In the present paper, we evaluate three definite integrals which are then used to establish two Fourier sine series and one infinite series for the multivariable H-function due to H. M. Srivastava and R. Panda [10] .

1. INTRODUCTION

The multivariable H-function defined by Srivastava and Panda [10, p. 271, Eq. (4. 1)] (see also Prasad and Singh [7] and Srivastava and Panda [11]) will be represented here in the manner already detailed by Srivastava, Gupta and Goyal [9, p. 251, Eq. (C. 1)]. The object of th is paper is to evaluate three definite integrals which are then used to establish a number of expansion formulas involvlng these multivariable H-functions.

2. Integral Formulas

In this section, we evaluate the following definite integrals

$$\int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{\omega(u+v)x} \; H^{0,0:(1,N);\ldots;}_{P,Q:[P,Q'+1];\ldots;}$$

$$(1, N^{(r)})$$
$$[P^{(r)}, Q^{(r)}+1]\left[Z_1 (\sin x\, e^{\omega x})^{\sigma_1} ;\ldots; Z_r (\sin x\, e^{\omega x})^{\sigma_r} \right|$$

$$\{(e_p; E_p' ;\ldots; E_p^{(r)})\} : \{(C_p', \gamma_p')\} ;\ldots;$$
$$\{(f_Q; F_Q' ,\ldots, F_Q^{(r)})\} : (D_0', \delta_0') \{(D_Q', \delta'_Q)\} ;\ldots;$$

$$\{(C^{(r)}{}_{p^{(r)}} , \gamma^{(r)}{}_{p^{(r)}})\}$$
$$(D_0^{(r)} , \delta_0^{(r)}) , \{(D^{(r)}{}_{Q^{(r)}} , \delta^{(r)}{}_{Q^{(r)}} \Big]$$

$$H^{0,\,0 : (V', W') ;\ldots; (V^{(r)}, W^{(r)})}_{p,\,q: [X', Y'] ;\ldots; [X^{(r)}, Y^{(r)}]} \left[x_1 (\sin x\, e^{w_x})^{h_1} ;\ldots;\right.$$

$$x_r (\sin x\, e^{\omega x})^{h_r} \left| \begin{array}{l} \{(a_p; \alpha_p' ,\ldots, \alpha_p^{(r)})\} : \{(A'_{X'} , \eta'_{X'})\} ;\ldots; \\ \{(b_q; \beta_q' ,\ldots, \beta_q^{(r)})\} : \{(B'_{Y'} , \xi'_{Y'})\} ;\ldots; \end{array}\right.$$

$$\left.\begin{array}{l} \{(A^{(r)}{}_{X^{(r)}} , \eta^{(r)}{}_{X^{(r)}})\} \\ \{(B^{(r)}{}_{Y^{(r)}} , \xi^{(r)}{}_{Y^{(r)}})\} \end{array}\right] dx.$$

$$= \frac{e^{\omega u \pi/2}\, \Gamma(v)}{\delta_0' \ldots \delta_0^{(r)}} \sum_{\nu_1,\ldots \nu_r = 0}^{\infty} \phi(\rho_{\nu_1},\ldots, \rho_{\nu_r})$$

$$e^{\omega (\sum_{i=1}^{r} \sigma_i \rho_{\nu_i})\pi/2} \prod_{i=1}^{r} \{\theta_i (\rho_{\nu_i})$$

$$\left. \frac{(-1)^{\nu_i} Z_i\, \rho_{\nu_i}}{\nu_i\,!} \right\} H^{0,\,1 : (V', W') ;\ldots; (V^{(r)}, W^{(r)})}_{p+1,\,q+1 : [X', Y'] ;\ldots; [X^{(r)} , Y^{(r)}]}$$

$$\left[x_1\, e^{\omega h_1 \pi/2} ,\ldots, x_r\, e^{\omega h_r \pi/2} \left| \begin{array}{l} (1 - u - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} ; h_1 ,\ldots, h_r) \\ \{(b_q; \beta'_q ,\ldots, \beta^{(r)}{}_q)\}, \end{array}\right.\right.$$

$$\{(a_p;\ \alpha'_p,\ldots,\alpha^{(r)}{}_p)\} : \{(A'_{X'},\ \eta'_{X'})\}\ ;\ldots;$$

$$(1-u-v-\sum_{i=1}^{r}\sigma_i\,\rho_{\nu_i}\ ;\ h_1,\ldots,h_r) : (B'_{Y'},\ \xi'_{Y'})\ ;\ldots;$$

$$\left.\begin{matrix}\{(A^{(r)}{}_{X^{(r)}},\ \eta^{(r)}{}_{X^{(r)}})\}\\ \{(B^{(r)}{}_{Y^{(r)}},\ \xi^{(r)}{}_{Y^{(r)}})\}\end{matrix}\right] , \tag{2.1}$$

provided that $\omega = \sqrt{-1}$, $h_1, h_2,\ldots, h_r > 0$, $Re\,(u + \sum_{i=1}^{r}(\sigma_i\,\rho_{\nu_i}$

$+ h_i\,\alpha_i)) > 0$, $R_e(v) > 0$, $|\arg x_i| < \frac{1}{2}\,U_i\pi$, $U_i > 0$,

$|\arg Z_i| < \frac{1}{2}\,V_i\pi$, $V_i > 0\ (i = 1,\ldots, r)$,

where α_i, U_i and V_i are as follows :

$$\alpha_i = \min R_e\,(B_j^{(i)}\ /\ \xi_j^{(i)}),\ j = 1,\ldots, V^{(i)},\ i = 1,\ldots, r, \tag{2.2}$$

$$U_i = -\sum_{j=1}^{p}\alpha_j^{(i)} - \sum_{j=1}^{q}\beta_j^{(i)} + \sum_{j=1}^{V^{(i)}}\xi_j^{(i)} - \sum_{j=V^{(i)}+1}^{Y^{(i)}}\xi_j^{(i)} +$$

$$\sum_{j=1}^{W^{(i)}}\eta_j^{(i)} - \sum_{j=W^{(i)}+1}^{X^{(i)}}\eta_j^{(i)} > 0,\ i = 1,\ldots, r, \tag{2.3}$$

rnd

$$V_i = -\sum_{j=1}^{P}E_j^{(i)} - \sum_{j=1}^{Q}F_j^{(i)} + \delta_0\sum_{j=1}^{Q^{(i)}}\delta_j^{(i)} + \sum_{j=1}^{N^{(i)}}\gamma_j^{(i)}$$

$$-\sum_{j=N^{(i)}+1}^{P^{(i)}}\gamma_j^{(i)} > 0,\ i = 1,\ldots, r. \tag{2.4}$$

$$\int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}\ e^{\omega(u+v)x}\ H\begin{matrix}0, 0 : (1 + N')\ ;\ldots;\\ P, Q : [P', Q' + 1]\ ;\ldots;\end{matrix}$$

$$\begin{matrix}(1, N^{(r)}) \\ [P^{(r)}, Q^{(r)}+1]\end{matrix}\Bigg[Z_1 (\tan x)^{\sigma_1}, \ldots, Z_r (\tan x)^{\sigma_r}$$

$$\Bigg| \begin{matrix}\{(e_P ; E'_{P'}, \ldots, E^{(r)}_{P^{(r)}})\} : \\ \{(f_Q ; F'_{Q'}, \ldots, F^{(r)}_{Q^{(r)}})\} :\end{matrix}$$

$$\begin{matrix}\{(C'_{P'}, \gamma'_{P'})\} ; \ldots ; \{(C^{(r)}_{P^{(r)}}, \gamma^{(r)}_{P^{(r)}})\} \\ \{(D'_0, \delta'_0), \{(D'_{Q'}, \delta'_{Q'})\} ; \ldots ; (D_0^{(r)}, \delta_0^{(r)}) \{(D^{(r)}_{Q^{(r)}}, \delta^{(r)}_{Q^{(r)}})\}\end{matrix}\Bigg]$$

$$H^{\begin{matrix}0, 0 : (V', W') ; \ldots ; (V^{(r)}, W^{(r)}) \\ p. q : [X', Y'] ; \ldots ; [X^{(r)}, Y^{(r)}]\end{matrix}}\Bigg[x_1 (\tan x)^{h_1}, x_r (\tan x)^{h_r} \Bigg|$$

$$\begin{matrix}\{(a_p ; \alpha'_p, \ldots, \alpha_p^{(r)})\} : \{(A'_{X'}, \eta'_{X'})\} ; \ldots ; \{(A^{(r)}_{X^{(r)}}, \eta^{(r)}_{X^{(r)}})\} \\ (b_q ; \beta'_q, \ldots, \beta_q^{(r)}) : \{(\beta'_{Y'}, \xi'_{Y'})\} ; \ldots ; \{(B^{(r)}_{Y^{(r)}}, \xi^{(r)}_{Y^{(r)}})\}\end{matrix}\Bigg] dx.$$

$$= \frac{e^{\omega u \pi/2}}{\Gamma(u+v)\, \delta_0' \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \phi(\rho_{\nu_1}, \ldots, \rho_{\nu_r})$$

$$e^{\omega(\sum_{i=1}^{r} \sigma_i \rho_{\nu_i})\pi/2} \prod_{i=1}^{r} \{\theta_i (\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i !} Z_i^{\rho_{\nu_i}}\}$$

$$H^{\begin{matrix}1, 1 : (V', W') ; \ldots ; (V^{(r)}, W^{(r)}) \\ p+1, q+1 : [X', Y'] ; \ldots ; [X^{(r)}, Y^{(r)}]\end{matrix}}\Bigg[x_1 e^{\omega h_1 \pi/2}, \ldots, x_r e^{\omega h_r \pi/2} \Bigg|$$

$$(1 - u - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} ; h_1 ; \ldots ; h_r), \{(a_p ; \alpha'_p, \ldots, \alpha^{(r)}_p)\} : \{(A'_{X'}, \eta'_{X'})\} ;$$

$$\{(b_q ; \beta'_q, \ldots, \beta^{(r)}_q)\}, (v - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} ; h_1, \ldots, h_r : \{(B'_{Y'}, \xi'_{Y'})\} ;$$

$$\left.\begin{matrix} \ldots; \{(A^{(r)}{}_{X^{(r)}}, \eta^{(r)}{}_{X^{(r)}})\} \\ \ldots; \{(B^{(r)}{}_{Y^{(r)}}, \xi^{(r)}{}_{Y^{(r)}})\} \end{matrix}\right], \qquad \omega = \sqrt{-1}, \tag{2.5}$$

provided that $h_1, h_2, \ldots, h_r > 0$, $R_e\,(u + \sum_{i=1}^{r} (\sigma_i\, \rho_{\nu_i} + h_i \alpha_i)) > 0$,

$R_e\,(v - \sum_{i=1}^{r} (\sigma_i\, \rho_{\nu_i} + h_i\, \beta_i)) > 0$, $|\arg x_i| < \frac{1}{2}\, U_i \pi$, $U_i > 0$,

$|\arg Z_i| < \frac{1}{2}\, V_i \pi\, V_i > 0$ $(i = 1, \ldots, r)$, where α_i, U_i and V_i are given by the Equation (2.2), (2.3) and (2.4) respectively and

$$\beta_i = \max R_e\,((A_j^{(i)} - 1)/\eta_j^{(i)}),\ j = 1, \ldots, W^{(i)},\ i = 1, \ldots, r. \tag{2.6}$$

$$\int_0^\infty x^\rho e^{-x}\ L_k^{(\sigma)}(x)\, H^{\,0,\,0\,:\,(1,\,N')\,;\ldots;\,(1,\,N^{(r)})}_{\,P,\,Q\,:\,[P',\,Q'+1]\,;\ldots;\,[P^{(r)},\,Q^{(r)}+1]}$$

$$\left[Z_1\,(x)^{\sigma_1}, \ldots, Z_r\,(x)^{\sigma_r} \left| \begin{matrix} \{(e_P\,;\, E'_P, \ldots, E^{(r)}{}_P)\} : \{(C'_{P'}, \gamma'_P)\}; \ldots; \\ \{(f_Q;\, F'_Q, \ldots, F_Q^{(r)})\} : (D'_O, \delta'_O), \{(D'_Q, \delta'_Q)\}; \ldots; \end{matrix} \right.\right.$$

$$\left.\left. \begin{matrix} \{(C^{(r)}{}_{P^{(r)}}, \gamma^{(r)}{}_{P^{(r)}})\} \\ (D_O{}^{(r)}, \delta_O{}^{(r)}), \{(D^{(r)}{}_{Q^{(r)}}, \delta^{(r)}{}_{Q^{(r)}})\} \end{matrix} \right]\right.$$

$$H^{\,0,\,0\,:\,(V',\,W')\,;\ldots;\,(V^{(r)},\,W^{(r)})}_{\,p,\,q:\,[X',\,Y']\,;\ldots;\,[X^{(r)},\,Y^{(r)}]} \left[C_1\,(x)^{\mu_1}\,;\ldots;\, C_r\,(x)^{\mu_r} \right|$$

$$\left| \begin{matrix} \{(a_p;\, \alpha_p', \ldots, \alpha_p^{(r)})\} : \{(A'_{X'}, \eta'_{X'})\}; \ldots; \{(A^{(r)}{}_{X^{(r)}}, \eta^{(r)}{}_{X^{(r)}})\} \\ \{(b_q;\, \beta_q', \ldots, \beta_q^{(r)})\} : \{(B'_{Y'}, \xi'_{Y'})\}\,; \ldots; \{(B^{(r)}{}_{Y^{(r)}}, \xi^{(r)}{}_{Y^{(r)}})\} \end{matrix} \right] dx.$$

$$= \frac{1}{k!\delta_0' \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots \nu_r = 0}^{\infty} \phi(\rho_{\nu_1}, \ldots, \rho_{\nu_r})$$

$$\prod_{i=1}^{r} \{\theta_i(\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i!} Z_i^{\rho_{\nu_i}}\}$$

$$H^{1,1:(V',W');\ldots;(V^{(r)},W^{(r)})}_{p+2,q+1:[X',Y'];\ldots;[X^{(r)},Y^{(r)}]} \left[C_1, \ldots, C_r \middle| \right.$$

$$(\sigma - \rho - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i}; \mu_1; \ldots; \mu_r), \{(a_p; \alpha'_p, \ldots, \alpha^{(r)}_p)\},$$

$$(-\sigma - \rho - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} + k; \mu_1, \ldots, \mu_r),$$

$$(-\rho - \sum_{i=1}^{r} \sigma_i \rho_{\nu_i} : \mu_1, \ldots, \mu_r):$$

$$\{(b_q; \beta'_q, \ldots, \beta^{(r)}_q)\} : \{(B'_{Y'}, \xi'_Y)\} :$$

$$\left. \begin{array}{r} \{(A'_{X'}, \eta'_{X'})\}; \ldots; \{(A^{(r)}_{X^{(r)}}, \eta^{(r)}_{X^{(r)}})\} \\ ; \ldots; \{(B^{(r)}_{Y^{(r)}}, \xi^{(r)}_{Y^{(r)}})\} \end{array} \right], \tag{2.7}$$

provided that $Re(\rho + 1 + \sum_{i=1}^{r} (\sigma_i \rho_{\nu_i} + \mu_i \alpha_i)) > 0$,

$|\, are\ C_i \,| < \frac{1}{2} U_i \pi$, $U_i > 0$, $|\arg Z_i| < \frac{1}{2} V_i \pi$, $V_i > 0$ $(i = 1, \ldots, r)$,

where α_i, U_i and V_i are given by the Equations (2.2), (2.3) and (2.4) respectively.

Proof. To prove (2.1), we write the series expansion of $H^{(1)}[Z_1(\sin x\, e^{\omega x})^{\sigma_1}, \ldots, Z_r(\sin x\, e^{wx})^{\sigma_r}$ with the help of the result

given by Saxena [8] and Mukherjee and prasad [6,p.6], as follows :

$$H^{(1)}\,[Z_1\,(\sin x\, e^{\omega x}\,)^{\sigma_1},\ldots,\, Z_r\,(\sin x\; e^{\omega x}\;)^{\sigma_r}]$$

$$= \frac{1}{\delta_0{}'\ldots\,\delta_0{}^{(r)}} \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \phi(\rho_{\nu_1},\ldots,\,\rho_{\nu_r})$$

$$\prod_{i=1}^{r} \{\theta_i(\rho_{\nu_i})\frac{(-1)^{\nu_i}}{\nu_i\,!} Z_i^{\rho_{\nu_i}}\;(\sin x\; e^{\omega x}\;)^{\sigma_i\rho_{\nu_i}}\},\; \rho_{\nu_i} = \frac{D_0{}^{(i)} + \nu_i}{\delta_0{}^{(i)}},$$

$$\omega = \sqrt{-1}, \tag{2.8}$$

where

$$\theta_i\,(s_i) = \frac{\prod_{j=1}^{N^{(i)}} \Gamma(1 - C_j{}^{(i)} + \gamma_j{}^{(i)}\, s_i)}{\prod_{j=1}^{Q^{(i)}} \Gamma(1 - D_j{}^{(i)} + \delta_j{}^{(i)} s_i) \prod_{j=N^{(i)}+1}^{P^{(i)}} \Gamma(C_j{}^{(i)} - \gamma_j{}^{(i)} s_i)}$$

$$i = 1\,,\ldots,\, r, \tag{2.9}$$

and

$$\phi(s_1\,,\ldots,\; s_r) = \left[\prod_{j=1}^{P} \Gamma(e_j - \sum_{i=1}^{r} E_j{}^{(i)}\; s_i) \prod_{j=1}^{Q} \Gamma\,(1 - f_j + \sum_{i=1}^{r} F_j{}^{(i)}\; s_i)\right]^{-1}, \tag{2.10}$$

where $(\sigma_i) > 0$, $|\arg Z_i| < 1/2\, V_i\, \pi$, $V_i > 0$ $(i = 1\,,\ldots,\, r)$,

where V_i is given by Equation (2.4); take the contour integral form for

$$H^{(2)}[x_1(\sin x\, e^{\omega x})^{h_1},\ldots,\, x_r\,(\sin x\, e^{\omega x})^{h_r}\,] = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(s_1)$$

$$\ldots \phi_r(s_r)\,\Psi(s_1,\ldots,s_r)\, x_1^{s_1}(\sin x\; e^{\omega x})^{h_1 s_1} \ldots x_r^{s_r}(\sin x\; e^{\omega x})^{h_r s_r}$$

$$ds_1 \ldots ds_r\,,\ \omega = \sqrt{-1}, \tag{2.11}$$

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{V^{(i)}} \Gamma(B_j^{(i)} - \xi_j^{(i)} s_i) \prod_{j=1}^{W^{(i)}} \Gamma(1 - A_j^{(i)} + \eta_j^{(i)} s_i)}{\prod_{j=V^{(i)}+1}^{Y^{(i)}} \Gamma(1 - B_j^{(i)} + \xi_j^{(i)} s_i) \prod_{j=W^{(i)}+1}^{X^{(i)}} \Gamma(A_j^{(i)} - \eta_j^{(i)} s_i)}$$

$$,\ i = 1, \ldots, r, \tag{2.12}$$

$$\Psi(s_1, \ldots, s_r) = \left[\prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i) \right.$$

$$\left. \prod_{j=1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i) \right]^{-1}, \tag{2.13}$$

in the integrand of left hand side of (2.1) and change the order of integration, which is justifiable under the given conditions. The left-hand side of (2.1) reduces to

$$\frac{1}{\delta_0' \ldots \delta_0^{(r)}} \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} \phi(\rho_{\nu_1}, \ldots, \rho_{\nu_r}) \prod_{i=1}^{r} \{\theta_i(\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i\,!} Z_i^{\rho_{\nu_i}}\}$$

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r)\, \Psi(s_1, \ldots, s_r)\, x_1^{s_1} \ldots x_r^{s_r}$$

$$\left\{ \int_0^{\pi/2} (\sin x)^{u + \sum_{i=1}^{r} (\sigma_i \rho_{\nu_i} + h_i s_i) - 1} (\cos x)^{v-1} \right.$$

$$e^{\omega(u+v+\sum_{i=1}^{r}(\sigma_i \rho_{\nu_i}+h_i s_i))} dx \Big\} ds_1 \dots ds_r. \tag{2.14}$$

Now, evaluating the inner integral in (2.14) with the help of the result [4, p. 80, Eq. (7)]

$$\int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{\omega(u+v)x} dx = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} e^{\omega u \pi/2},$$

$$Re\,(u) > 0,\ Re\,(v) > 0,\ \omega = \sqrt{-1}\ , \tag{2.15}$$

we arrive at the right hand side of (2.1) stated as above.

The proof of (2.5) is similar to the proof of (2.1).

To prove (2.7), we write the series expansion of $H^{(1)}[Z_1(x)^{\sigma_1},\dots,Z_r(x)^{\sigma_r}]$ from the Equation (2.8) and contour integral form of $H^{(2)}[C_1(x)^{\mu_1}, \dots, C_r(x)^{\mu_r}]$ from the Equation (2.11) in the left-hand side of (2.7) and change the order of integration, which is justifiable under given conditions. The left-hand side of (2.7) reduces to

$$\frac{1}{\delta_0'\dots\delta_0^{(r)}} \sum_{\nu_1,\dots,\nu_r=0}^{\infty} \phi(\rho_{\nu_1},\dots,\rho_{\nu_r}) \prod_{i=1}^{r} \{\theta_i(\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i\,!} Z_i^{\rho_{\nu_i}}\}$$

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi_1(s_1) \dots \phi_r(s_r)\, \Psi(s_1,\dots,s_r)\, C_1^{s_1} \dots C_r^{s_r}$$

$$\Big\{ \int_0^{\infty} x^{\rho+\sum_{i=1}^{r}(\sigma_i \rho_{\nu_i}+\mu_i s_i)} e^{-x} L^{(\sigma)}{}_k(x)\, dx \Big\} ds_1,\dots, ds_r. \tag{2.16}$$

Now, on evaluating the inner integral of (2.16) with the help of the known result [3, p. 292, Eq. (1)]

$$\int_0^\infty x^{\beta-1} e^{-x} L_k^{(\alpha)}(x)\, dx = \frac{\Gamma(\alpha-\beta+k+1)\,\Gamma(\beta)}{k!\,\Gamma(\alpha-\beta+1)}, \; Re(\beta) > 0,$$

we arrive at the right-hand side of (2.7).

3. INFINITE SERIES

In this section, we establish an infinite series for the product of two multivariable H-functions. The series is

$$\sum_{t=0}^{\infty} \frac{x^{2a_1-2-t}\, W^t}{t!}\, H^{0,0:(1,N');\ldots;(1,N^{(r)})}_{P,Q:[P',Q'+1];\ldots;[P^{(r)},Q^{(r)}+1]}$$

$$\left[Z_1(x)^{2E_1'},\ldots,Z_r(x)^{2E_1^{(r)}} \left| \begin{matrix} \{(e_P;E'_P,\ldots,E^{(r)}_P)\}:\{(C'_{P'},\gamma'_P)\};\ldots; \\ \{(f_Q;F'_Q,\ldots,F_Q^{(r)})\}:(D'_O,\delta'_O)\{(D'_{Q'},\delta'_{Q'})\};\ldots \end{matrix}\right.\right.;$$

$$\left.\left. \begin{matrix} \{(C^{(r)}_{P^{(r)}},\gamma^{(r)}_{P^{(r)}})\} \\ (D_O^{(r)},\delta_O^{(r)}),\{(D^{(r)}_{Q^{(r)}},\delta^{(r)}_{Q^{(r)}})\} \end{matrix}\right]\right.$$

$$H^{0,n:(V',W');\ldots;(V^{(r)},W^{(r)})}_{p,q:[X',Y'];\ldots;[X^{(r)},Y^{(r)}]} \left[C_1(x)^{-2a_1'};\ldots;C_r(x)^{-2a_1^{(r)}}\right|$$

$$(a_1-t;\alpha_1',\ldots,\alpha_1^{(r)})(a_i;\alpha_i',\ldots,\alpha_i^{(r)})_{2:p}:$$

$$\{(b_q;\beta_q',\ldots,\beta_q^{(r)})\}:$$

$$\left.\left. \begin{matrix} \{(A'_{X'},\eta'_{X'})\};\ldots;\{(A^{(r)}_{X^{(r)}},\eta^{(r)}_{X^{(r)}})\} \\ \{(B'_{Y'},\xi'_{Y'})\};\ldots;\{(B^{(r)}_{Y^{(r)}},\xi^{(r)}_{Y^{(r)}})\} \end{matrix}\right]\right.$$

$$= (x^2 - Wx)^{a_1-1} \frac{1}{\delta_0' \cdots \delta_0^{(r)}} \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \phi(\rho_{\nu_1},\ldots,\rho_{\nu_r})$$

$$\prod_{i=1}^{r} \{\theta_i(\rho_{\nu_i}) \frac{(-1)^{\nu_i}}{\nu_i!} Z_i^{\rho_{\nu_i}} (x)^{2E_1^{(i)}}\}$$

$$H^{0,n:(V',W');\ldots;(V^{(r)},W^{(r)})}_{p\ q:[X',Y'];\ldots;[X^{(r)},Y^{(r)}]}\left[C_1(x^2-Wx)^{-\alpha_1'},\right.$$

$$\ldots, C_r(x^2-Wx)^{-\alpha_1^{(r)}} \left| \begin{matrix} \{(a_p;\alpha'_p,\ldots,\alpha^{(r)}_p)\}:\{(A'_{X'},\eta'_{X'})\}, \\ \{(b_q;\beta'_q,\ldots,\beta^{(r)}_q)\}:\{(B'_{Y'},\xi'_{Y})\}: \end{matrix}\right.$$

$$\left.\begin{matrix} \ldots;\{(A^{(r)}_{X^{(r)}},\eta^{(r)}_{X^{(r)}})\} \\ ;\ldots;\{(B^{(r)}_{Y^{(r)}},\xi^{(r)}_{Y^{(r)}})\} \end{matrix}\right], \tag{3.1}$$

provided that $|w/x| < 1$, $|\arg C_i| < \frac{1}{2} U_i \pi$, $U_i > 0$, $|\arg Z_i| < \frac{1}{2} V_i\pi$, $V_i > 0$, where U_i and V_i are given by the Equations (2.3) and (2.4) respectively, and $(a_i;\alpha_i',\ldots,\alpha_i^{(r)})_{2,p}$ stands for the set of parameters $(a_2;\alpha_2',\ldots,\alpha_2^{(r)}),\ldots,(a_p;\alpha_p';,\ldots,\alpha_p^{(r)})$.

Proof : To prove the result (3.1), we write the series, expansion of $H^{(1)}[Z_1(x)^{2E'_1},\ldots,Z_r(x)^{2E_1^{(r)}}]$ with the help of equation (2.8) and write $H^{(2)}[C_1(x)^{-2\alpha_1'},\ldots,C_r(x)^{-2\alpha_1^{(r)}}]$ into its contour form, change the order of integration and summation and use the formula ${}_1F_0(a;-;x) = (1-x)^{-a}$, and finally interpret the result in the light of multivariable H-function.

PARTICULAR CASES

(1) On taking $r = 1, P = Q = 0 = P' = Q' = D_0'$, $\delta_0' = 1 = E_1'$

$Z_1 \to 0$ in $H^{(1)}[Z_1 (x)^{2E_1'}, \ldots, Z_r (x)^{2E_1^{(r)}}]$ in the Equation (3.1). we arrive at the result recently studied by Maurya [5, p. 255].

(2) On taking $r = 2$ in the particular case (1), we arrive at the result given by Chaurasia [2].

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FINITE INTEGRALS INVOLVING THE PRODUCTS OE THE H-FUNCTIONS OF TWO AND MORE VARIABLES

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ABSTRACT

In this paper we evaluate two finite integrals associated with the product of the H-function of two variables and Srivastava and Panda's multivariable H-function. These integrals are quite general in character and a number of interesting (known or new) integrals can be deduced as particular cases. Some special cases are also discussed briefly.

1. INTRODUCTION

The H-function of two variables occurring in this paper is due to Mittal and Gupta [6]. The details of this function can be found in a recent book by Srivastava, Gupta and Goyal [11].

The following expansion formula ([4], p. 17, Eq. (2.1)) for a special H-function of two variables will be required in the sequel :

$$(1.1)\quad H^*[ax, bx] \equiv H^{\,0,\, n_1' : 1,\, n_2' : 1,\, n_3'}_{\,p_1',\, q_1' : p_2',\, q_2' + 1 ;\, p_3',\, q_3' + 1} \left[\begin{matrix} ax \\ bx \end{matrix} \middle| \begin{matrix} (u_j ; \mu_j, U_j)_{1,p_1'} : (g_j, G_j)_{1,p_2'} ; (k_j, K_j)_{1,p_3'} \\ (v_j ; \eta_j, V_j)_{1,q_1'} : (0, 1), (h_j, H_j)_{1,q_2'} ; (0,1), (l_j, L_j)_{1,q_3'} \end{matrix}\right]$$

$$= \sum_{M=0}^{\infty} h_M \frac{(-ax)^M}{M!}$$

where

$$(1.2) \quad h_M = \sum_{N=0}^{M} \phi'(M-N, N)\, \theta_1'(M-N)\, \theta_2'(N)\, (b/a)^N \binom{M}{N}$$

$$(1.3) \quad \phi'(\xi, \lambda) = \sum_{j=1}^{n_1'} \Gamma(1 - u_j + \mu_j \xi + U_j \lambda)$$

$$\left[\prod_{j=n_1'+1}^{p_1'} \Gamma(u_j - \mu_j\xi - U_j\lambda) \prod_{j=1}^{q_1'} \Gamma(1 - v_j + \eta_j\xi + V_j\lambda) \right]^{-1}$$

$$(1.4) \quad \theta_1'(\xi) = \prod_{j=1}^{n_2'} \Gamma(1 - g_j + G_j\xi)$$

$$\left[\prod_{j=n_2'+1}^{p_2'} \Gamma(g_j - G_j\xi) \prod_{j=1}^{q_2'} \Gamma(1 - h_j + H_j\xi) \right]^{-1}$$

and with $\theta_2'(\lambda)$ defined analogously to $\theta_1(\xi)$ in terms of the parameter sets $(k_j, K_j)_{1,p_3'}$, $(l_j, L_j)_{1,q_3'}$.

The H-function of several complex variables (or the multivariable H-function) was introduced and studied systematically in a series of papers by Srivastava and Panda (see, e. g., [12] and [13]; see also [11]). We shall employ the following contracted notation of Srivastava and Panda ([13], p. 130, Eq. (1.3)) :

$$(1.5) \quad H[z_1, \ldots, z_r] \equiv H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right]$$

$$\left.\begin{matrix}(a_j\,;\,\alpha_j',\ldots,\alpha_j^{(r)})_{1,p}:(c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\(b_j\,;\,\beta_j',\ldots,\beta_j^{(r)})_{1,q}:(d_j',\delta_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

to denote the H-function of r complex variables $z_1,\ldots,z_r$. [See Srivastava, Gupta and Goyal ([11], p. 251, Eq. (4.1)) for details of this function.]

With a view to facilitating the derivation of our main integral (2.1) in the next section, we give here an elementary integral contained in the following

Lemma. *If*

$$\text{(i)}\quad \min_{1\leqslant j\leqslant r}\left\{\sigma_j,\ \rho_j,\ Re\,(\alpha),\ Re\,(\beta)\right\}>0,$$

$$Re\,(\alpha)+\sum_{j=1}^{r}\sigma_j\,\xi_j>0,\ Re\,(\beta)+\sum_{j=1}^{r}\rho_j\,\xi_j>0,$$

where

$$(1.6)\quad \xi_j=\min_{1\leqslant j\leqslant m_k}\left[Re\,(d_j^{(k)}/\delta_j^{(k)})\right]\qquad \forall\ k\in\{1,\ldots,r\},$$

and

$$\text{(ii)}\quad U_k>0,\ |\arg z_k|>\tfrac{1}{2}\,U_k\,\pi,$$

where

$$(1.7)\quad U_k=-\sum_{j=n+1}^{p}\alpha_j^{(k)}-\sum_{j=1}^{q}\beta_j^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}$$

$$-\sum_{j=n_k+1}^{p_k}\gamma_j^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}\delta_j^{(k)}$$

$$\forall\ k\in\{1,\ldots,r\},$$

then

$$(1.8)\quad \int_0^{\pi/2} \exp\{i(\alpha+\beta)\theta\} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta$$

$$H\ [z_1 \exp\{i(\sigma_1+\rho_1)\theta\} \sin^{\sigma_1} \cos^{\rho_1}\theta, \ldots,$$

$$z_r \exp\{i(\sigma_r+\rho_r)\theta\} \sin^{\sigma_r}\theta \cos^{\rho_r}\theta\ ]\, d\theta$$

$$= \exp(i\pi\alpha/2)\ H^{(1)}_{\alpha,\beta}[z_1 \exp(i\pi\sigma_1/2), \ldots, z_r \exp(i\pi\sigma_r/2)]$$

where

$$(1.9)\quad H^{(1)}_{\alpha,\beta}[z_1, \ldots, z_r] \equiv H^{\,0,\, n+2 : m_1, n_1 ; \ldots ; m_r, n_r}_{\,p+2,\, q+1 : p_1, q_1 ; \ldots ; p_r, q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-\alpha\ ;\ \sigma_1, \ldots, \sigma_r), (1-\beta\ ;\ \rho_1, \ldots, \rho_r), \\ (1-\alpha-\beta\ ;\ \sigma_1+\rho_1, \ldots, \sigma_r+\rho_r), \end{matrix}\right.$$

$$\left.\begin{matrix} (a_j\ ;\ \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (c_j', \gamma_j')_{1,p_1}, \ldots, (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j\ ;\ \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (d_j', \delta_j')_{1,q_1} ; \ldots ; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

Proof. To establish (1.8), we first write the Mellin-Barnes contour integral ([11], p. 251, Eq. (4.1)) and change the order of integrations therein. We then apply MacRobert's result ([5], p. 450, Eq (4)):

$$(1.10)\quad \int_0^{\pi/2} \exp\{i(\alpha+\beta)\theta\} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta\, d\theta$$

$$= \exp(i\pi\alpha/2)\, B(\alpha, \beta),$$

$$\{Re(\alpha) > 0,\ Re(\beta) > 0\},$$

in order to evaluate the θ-integral, and interpret the resulting contour

integral as the multivariable H-function; the integral (1.8) follows at once.

2. THE MAIN INTEGRALS

We shall establish the following two general and finite integrals :

$$(2.1)\quad \int_0^{\pi/2} \exp\{ i(\alpha + \beta)\theta\} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta$$

$$H^* [a \exp\{ i(\sigma + \rho)\theta\} \sin^{\sigma}\theta \cos^{\rho}\theta ,$$

$$b \exp\{ i(\sigma + \rho)\theta\} \sin^{\sigma}\theta \cos^{\rho}\theta]$$

$$H [z_1 \exp\{ i(\sigma_1 + \rho_1)\theta\} \sin^{\sigma_1}\theta \cos^{\rho_1}\theta ,\ldots,$$

$$z_r \exp\{ i(\sigma_r + \rho_r)\theta\} \sin^{\sigma_r}\theta \cos^{\rho_r}\theta]\, d\theta$$

$$= \sum_{M=0}^{\infty} h_M \frac{(-a)^M}{M!} \exp\{i\pi(\alpha + \sigma M)/2\}$$

$$H^{(1)}_{\alpha+\sigma M,\, \beta+\rho M} [z_1 \exp(i\pi\sigma_1/2) ,\ldots, z_r \exp(i\pi\sigma_r/2)]$$

The integral (2.1) is valid if the following (sufficient) conditions are satisfied :

(i) The sets of conditions (i) and (ii) mentioned with the Lemma hold true

(ii) $\rho > 0$, $\sigma > 0$, $U \geqq 0$, and $V \geqq 0$ where

$$(2.2)\quad U = 1 + \sum_{j=1}^{q_1'} \eta_j + \sum_{j=1}^{q_2'} H_j - \sum_{j=1}^{p_1'} \mu_j - \sum_{j=1}^{p_2'} G_j$$

$$(2.3)\quad V = 1 + \sum_{j=1}^{q_1'} V_j + \sum_{j=1}^{q_3'} L_j - \sum_{j=1}^{p_1'} U_j - \sum_{j=1}^{p_3'} K_j$$

where each of the equalities holds when the variables are suitably constrained.

(iii) The series occurring on the rlght-hand side of (2.1) is absolutely convergent.

$$(2.4)\quad \int_0^t x^{\alpha-1}(t-x)^{\beta-1}(1+Bx)^{-\alpha-\beta}$$

$$H^*\left[\frac{a\,x^{\sigma}(t-x)^{\rho}}{(1+Bx)^{\sigma+\rho}},\ \frac{b\,x^{\sigma}(t-x)^{\rho}}{(1+Bx)^{\sigma+\rho}}\right]$$

$$H\left[z_1\left(\frac{x}{1+Bx}\right)^{\sigma_1}\left(\frac{t-x}{1+Bx}\right)^{\rho_1},\ldots,z_r\left(\frac{x}{1+Bx}\right)^{\sigma_r}\left(\frac{t-x}{1+Bx}\right)^{\rho_r}\right]dx$$

$$=\frac{t^{\alpha+\beta-1}}{(1+Bt)^{\alpha}}\sum_{M=0}^{\infty} h_M\,\frac{(-a)^M}{M!}\left(\frac{t^{\sigma+\rho}}{(1+Bt)^{\sigma}}\right)^M$$

$$H^{(1}_{\alpha+\sigma M,\,\beta+\rho M}\left[z_1\frac{t^{\sigma_1+\rho_1}}{(1+Bt)^{\sigma_1}},\ldots,z_r\frac{t^{\sigma_r+\rho_r}}{(1+Bt)^{\sigma_r}}\right]$$

The integral formula (2.4) is valid under the following conditions :

(i) $Bt > -1$

(ii) conditions (i) and (ii) given with the Lemma hold true

(iii) The series occurring on the right-hand side of (2.4) is absolutely convergent.

In(2. 1) and (2.4), H^*-function, $H^{(1)}_{\alpha,\beta}$-function and h_M are defined by (1.1), (1.9) and (1.2), respectively.

Derivation of the integrals. To establish the integral (2.1), we use the series expansion (1.1) for H^*-function occurring in the integrand, change the order of integration and summation (which is justified under the conditions given with (2.1) and evaluate the resulting θ-integral with the help of (1.8), we easily arrive at the right hand side of (2.1).

The proof of the integral formula (2.4) is similar to that of the integral (2.1) with the only difference that here we use a known integral due to Agal and Koul ([1], p. 14, Eq (2.1)) instead of (1.8).

3 Special Cases The multivariable H-function is an extension of the G-function of several complex variables, and includes Fox's H-function of one and two variables, Meijer's G-function of one and two variables, the generalized Lauricella function of Srivastava and Daoust [10], Appell functions, the Whittaker functions, etc. (cf.[11]). Therefore, our integrals can be suitably applied to evaluate the corresponding integrals involving these functions simply by specializing the parameters of the multivariable H-functions.

On the other hand, on specialization of the parameters of the $H^*[ax, bx]$ suitably, several integrals involving the various products of elementary functions of one or two variables and the multivariable H-function can be easily obtained. Thus, for example, if we put $n_1' = p_1'$, $n_2' = p_2'$, $n_3' = p_3'$ in (2 1) and use the known result ([11], p. 88, Eq. (6. 4. 2)), we get

$$(3.1)\quad \int_0^{\pi/2} \exp\{ i(\alpha+\beta)\theta\} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta,$$

$$S[a \exp\{i(\sigma+\rho)\theta\} \sin^{\sigma}\theta \cos^{\rho}\theta, b \exp\{i(\sigma+\rho)\theta\} \sin^{\sigma}\theta \cos^{\rho}\theta]$$

$$H[z_1 \exp\{i(\sigma_1+\rho_1)\theta\} \sin^{\sigma_1}\theta \cos^{\rho_1}\theta, \ldots,$$

$$z_r \exp\{i(\sigma_r+\rho_r)\theta\} \sin^{\sigma_r}\theta \cos^{\rho_r}\theta] d\theta$$

$$= \sum_{M=0}^{\infty} h'_M \frac{a^M}{M!} \exp\{i\pi(\alpha+\sigma M)/2\}$$

$$H^{(1)}_{\alpha+\sigma M,\, \beta+\rho M} [z_1 \exp(i\pi\sigma_1/2), \ldots, z_r \exp(i\pi\sigma_r/2)]$$

where

$$(3.2)\quad S[x, y] \equiv S^{\,p_1' : p_2' ; p_3'}_{\,q_1' : q_2' ; q_3'} \left[\begin{matrix} (u_j ; \mu_j, U_j)_{1, p_1'} : \\ (v_j ; \eta_j, V_j)_{1, q_1'} : \end{matrix} \right.$$

$$\left. \begin{matrix} (g_j, G_j)_{1, p_2'} \; (k_j, K_j)_{1, p_3'} ; \\ (h_j, H_j)_{1, q_2'} \; (l_j, L_j)_{1, q_3'} ; \end{matrix} \; x\, y \right]$$

stands for the (Srivastava and Daoust) generalized Kampé de Fériet function ([9], p. 199, Eq. (2.1)),

$$(3.3)\quad h_M' = \sum_{N=0}^{M} \phi''(M-N, N)\, \theta_1''(M-N)\, \theta_2''(N)\, (b/a)^N \binom{M}{N},$$

$$\phi''(\xi, \lambda) = | \phi'(\xi, \lambda) |_{n_1' = p_1'},$$

$$\theta_1''(\xi) = | \theta_1'(\xi) |_{n_2' = p_2'}$$

and

$$\theta_2''(\lambda) = |\,\theta_2'(\lambda)\,|_{n_3' = p_3'}\,.$$

The condition of validity for the integral are the same as mentioned with the main integral (2.1).

Again, setting $n_1' = p_1' = q_1' = 0$ in (2.1) and using the known result ([10], p. 90, Eq. (6. 4. 15)), we arrive at the following interesting integral :

$$(3.4)\quad \int_0^{\pi/2} \exp\{\,i(\alpha+\beta)\,\theta\}\,\sin^{\alpha-1}\theta\,\cos^{\beta-1}\theta$$

$$H^{1,\,n_2'}_{p_2',\,q_2'+1}\left[a\exp\{i(\sigma+\rho)\theta\}\sin^{\sigma}\cos^{\rho}\theta\,\middle|\,\begin{matrix}(g_j,\,G_j)_{1,p_2'}\\(0,1),(h_j,H_j)_{1,q_2'}\end{matrix}\right]$$

$$H^{1,\,n_3'}_{p_3',\,q_3'+1}\left[b\exp\{i(\sigma+\rho)\theta\}\sin^{\sigma}\cos^{\rho}\theta\,\middle|\,\begin{matrix}(k_j,\,K_j)_{1,p_3'}\\(0,1),(l_j,L_j)_{1,q_3'}\end{matrix}\right]$$

$$H\,[z_1\exp\{\,i(\sigma_1+\rho_1)\,\theta\}\,\sin^{\sigma_1}\theta\,\cos^{\rho_1}\theta\,,\ldots,$$

$$z_r\exp\{\,i(\sigma_r+\rho_r)\,\theta\}\,\sin^{\sigma_r}\theta\,\cos^{\rho_r}\theta]\,d\,\theta$$

$$=\sum_{M=0}^{\infty} h_M''\,\frac{(-a)^M}{M\,!}\exp\{i\,\pi\,(\alpha+\sigma M)/2\}$$

$$H^{(1)}_{\alpha+\sigma M,\,\beta+\rho M}\,[z_1\exp(i\,\pi\,\sigma_1/2)\,,\ldots,\,z_r\exp(i\,\pi\,\sigma_r/2)\,]$$

where

$$(3.5)\quad h_M'' = \sum_{N=0}^{M}\theta_1'(M-N)\,\theta_2(N)\,(b/a)^N\,(M/N)$$

Similar type of integrals can be obtained from the integral (2.4) also.

Lastly, if we put $n_3' = p_3' = q_3' = 0$ in (3.4) and let $b \to 0$ therein, we arrive at the known result due to Garg ([3], p. 136, Eq. (2.1)). The results established earlier by Rathie ([8], p. 237, Eq. (2.3)), Vasishat and Goyal [14], p. 13, Eq. (1.1)), and others, can also be deduced as special cases of (2.1). Also, the known integrals due to Garg ([2], p. 23, Eq. (2 1)), Agal and Koul ([1], p. 14, Eq. (2.1)), Prasad and Singh ([7], p. 126, Eq. (2.1), and Srivastava and Panda ([13], p. 131, Eq. (2.2)), follow easily as special cases of our second integral (2 4).

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CERATIN APPLICATIONS OF JACKSON'S SUMMATION FORMULA

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ABSTRACT

In this note we show that the q-Watson and q-Whipple summations due to Andrews are the particular cases of the well known Jackson's formula for summing ${}_8\Phi_7$. We also obtain the identities of Rogers-Ramanujan type related to the modulus 23.

1. INTRODUCTION

If we let

$$|q| < 1, [a; q]_n = (1-a)(1-aq) \ldots\ldots (1-aq^{n-1}), [a; q]_0 = 1, \tag{1.1}$$

$$[a; q]_\infty = \prod_{j=0}^{\infty} (1-aq^j) \tag{1.2}$$

then the basic hypergeometric series may be defined by

$${}_{p+1}\Phi_{p+r}\left[\begin{matrix} a_1, \ldots, a_{p+1}; q; x \\ b_1, \ldots, b_{p+r} \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[a_1; q]_n \ldots [a_{p+1}; q]_n (-)^{nr} x^n}{[q;q]_n [b_1; q]_n \ldots [b_{p+r}; q]_n} \cdot q^{rn(n-1)/2} \tag{1.3}$$

where the series converges for all positive integral values of r and for

all x, except that, when $r=0$, it converges only for $|x| < 1$. Further, we shall simply write

$$\Pi \begin{bmatrix} a_1, \ldots, a_r; q \\ b_1, \ldots, b_s \end{bmatrix} \quad \text{for} \prod_{j=0}^{\infty} \frac{(a - a_1 q^j) \ldots (1 - a_r q^j)}{(1 - b_1 q^j) \ldots (1 - b_s q^j)}.$$

On the other hand, the well-poised series

$${}_{p+3}\Phi_{p+2} \begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, \ldots, b_p; q; z \\ \sqrt{a}, -\sqrt{a}, aq/b_1, \ldots, aq/b_p \end{bmatrix}$$

will be abbreviated by

$${}_{p+3}W_{p+2}\,[a; b_1, \ldots, b_p; q; z].$$

F. H. Jackson studied the concept of the basic number at length. He gave the basic analogues of most of the special summation theorems. In particular, he proved the basic analogues of Dougall's theorem as early as 1909. But he did not give the q-analogues of Watson's and Whipple's summation formulas. Andrews [1] used Watson's q-analogue of Whipple's theorem [10] and obtained the q-analogues of the terminating versions of Watson's and Whipple's summation theorems. We show here that the q-Watson and q-Whipple formulas of Andrews [1] are special cases of Jackson's theorem [7; 3. 3. 1. 1] for ${}_8\Phi_7$. We also obtain a natural extension of a formula of Verma and Jain [8; 1. 3] in the form.

$${}_{12}W_{11}\,[a; a^2 q/b^2, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd,$$
$$q^{-m}, -q^{-m}, q; q]$$

$$= \frac{[a^2 q^2; q^2]_m\,[b^2 q^2/c^2; q^2]_m\,[b^2 q^2/d^2; q^2]_m\,[a^2 q^2/c^2 d^2; q^2]_m}{[b^2 q^2; q^2]_m\,[a^2 q^2/c^2; q^2]_m\,[a^2 q^2/d^2; q^2]_m\,[b^2 q^2/c^2 d^2; q^2]_m}$$

$$\cdot\, {}_{10}W_9\, [b^2;\, -b^2/a,\, -b^2 q/a,\, a^2 q^2/b^2,\, c^2,\, d^2,\, \frac{a^2 b^2 b^{2+2m}}{c^2 d^2},$$

$$q^{-2m};\, q^2;\, \frac{b^2 q}{a^2}\,] \tag{1.4}$$

which is a q-analogue of a known formula [2; 8. 11] (on replacing a by $-a$). In fact, writing the series in reverse order on both the sides of (1.1) one may get a q-analogue of a well-known formula [2; 8. 1] which will be different in nature from a transformation of Bailey [3]. Similarly, we also obtain the q-analogues of the formulas [2; 8. 21, 8. 31, 8. 41] which are different in nature than the known ones [3] and [5; 4. 6]. Lastly, as applications of (1.1), we deduce identities of Rogers-Ramanujan type related to the modulus 23.

2. SPECIALIZATIONS OF JACKSON'S FORMULA

For $c = -d = \sqrt{a}$, Jackson's summation formula [7; 3. 3. 1. 1]

$${}_8W_7\, [a;\, b,\, c,\, d,\, a^2 q^{1+n}/bcd,\, q^{-n},\, q;\, q]$$

$$= \frac{[aq;\, q]_n\, [aq/bc;\, q]_n\, [aq/bd;\, q]_n\, [aq/cd;\, q]_n}{[aq/b;\, q]_n\, [aq/c;\, q]_n\, [aq/d;\, q]_n\, [aq/bcd;\, q]_n} \tag{2.1}$$

reduces to

$${}_4\Phi_3 \begin{bmatrix} a,\, b,\, -aq^{1+n}/b,\, q^{-n};\, q;\, q \\ aq/b,\, -bq^{-n},\, aq^{1+n} \end{bmatrix} = \frac{[-q;\, q]_n\, [aq;\, q]_n\, [aq^2/b^2;\, q^2]_n}{[-q/b;q]_n\, [aq^2;\, q^2]_n\, [aq/b;q]_n} \tag{2.2}$$

(2.2) is a q-analogue of the terminating version of Dixon's theorem [7; 2. 3. 3. 6].

Transforming the ${}_4\Phi_3$ on the left-hand side of (2.2) by the formula [6]

$$
{}_4\Phi_3\left[\begin{matrix} a, b, c, q^{-n}; q; q \\ d, e, f \end{matrix}\right] = \frac{[e/c; q]_n\,[de/ab; q]_n}{[e; q]_n\,[de/abc; q]_n}
$$

$$
{}_4\Phi_3\left[\begin{matrix} d/a, d/b, c, q^{-n}; q; q \\ d, \dfrac{cq^{1-n}}{e}, \dfrac{cq^{1-n}}{f} \end{matrix}\right] \tag{2.3}
$$

where $abcq^{1-n} = def$ (with $d \to aq^{1+n}$, $e \to aq/b$, $f \to bq^{-n}$, $c \to b$, $b \to -aq^{1+n}/b$) we get

$$
{}_4\Phi_3\left[\begin{matrix} q^{-n}, q^{1+n}, b, -b; q; q \\ aq^{1+n}, -q, b^2 q^{-n}/a \end{matrix}\right] = \frac{[aq; q]_n\,[aq^2/b^2; q^2]_n}{[aq^2; q^2]_n\,[aq/b^2; q]_n} \tag{2.4}
$$

(2.4) is q-Whipple's theorem due to Andrews [1].

Next, transforming the ${}_4\Phi_3$ on the left-hand side of (2.2) by (2.3) (with $a \to -aq^{1+n}/b$, $c \to a$, $d \to aq/b$, $e \to aq^{1+n}$, $f \to -bq^{-n}$) we get a known formula [4; 3. 17]. On the other hand transforming the ${}_4\Phi_3$ (with $b \to aq^{1+n}/b$, $c \to b$, $d \to aq/b$, $e \to aq^{1+n}$, $f \to bq^{-n}$) one may obtain another known formula [4; 3. 19].

Now, in the non-terminating version of (2.1) i. e. [7; p. 248(15)], setting $c = -d = \sqrt{a}$, we get a non-terminating version of (2.2) in the form

$$
{}_4\Phi_3\left[\begin{matrix} a, b, -aq/bf, f; q; q \\ aq/b, -bf, aq/f \end{matrix}\right] + \Pi\left[\begin{matrix} a, f, b/a, bq/f, -aq/bf, -b^2f/a; q \\ a/b, aq/f, bf/a, b^2/a, -bf, -q/f \end{matrix}\right] \cdot
$$

$$
\cdot\, {}_4\Phi_3\left[\begin{matrix} b^2/a, b, -q/f, bf/a; q; q \\ bq/a, -b^2f/a, bq/f \end{matrix}\right] =
$$

$$
= \Pi\left[\begin{matrix} b/a, -b, -q; q \\ aq/f, bf/a, -bf, -q/f \end{matrix}\right] \Pi\left[\begin{matrix} b^2f^2/a, aq^2/f^2, aq; q^2 \\ b^2/a \end{matrix}\right] \tag{2.5}
$$

(2.5) for $a = q^{-n}$ reduces to

$$ {}_4\Phi_3 \left[\begin{matrix} q^{-n}, b, -q^{1-n}/bf, f; q; q \\ q^{1-n}/b, -bf, q^{1-n}/f \end{matrix} \right] = \frac{[bf; q]_n (-f)^n q^{n(n-1)/2}}{[b; q]_n [f; q]_n} \cdot $$

$$ \cdot \Pi \left[\begin{matrix} b^2, b^2f^2q^n, q^{2-n}/f^2, q^{1-n}; q^2 \\ q, b^2f^2, q^2/f^2, b^2q^n \end{matrix} \right] \tag{2.6} $$

Transforming the ${}_4\Phi_3$ on the left hand side by (2.3) (with $a \to -q^{1-n}/bf$, $c \to f$, $d \to q^{1-n}/b$, $e \to -bf$, $f \to q^{1-n}/f$) we get q-Watson s summation formula due to Andrews [1].

3. PROOF OF (1. 4)

Multiply to the following form of (2.1)

$$ {}_8W_7 [a; a^2q/b^2, bq^n, -bq^n, -q^{-n}, q^{-n}; q; q] $$

$$ = \frac{b^{2n} [a^2q^2; q^2]_n [a^2q^2/b^2; q^2]_n [-b^2/a; q]_{2n}}{(qa^2)^n [-aq; q]_{2n} [b^4/a^2; q^2]_n [b^2/a^2; q^2]_n} \tag{3.1} $$

by

$$ \frac{[b^2; q^2]_n [b^2/a^2; q^2]_n [c^2; q^2]_n [d^2; q^2]_n [b^2q^2; q^4]_n [q^{-2m}; q^2]_n}{[q^2; q^2]_n [a^2q^2; q^2]_n [b^2q^2/c^2; q^2]_n [b^2q^2/d^2; q^2]_n [b^2; q^4]_n} \cdot $$

$$ \cdot \frac{[a^2b^2q^{2+2m}/c^2d^2; q^2]_n q^{2n}}{[c^2d^2q^{-2m}/a^2; q^2]_n [b^2q^{2+2m}; q^2]_n} $$

and summing with respect to n from 0 to m, we get

$$ {}_{10}W_9 [b^2; -b^2/a, -b^2q/a, a^2q^2/b^2, c^2, d^2, a^2b^2q^{2+2m}/c^2d^2, q^{-2m}; q^2; b^2q/a^2] $$

$$ = \sum_{r=0}^{m} \frac{[a; q]_r [aq^2; q^2]_r [a^2q/b^2; q]_r [b^2q^2; q^2]_{2r} [c^2; q^2]_r}{[q; q]_r [a; q^2]_r [b^2/a; q]_r [a^2q^2; q^2]_{2r} [b^2q^2/c^2; q^2]_r} \cdot $$

$$\cdot \frac{[d^2; q^2]_r\, [a^2b^2q^2q^{2+2m}/c^2d^2; q^2]_r\, [q^{-2m}; q^2]_r\, q^r\, b^{2r}}{[b^2q^2/d^2; q^2]_r\, [c^2d^2q^{-2m}/a^2; q^2]_r\, b^2q^{2+2m}; q^2]_r\, a^{2r}}.$$

$$\cdot\, {}_8W_7\, [b^2q^{4r}; b^2/a^2, c^2q^{2r}, d^2q^{2r}, a^2b^2q^{2+2m+2r}/c^2d^2, q^{-2m+2r}; q^2; q^2]$$

on summing the inner ${}_8W_7$ by (2.1), we have (1.1) on some simplification.

In (1.1) setting $c^2 = -aq^2$, $d = -aq$ and then summing the resulting ${}_6\Phi_5$ ou the right hand side, we get (on replacing b by $b\sqrt{q}$) $(1 = \sqrt{-1})$

$${}_{10}W_9\, [a; a^2/b^2, iq\sqrt{a}, -iq\sqrt{a}, bq^m, -bq^m, -q^{-m}, q^{-m}; q; q]$$

$$= \frac{[a^2q^2; q^2]_m\, [a^2/b^2; q^2]_m\, [-b^2q/a; q]_{2m}\, b^{2m}}{[b^4q^2/a^2; q^2]_m\, [b^2/a^2; q^2]_m\, [-a; q]_{2m}\, a^{2m}\, q^m} \tag{3.2}$$

(3.2) is a q-analogue of the formula [2; § 8 (B)].

Next, we prove the following three transformations

$${}_{12}W_{11}\, [a; a^2/b^2, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, -q^{-m}, q^{-m}; q; q^2]$$

$$= \frac{[a^2b^2; q^2]_m\, [b^2q^2/c^2; q^2]_m\, [b^2q^2/d^2; q^2]_m\, [a^2q^2/c^2d^2; q^2]_m}{[b^2q^2; q^2]_m\, [a^2q^2/c^2; q^2]_m\, [a^2q^2/d^2; q^2]_m\, [b^2q^2/c^2d^2; q^2]_m}.$$

$$\cdot\, {}_{10}W_9\, [b^2; a^2/b^2, -b^2/a, -b^2q/a, c^2, d^2, a^2b^2q^{2+2m}/c^2d^2, q^{-2m}; q^2; \frac{b^2}{a^2}\, q^3] \tag{3.3}$$

$${}_{14}W_{13}\, [a; a^2/b^2, iq\sqrt{a}, -iq\sqrt{a}, c, -c, d, -d, abq^{1+m}/cd, -abq^{1+m}/cd, -q^{-m}, q^{-m}; q; q]$$

$$= \frac{[a^2q^2;\, q^2]_m\, [b^2q^2/c^2;\, q^2]_m\, [b^2q^2/d^2;\, q^2]_m\, [a^2q^2/c^2d^2;\, q^2]_m}{[b^2q^2;\, q^2]_m\, [a^2q^2/c^2;\, q^2]_m\, [a^2q^2/d^2;\, q^2]_m\, [b^2q^2/c^2d^2;\, q^2]_m} \cdot$$

$$\cdot\ {}_{10}W_9\, [b^2;\, -b^2q/a,\, -b^2q^2/a,\, a^2/b^2,\, c^2,\, d^2,\, a^2b^2q^{2+2m}/c^2d^2,\, q^{-2m};\, q^2; b^2q/a^2] \tag{3.4}$$

and

$${}_{14}W_{13}\, [a;\, a^2/b^2q,\, iq\sqrt{a},\, -iq\sqrt{a},\, c,\, -c,\, d,\, -d,\, abq^{1+m}/cd,\, -abq^{1+m}/cd,$$

$$-q^{-m},\, q^{-m};\, q;\, q^2]$$

$$= \frac{[a^2q^2;\, q^2]_m\, [b^2q^2/c^2;\, q^2]_m\, [b^2q^2/d^2;\, q^2]_{2m}\, [a^2q^2/c^2d^2;\, q^2]_m}{[b^2q^2;\, q^2]_m\, [a^2q^2/c^2;\, q^2]_m\, [a^2q^2/d^2;\, q^2]_m\, [b^2q^2/c^2d^2;\, q^2]_m} \cdot$$

$$\cdot\ {}_{10}W_9\, [b^2;\, a^2/b^2q^2,\, -b^2q/a,\, -b^2q^2/a, c^2,\, d^2,\, a^2b^2q^{2+2m}/c^2d^2,\, q^{-2m};$$

$$q^2;\, \frac{b^2q^2}{a^2}\,] \tag{3.5}$$

Proof of (3.3) : We have a summation formula [9; 3. 1]

$${}_8W_7\, [a;\, a^2/b^2,\, bq^n,\, -bq^n,\, -q^{-n},\, q^{-n};\, q;\, q^2]$$

$$= \frac{[a^2q^2;\, q^2]_n\, [a^2/b^2;\, q^2]_n\, [-b^2/a;\, q]_{2n}\, b^{2n}\, q^n}{[b^2/a^2;\, q^2]_n\, [b^4q^2/a^2;\, q^2]_n\, [-aq;\, q]_{2n}\, a^{2n}} \tag{3.6}$$

Proof of (3.3) follows on the lines of the proof of (1.1) on using (3.6) instead of (4.1).

However, from (3.3) one may obtain the summation formula

$${}_{10}W_9\, [a;\, a^2/b^2q,\, iq\sqrt{a},\, -iq\sqrt{a},\, bq^n,\, -bq^n,\, -bq^n,\, -q^{-n},\, q^{-n};\, q;\, q^2]$$

$$= \frac{[a^2q^2;\, q^2]_n\, [a^2/b^2q^2;\, q^2]_n\, [-b^2q/a;\, q]_{2n}\, b^{2n}\, q^n}{[b^2/a^2;\, q^2]_n\, [b^4q^4/a^2;\, q^2]_n\, [-a;\, q]_{2n}\, a^{2n}} \tag{3.7}$$

Proofs of (3.4) and (3.5) follow on the lines of the proof of (1.1) on using (3.2) and (3.7) respectively instead of (3.1).

(3.3)-(3.5) are q-analogues of the formulae of Bailey [2; 8. 31, 8. 21, 8. 41].

4. FURTHER TRANSFORMATIONS

We begin this section by proving the following transformations:

$$[a^4q^4;\, q^4]_\infty \sum_{n,\, m,\, r=0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2-4p(n+m+r)}\; a^{4n+8m+10r}}{[q^4;\, q^4]_n\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-aq;\, q]_{2r}\, [-a^2q^2;\, q^2]_{2m+2r}}$$

$$= \sum_{j=0}^{p} \frac{[q^{-4p};\, q^4]_j\, (-)^j\, a^{4j}\, q^{2j(j+1)}}{[q^4;\, q^4]_j} \sum_{n=0}^{\infty} \frac{[a;\, q]_n\, (1-aq^{2n})^n\, (-)^n\, a^{11n}}{[q;\, q]_n\, (1-a)}$$

$$q^{n(23n-1-8p+16j)/2} \tag{5.1}$$

and

$$[a^4q^4;\, q^4]_\infty \sum_{n,\, m,\, r=0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2\,+\,4(m+r)^2\,+\,2r^2\,-\,4p(n+m+r)\,+\,4(n+2m+2r)}}{[q^4;\, q^4]_n\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-aq;\, q]_{2r}}$$

$$\cdot \frac{a^{4n+8m+10r}}{[-a^2q^2;\, q^2]_{2m+2r}}$$

$$= \sum_{j=0}^{p} \frac{[q^{-4p}; q^4]_j (-)^j a^{4j} q^{2j(j+3)}}{[q^4; q^4]_j} \cdot \sum_{n=0}^{\infty} \frac{[aq; q]_n (1-a^4 q^{8n+4})}{[q; q]_n}$$

$$\cdot a^{11n} (-)^n q^{n(23n+15-8p+16j)/2} \tag{4.2}$$

Proof of (4.1) (1.1) for $d^2 = -q^{-2m}$, $b \to \infty$, $c \to \infty$ yields

$$\sum_{r=0}^{m} \frac{[a; q]_r (1-aq^{2r}) (-)^r a^{3r} q^{r(7r-1)/2}}{[q; q]_r (1-a) [a^4q^4; q^4]_{m+r} [q^4; q^4]_{m-r}}$$

$$= \frac{1}{[-a^2q^2; q^2]_{2r}} \sum_{r=0}^{m} \frac{a^{2r} q^{2r^2}}{[q^2; q^2]_r [-aq; q]_{2r} [q^4; q^4]_{m-r}} \tag{4.3}$$

Multiply to (4.3) by $\dfrac{[x; q^4]_m [y; q^4]_m [q^{-4n}; q^4]_m q^{4m}}{[xyq^{-4n}/a^4; q^4]_m}$ and sum with respect to m from 0 to n, we get on using q-analogue of Saalschutz summation theorem [7; 3. 3. 2. 2] and then letting $x, y \to \infty$:

$$\sum_{r=0}^{n} \frac{[a; q]_r (1-aq^{2r}) (-)^r a^{7r} q^{r(15r-1)/2}}{[q; q]_r (1-a) [a^4q^4; q^4]_{n+r} [q^4; q^4]_{n-r}}$$

$$= \sum_{r, m \geqslant 0} \frac{a^{4m+6r} q^{4(m+r)^2+2r^2}}{[q^2; q^2]_r [q^4; q^4]_m [-aq; q]_{2r} [-a^2q^2; q^2]_{2m+2r}} \cdot$$

$$\cdot \frac{1}{[q^4; q^4]_{n-m-r}} \tag{4.4}$$

Using (4 4) in the formula [8, 2 15]

$$[aq; q]_\infty \sum_{n=0}^{\infty} a^n q^{n(n-p)} \beta_n = \sum_{j=0}^{p} \frac{[q^{-p}; q]_j (-a)^j q^{j(j+1)/2}}{[q; q]_j} \cdot$$

$$\cdot \sum_{n=0}^{\infty} a^n q^{n(n-p+2j)} \alpha_n$$

where

$$\beta_n = \sum_{r=0}^{n} \frac{a_r}{[q;\, q]_{n-r}\, [aq;\, q]_{n+r}}$$

for evaluating $< \beta_n >$ with a and q replaced by a^4, q^4 respectively and

$$a_r = \frac{[a;\, q]_r\, (1-aq^{2r})\, (-)^r\, a^{7r}\, q^{r(15r-1)/2}}{[q;\, q]_r\, (1-a)},$$

we get (4 1).

Proof of (4.2) : However (4.3) may be written in the form (see [8] for details)

$$\sum_{r=0}^{m} \frac{[aq;\, q]_r\, (1-a^4\, q^{8r+4})\, (-)^r\, a^{3r}\, q^{r(7r-1)/2}}{[q;\, q]_r\, [q^4;\, q^4]_{m-r}\, [a^4\, q^8;\, q^4]_{m+r}}$$

$$= \frac{(1-a^4q^4)}{[-a^2q^2;\, q^2]_{2m}} \sum_{r=0}^{m} \frac{a^{2r}\, q^{2r^2}}{[q^2;\, q^2]_r\, [-aq;\, q]_{2r}\, [q^4;\, q^4]_{m-r}} \tag{4.5}$$

Proof of (4.2) follows on the lines of the proof of (4.1) on using (4.5) instead of (4.3).

Identities of Roger S-Ramanujan Type Related to Modulus 23

(4.1) for $p = 0$, $a = 1$, q. q^2, q^3, q^4 and q^5 yields the following identities

$$\frac{[q^4;q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty} \frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2}}{[q^4;\, q^4]_r\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-q;\, q]_{2r}\, [-q^2;\, q]_{2m+2r}}$$

$$= \prod_{n \not\equiv 0,\, 11,\, 12 \pmod{23}} (1-q^n)^{-1} \tag{4.6}$$

$$\frac{[q^4;\, q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty} \frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+4n+8m+10r}}{[q^4;\, q^4]_n\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-q;\, q]_{2r+1}\, [-q^2;\, q^2]_{2m+2r+1}}$$

$$= \prod_{n \not\equiv 0,\, 1,\, 22 \pmod{23}} (1-q^n)^{-1} \tag{4.7}$$

$$\frac{[q^4;\, q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty} \frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+2r}}{[q^4;q^4]_n\, [q^4;q^4]_m\, [q^2;q^2]_r\, [-q;q]_{2r}\, [-q^2;q^2]_{2m+2r}}$$

$$= \prod_{n \not\equiv 0,\, 10,\, 13 \pmod{23}} (1-q^n)^{-1}, \tag{4.8}$$

$$\frac{[q^4;\, q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty} \frac{q^{(n+m+r)^2+4(m+r)^2+2r^2+4n+8m+10r}}{[q^4;\, q^4]_n\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-q;\, q]_{2r}\, [-q^2;\, q^2]_{2m+2r+1}}$$

$$= \prod_{n \not\equiv 0,\, 2,\, 21 \pmod{23}} (1-q^n)^{-1} \tag{4.9}$$

$$\frac{[q^4;\, q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty} \frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+20r+8n+16m+9}}{[q^4;\, q^4]_n\, [q^4;\, q^4]_m\, [q^2;\, q^2]_r\, [-q;\, q]_{2r+1}\, [-q^2;\, q^2]_{2m+2r+2}}$$

$$= \prod_{n \not\equiv 0,\, 10,\, 13 \pmod{23}} (1-q^n)^{-1} \qquad = \prod_{n \not\equiv 0,\, 9,\, 14 \pmod{23}} (1-q^n)^{-1} \tag{4.10}$$

$$\frac{[q^4;\, q^4]_\infty}{[q;\, q]_\infty} \sum_{n,\, m,\, r=0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+8m+4n+12r}(1+q^2+q^{2r+1})}{[q^4;q^4]_n\,[q^4;q^4]_m\,[q^2;q^2]_r\,[-q;q]_{2r+1}\,[-q^2;q^2]_{2m+2r+1}}$$

$$= \prod_{n \not\equiv 0,\,3,\,20 \pmod{23}} (1-q^n)^{-1} \tag{4.11}$$

Next, (4.2) for $a = 1$ and $p = 0, 1$ gives

$$\frac{[q^4;q^4]_\infty}{[q;q]_\infty} \sum_{n,\,m,\,r\,=\,0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+4n+8m+8r}}{[q^4;q^4]_n\,[q^4;q^4]_m\,[q^2;q^2]_r\,[-q;q]_{2r}\,[-q^2;q^2]_{2m+2r}}$$

$$= \prod_{n \not\equiv 0,\,4,\,19 \pmod{23}} (1-q^n)^{-1} \tag{4.12}$$

$$\frac{[q^4;q^4]_\infty}{[q;q]_\infty} \sum_{n,\,m,\,r\,=\,0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2+4m+4r}}{[q^4;q^4]_n\,[q^4;q^4]_m\,[q^2;q^2]_r\,[-q;q]_{2r}\,[-q^2;q^2]_{2m+2r}}$$

$$= \prod_{n \not\equiv 0,\,8,\,15 \pmod{23}} (1-q^n)^{-1} \tag{4.13}$$

Furthermore (4.1) for $a = 1, p = 1$ gives the following identity (on using (4. 12))

$$\frac{[q^4;q^4]_\infty}{[q;q]_\infty} \sum_{n,\,m,\,r\,=\,0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2+4(m+r)^2+2r^2-4(n+m+r)}(1-q^{4n+8m+8r})}{[q^4;q^4]_r\,[q^4;q^4]_m\,[q^2;q^2]_r\,[-q;q]_{2r}\,[-q^2;q^2]_{2m+2r}}$$

$$= \prod_{n \not\equiv 0,\,7,\,16 \pmod{23}} (1-q^n)^{-1} \tag{4.14}$$

(4.1) for $a = q, p = 1$ reduces to

$$\frac{[q^4; q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r = 0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2+4n+6r}}{[q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r+1} [-q^2; q^2]_{2m+2r+1}}$$

$$= \prod_{n \not\equiv 0, 5, 18 \pmod{23}} (1-q^n)^{-1} \quad -q \quad = \prod_{n \not\equiv 0, 3, 20 \pmod{23}} (1-q^n)^{-1} \tag{4.15}$$

Lastly, (4.11) for $a = q^3, p = 1$ gives the following identity (on using (4.15))

$$\frac{[q^4; q^4]_\infty}{[q; q]_\infty} \sum_{n, m, r = 0}^{\infty}$$

$$\frac{q^{4(n+m+r)^2 + 4(m+r)^2 + 2r^2+4m+6r} (1+q^{4m+4n+4r+2})}{[q^4; q^4]_n [q^4; q^4]_m [q^2; q^2]_r [-q; q]_{2r} [-q^2; q^2]_{2m+2r+1}}$$

$$= \prod_{n \not\equiv 0, 6, 17 \pmod{23}} (1-q^n)^{-1} \tag{4.16}$$

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UNIFORM SPACE VALUED UNIFORMLY ALMOST PERIODIC FUNCTIONS DEPENDING ON PARAMETERS

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ABSTRACT

Sharma, Reddy and Funakoshi [3] studied some fundamental properties of uniform space valued almost periodic functions depending on parameters. The purpose of this paper is to generalize some results given in [1] on uniform spaces. For uniform spaces and the related nations, see [2]

1. INTRODUCTION AND DEFINITIONS

Let T and E be uniform spaces. We shall study here only, E-valued functions defined on $T \times R$, where R is the real line, and assume that any function in theorems or definition is continuous.

Definition 1 : Let $f(t, x)$ be a real or complex function defined for all real values of x. A number τ is called a U translation number of $f(t, x)$ if

$$\underset{-\infty < x < \infty}{u.\,b.} \quad (f(t, x+\tau), f(t, x)) \in U,$$

Whenever $t \in T$ and $x \in R$.

We denote the set of all U translation numbers of a function $f(t, x)$ by $E\{U, f(t, x)\}$.

Definition 2 : A continuous function $f(t, x)$ is called uniformly almost periodic in x, uniformly with respect to $t \epsilon T$, if to each entourage U of R the set $E\{U, f(t, x)\}$ is relatively dense.

2. MAIN RESULTS

We first state and prove

Theorem 1. *If T is a compact uniform space and $f(t, x)$ is an uniformly almost periodic function in x uniformly with respect to $t \epsilon T$, when the set $\{f(t, x)/t \epsilon T, x \epsilon R\}$ is precompact.*

Proof. Let V be an entourage of E. Then there is a symmetric entourage U of E with $U^2 \subset V$. Let (t_0, x_0) be an arbitrary element of $T \times R$. By the definition of uniformly almost periodicity of $f(t, x)$, there is a U translation number τ of the set $E\{U, f(t_0, x_0)\}$. Obviously, the set

$$S = \{f(t, x)/t \epsilon T, x \epsilon [0, k(U)]\}$$

where $k(U)$ is a positive number, is compact. So there are $a_1, \ldots, a_n \epsilon S = \bigcup_{i=1}^{n} U(a_i)$ for which we can find a_n $m \epsilon \{1, \ldots, n\}$ such that

$f(t_0, x_0 + \tau) \epsilon U(a_m)$, since $x_0 + \tau \epsilon [0, k(u)]$,

or, equivalently,

$$(f(t_0, x_0 + \tau), a_m) \epsilon U.$$

Also, since $(f(t_0, x_0 + \tau), f(t_0, x_0)) \epsilon U$,

it follows from the above that

$$(f(t_0, x_0), a_m) \epsilon U^2 \subset V \text{ or } f(t_0, x_0) \epsilon V(a_m).$$

Therefore, we have

$$\{f(t, x)/t \epsilon T, X \epsilon R\} \subset \bigcup_{i=1}^{n} V(a_i)$$

which proves the above theorem.

Next we state

Theorem 2. *If T is a compact uniform space and $f(t, x)$ is an uniformly almost periodic function in x uniformly with respect to $t \epsilon T$, then the function $f(t, x)$ is uniformly continuous on $T \times R$.*

Proof. Let V be an entourage of E. Then there exists a symmetric entourage U of E with $U^3 \subset V$. Let $k(U)$ be the length associated with U such that $(f(t_1, y_r), f(t_2, y_2)) \epsilon U$ for any $(t_1, y_1), (t_2, y_2)$ belonging to the interval $T \times [0, 1 + k(U)]$ if only $(t_2, t_1) \epsilon W$, $| y_2, y_1 | < \delta$ where δ is a positive number $\delta = \delta(U) < 1$. Let now $(t_1, x_1), (t_2, x_2)$ be any two points of $T \times R$ with $(t_2, t_1) \epsilon W$, where W is an entourage of T and $| x_1, x_2 | < \delta$. There exists a U translation number τ of the set $E\{U, f(t, x)\}$ such that both the numbers $(t_1, x_1, + \tau)$, $(t_2, x_2 + \tau)$ belong to the interval $T \times [0, 1 + k(U)]$.
we have then

$$(f(t_1, x_1 + \tau), f(t_2, x_2 \times \tau)) \epsilon U.$$

On the other hand

$$(f(t, x + \tau), f(t, x)) \epsilon U, \text{ for any } X \epsilon R.$$

Thus

$$(f(t_1, x_1), f(t_2, x_2)) \epsilon U^3 \subset V.$$

Therefore, $f(t, x)$ is uniformly continuous on $T \times R$.

This proves the theorem.

3. FURTHER RESULTS

Theorem 3. *If $\{f_n(t, x)\}$ is a sequence of uniformly almost periodic functions in x uniformly with respect to $t \epsilon T$ which is uniformly convergent on $T \times R$ to a function $f(t, x)$ is also uniformly almost periodic in x uniformly with respect $t \epsilon T$.*

Proof. Let V be on entourage of E. Then there exists a symmetric entourage U of E with $U^3 \subset V$. Thus we can choose an n_0 such that

$$(f(t, x),\ f_{n_0}(t, x)) \epsilon U$$

for every $(t, x) \epsilon\ T \times R$. Non let τ be a translation number of $E\{U,\ f_{n_0}(t, x)\}$. Then

$$(f(t, x + \tau), f(t, x)) \epsilon\ U^3 \subset V,$$

which shows that

$$E\{U, f(t, x)\} \supset E\{U, f_{n_0}(t, x)\}.$$

Thus $E\{U, f(t,x)\}$ is relatively dense. Consequently, $f(t, x)$ is uniformly almost period.

Theorem 4. *The sum of two uniform space valued uniformly almost periodic functions $f_1(t, x)$, $f_2(t, x)$ depending on parameters is again uniformly almost periodic function depending on parameters.*

Proof. Let τ be any U translation number of the set

$$E\{U, f_1(t, x)\}, E\{U, f_2(t, x)\}.$$

then

$$(f_1(t, x + \tau) + f_2(t, x + \tau) - f_1(t, x) - f_2(t, x)) \epsilon U,$$

which shows that τ is a U translation number of the set

$E\{U, f_1(t, x) + f_2(t, x)\}$. Thus

$$E\{U, f_1(t, x) + f_2(t, x)\} \supset E\{U, f_1(t, x)\}. E\{U, f_2(t, x)\}.$$

Therefore, $E\{U, f_1(t, x) + f_2(t, x)\}$ is relatively dense. Consequently, $f_1(t, x) + f_2(t, x)$ is uniformly almost periodic depending on parameters, which proves the theorem.

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THE HARMONIC CESÀRO PRODUCT SUMMABILITY OF THE DERIVED FOURIER SERIES

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1. DEFINITIONS AND NOTATIONS

If σ_n denotes the $(C, 1)$ Transform of partial sum S_n of an infinite series $\sum_0^\infty U_n$ and

$$(1.1) \qquad \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\sigma_{n-k}}{k} = S,$$

where S is finite, then the series $\sum_0^\infty U_n$ is said to be summable $(H,1)$ $(C,1)$ to S, or symbolically

$$\sum_0^\infty U_n = S\ [(H,1)\ (C,1)]$$

Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and periodic with period 2π. Let the Fourier series associated with $f(x)$ be

$$(1.2) \qquad \frac{1}{2} a_n + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x)$$

The derived series of the Fourier series (1.2) is

$$(1.3) \qquad \sum_{n=1}^{\infty} n\,(b_n \cos nx - a_n \sin nx) \equiv \sum_{1}^{\infty} n\, B_n(x)$$

We shall use, for fixed x and s the following notations :

$$\phi(t) = f(x+t) + f(x-t) - 2s$$

$$g(t) = \frac{f(x+t) + f(x-t)}{4 \sin t/2}$$

2. INTRODUCTION

Siddiqi [3] and Hille and Tamarkin [1] have, respectively established the following theorems on the harmonic summability of the Fourier series :-

Theorem A (Siddiqi [3]) *If*

$$(2.1) \quad \Phi(t) \equiv \int_0^t |\Phi(u)|\, du = O\,(t/\log 1/t), \quad as\ t \to 0$$

then $\sum_{0}^{\infty} A_n(x) = S(H.\ 1)$.

Theorem B (Hille and Tamarkin [1]). *If*

$$(2.2) \quad \Phi(t) \equiv \int_0^t |\Phi(u)|\, du = O\,(t), \quad as\ t \to 0,$$

and

$$(2.3) \int_{\pi/n}^{\eta} \frac{|\Phi(t) - \Phi(t+\pi/n)|}{t} \log 1/t\, dt = O\,(\log n) \quad as\ n \to \infty,$$

where η is a positive constant, then

$$\sum_{0}^{\infty} A_n(x) = S(H, 1).$$

Since the derived Fourier series is neither $(C, 1)$ summable nor $(H, 1)$ under the hypothesis of Theorems A and B, respectively. Therefore it is natural to expect the extensions of Theorems A and B to the product summability $(H, 1)$ $(C, 1)$ of the derived Fourier series under analogous conditions with $\phi(t)$ replaced by $g(t)$, with this point of view, we prove here

Therefore 1. *If*

$$(2.4) \quad G(t) \equiv \int_0^t | g(u) | \, du = o \, | \, t \, | \, log 1/t), \quad as \; t \to 0,$$

then the derived Fourier series (1.3) *is summable* $(H. 1)$ $(C. 1)$ *to zero, at the point x.*

Theorem 2. *If*

$$(2\ 5) \quad G(t) \equiv \int_0^t | g(u) | \, du = o(t), \quad as \; t \to 0,$$

aud

$$(2.6) \quad \int_{\pi/n}^{\eta} \frac{| g(t) - g(t+\pi/n) |}{t} \; log \; 1/t \; dt = o(logn), \quad as \; n \to \infty.$$

Where η, is a positive constant, then the derived Fourier Series (1.3) *is summable* $(H. 1)$ $(C. 1)$ *to zero, at the point x.*

3. Proof of Theorem 1.

Following Sachan [2] the $(C, 1)$ transform $\{\sigma_{n+1}\}$

of the sequence $\{S_{n+1}\}$ of partial sums of the derived Fourier series is given by

$$\sigma_{n+1} = \frac{2}{\pi}\int_0^{\delta} g(t)\left[\frac{2\{1-\cos(n+1)t\}}{(n+1)t^2} + \frac{\sin(n+1)t}{(n+1)t}\right.$$

$$\left. - \frac{\sin(n+1)t}{t}\right] dt + o(1)$$

We choose $\delta > 0$ so small that for $t \in (0, \delta)$ the condition (2.3) is satisfied.

Denoting the harmonic transform of σ_{n+1}, *i. e.*, the $(H.\ 1)$, $(C.\ 1)$ transform of S_{n+1} by H_n, we have, by the regularity of the method of summation,

$$H_n = \frac{1}{\pi \log n}\int_0^{\delta} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1)}$$

$$\left[\frac{2(1-\cos rt)}{rt^2} + \frac{\sin rt}{rt} - \frac{\sin rt}{t}\right] dt + o(t)$$

$$= \frac{2}{\pi \log n}\int_0^{\pi/n} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1)}$$

$$\left[\frac{2(1-\cos rt)}{rt^2} + \frac{\sin rt}{rt} - \frac{\sin rt}{t}\right] dt$$

$$+ \frac{2}{\pi \log n}\int_{\pi/n}^{\delta} g(t) \sum_{r=1}^{n} \frac{1}{(n-r+1}\left[\frac{2(1-\cos rt)}{rt^2} + \frac{\sin rt}{rt}\right] dt$$

$$- \frac{2}{\pi \log n}\int_{\pi/n}^{\delta} g(t) \sum_{r=1}^{n} \frac{\sin rt}{(n-r+1)t}\, dt + o(t)$$

(3.1) $= 2/\pi\,[P + Q - R] + O(1)$, *say*

Since for $t > 0$,

$$\frac{2(1-\cos rt)}{rt^2} + \frac{\sin rt}{rt} - \frac{\sin tr}{t} = O(r) \qquad 4$$

We obtain

$$|P| = \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{(n-r+1)} \int_0^{\pi/n} |g(t)|\, O(r)\, dt$$

$$= \frac{1}{(\log n} O(n \log n) \int_0^{\pi/n} |g(t)|\, dt$$

(3.2) $= o(1)$, *as* $n \to \infty$.

By an integration by parts, we obtain

$$|Q| = \frac{1}{\log n} \left| \sum_{r=1}^{n} \frac{1}{r(n+1-r)} \int_{\pi/n}^{\delta} g(t) \left[\frac{2(1-\cos rt)}{t^2} + \frac{\sin rt}{t} \right] dt \right|$$

$$= O\left(\frac{1}{(n+1)\log n} \right) \sum_{r=1}^{n} \left\{ 1/r + \frac{1}{(n+1-r)} \right\} \int_{\pi/n}^{\delta} \frac{|g(t)|}{t^2}\, dt$$

(3.3) $= o(1)$, *as* $n \to \infty$.

Since for $t \in (\pi/n, \delta)$,

$$\sum_{1}^{n} \frac{\sin rt}{(n-r+1)} = O(\log 1/t)$$

By an integration by parts, we get

$$|R| = O\left(\frac{1}{\log n} \right) \int_{\pi/n}^{\delta} \frac{|g(t)|}{t} (\log 1/t)\, dt$$

$$= O\left(\frac{1}{logn}\left[\left\{\frac{G(t)}{t}\ log\ 1/t\right\}_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} G(t)\left(\frac{1+log\ 1/t}{t^2}\right)dt\right]\right.$$

$$= O\left(\frac{1}{logn}\right)\left[o(1) + o(1)\left\{\int_{\pi/n}^{\delta}\left(\frac{1}{t\ log\ 1/t} + 1/t\right)dt\right.\right]$$

$$= o(1) + O\left(\frac{1}{logn}\left[o(log\ logn) + o(logn)\right]\right.$$

(3.4) $\quad = o(1), \quad as\ n \to \infty.$

Finally, from (3.1), (3.2), (3.3) and (3.4), we find

$$H_n = o(1), \quad as\ n \to \infty$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

It may be noted that from the proof of Theorem 1 that P, $Q = o(1)$ under the hypothesis (2.5). Thus to prove Theorem 2, it is sufficient to show that $R = o(1)$, under the hypothesis (2.5) and (2.6)

We write

$$R = \frac{1}{logn}\int_{\pi/n}^{\delta} \frac{g(t)}{t}\ sin(n+1)t \sum_{r=1}^{n} \frac{cos\ rt}{r}\ dt$$

$$- \frac{1}{logn}\int_{\pi/n}^{\delta} \frac{g(t)}{t}\ cos\ (n+1)\ t \sum_{r=1}^{n} \frac{sin\ rt}{r}\ dt$$

$$= R_1 - R_2, \quad say$$

(2.6) and an integration by parts give

$$| R_2 | = \frac{1}{logn} \int_{\pi/n}^{\delta} \frac{| g(t) |}{t} O(1)\, dt$$

$$= o(1), \quad as\, for\ (3.3).$$

Further, we write

$$2\, R_1 = \frac{2}{logn} \int_{\pi/n}^{\delta} \frac{g(t)}{t} \sin (n+1)t \log (1/2 \sin t/2)\, dt$$

$$= \frac{2}{logn} \int_{\pi/n}^{\delta} \frac{g(t)}{t} \sin nt \log 1/t\, dt + o(1)$$

$$= \frac{1}{logn} \left[\int_{\pi/n}^{\delta-\pi/n} + \int_{\delta-\pi/n}^{\delta} \right] \frac{| g(t) |}{t} \log 1/t \sin nt\, dt$$

$$- \frac{1}{logn} \left[\int_{0}^{\pi/n} + \int_{\pi/n}^{\delta-\pi/n} \right] \frac{g(t+\pi/n)}{(t+\pi/n)} \log (1/t+\pi/n) \sin nt\, dt + o(1)$$

$$= \frac{1}{logn} \int_{\pi/n}^{\delta-\pi/n} \frac{g(t)-g(t+\pi/n)}{t} \log (1/t) \sin nt\, dt$$

$$+ \frac{1}{logn} \int_{\pi/n}^{\delta-\pi/n} \frac{g(t+\pi/n)}{t} \left[\log 1/t - \log 1/t + \pi/n \right] \sin nt\, dt$$

$$+ \frac{1}{logn} \int_{\pi/n}^{\delta-\pi/n} g(t + \pi/n) \left[1/t - \frac{1}{(t+\pi/n)} \right] \log(1/t+\pi/n) \sin nt\, dt$$

$$- \frac{1}{logn} \int_{0}^{\pi/n} \frac{g(t+\pi/n)}{(t+\pi/n)} \log (1/t + \pi/n) \sin nt\, dt$$

$$+ \frac{1}{logn} \int_{\delta-\pi/n}^{\delta} \frac{g(t)}{t} \log 1/t \sin nt\, dt + o(1)$$

$$= L_1 + L_2 + L_3 - L_4 + L_5 + o(1)$$

By (2.6), we get

$$| L_1 | = o(1), \quad as\ n \to \infty$$

An integration by parts, we have

$$| L_2 | = \frac{1}{logn} \int_{\pi/n}^{\delta-\pi/n} \frac{| g(t+\pi/n) |}{t} \log (1+\pi/nt)\, dt$$

$$= O\left(\frac{1}{\log n}\right) \int_{\pi/n}^{\delta-\pi/n} \frac{| g(t+\pi/n) |}{t}\, dt$$

$$= o(1), \quad as\ n \to \infty.$$

Again, on integration by parts, we obtain

$$| L_3 | < \int_{\pi/n}^{\delta-\pi/n} | g(t+\pi/n) | \frac{\pi/n}{t(t+\pi/n)}\, dt$$

$$= O(1/n) \int_{\pi/n}^{\delta-\pi/n} \frac{| g(t+\pi/n) |}{t^2}\, dt$$

$$= O\,(1/n) \left[\left\{ o(t+\pi/n)\, 1/t^2 \right\}_{\pi/n}^{\delta-\pi/n} + \int_{\pi/n}^{\delta-\pi/n} o(t+\pi/n)\, 1/t^3\, dt \right]$$

$$= O(1/n) \left[\left\{ o(1/t) \right\}_{\pi/n}^{\delta-\pi/n} + \int_{\pi/n}^{\delta-\pi/n} o(1/t^2)\, dt \right]$$

$$= o(1), \quad as\ n \to \infty.$$

Similarly, we fined

$$| L_4 | \leqslant \frac{1}{\log n} \int_{\pi/n}^{2\pi/n} \frac{| g(t) |}{t} \log 1/t\, dt$$

$$= O(n) \int_{\pi/n}^{2\pi/n} | g(t) | \, dt$$

$$= o(1), \quad as \; n \to \infty$$

and that

$$| L_5 | \leqslant \frac{| log \, (\delta - \pi/n) |}{(\delta - \pi/n) \, log \; n} \int_{\delta - \pi/n}^{\delta} | g(t) | \, dt$$

$$= o(1), \quad as \; n \to \infty$$

Thus, collecting the results, we obtain

$$R = o(1), \quad as \; n \to \infty,$$

and the proof of Theorem 2 is completed.

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