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A FIXED POINT THEOREM FOR L-CONTRACTIONS IN GENERALIZED METRIC SPACES

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ABSTRACT

The aim of this note is to extend a result concerning the existence of fixed-point for contractive mappings in generalized metric spaces to a more general class of mappings.

Before stating our result, we introduce, for sake of completeness, some standard notions [1].

Let E be a Banach space. A subset K of E is called a *cone* if it is closed, convex, $tK \subset K$ for $t \in R^+$ and $K \cap (-K) = 0$.

Given a cone K in E we define a *partial ordering* in E by writing $x < y$ if and only if $y - x \in K$.

Moreover, K is called *normal* if there exists $\sigma > 0$ such that

$$0 < x < y \quad \text{implies} \quad \| x \| \leqslant \sigma \| y \| .$$

We shall assume in the following that K is a *normal cone.*

Definition. A topological space X is said to be a *generalized metric space* if there exists a function d: $X \times X \rightarrow K$ such that:

(i) $d(x, y) = 0 \in K \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) < d(x, z) + d(z, y)$

("$<$" denotes the partial ordering induced by K).

Let $l\colon K \to R^+$ be a sublinear positively homogenous functional (that is, if $u, v \in R^+$ then

$l(u + v) \leqslant l(u) + l(v)$ and $l(tu) = t\,l(u)$ for $t \geqslant 0$) such that $l^{-1}(0) = 0$. Then, if $d^*\colon X \times X \to R^+$ is the function defined by

$$d^*(x, y) = l(d(x, y)), \tag{1}$$

we have that (x, d^*) is a metric space.

We shall say that a generalized metric space is a *complete generalized metric space* if it is complete in the metric defined by (1).

In a complete generalized metric space (x, d) the following result was proved in [1, Theorem 6. 2].

Let $T\colon X \to X$ be a map such that

$$d(T(x), T(y)) < Ld(x, y), \tag{2}$$

where L is a positive ($L(K) \subset K$) and bounded linear operator in E with spectral radius, $r(L)$, less than 1. If x is complete generalized metric space with $l(u) = \| u \|$, than T has a unique fixed point which is the limit of the successive approximations

$$x_{n+1} = Tx_n \qquad n = 0, 1, 2$$

for any initial value $x' = x_0 \in X$.

In the Theorem below we shall show that if $T\colon X \to X$ is not a necessarily continuous map which satisfies a relaxed condition than (2) then the same result holds.

More precisely we have the following

Theorem. *Let X be a generalized metric space which is complete in the metric* (1) *with* $l(u) = \| u \|$ ($\| u \|$ stands for the norm in E).

If $T: \to X$ satisfies

$$d(T(x), T(y)) < L[\, d(T(x), x) + d(T(y), y)\,], \tag{3}$$

where L is a bounded positive linear operator in E with spectral radius smaller than $\frac{1}{2}$, then the equation

$$x = Tx$$

has a unique solution in x, which is the limit of the successive approximations

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2$$

for any initial approximation $x' \in X$.

Proof. Since L is sublinear, it follows from (3) for $x = Tx'$ and $y = x'$

$$d(T(Tx'), T(x')) < Ld(T^2(x'), T(x')) + Ld(T(x'), x').$$

That is

$$(I - L)\, d(T^2(x'), T(x')) < Ld(T(x'), x).$$

Being $r(L) < \frac{1}{2} < 1$, we have that $(I - L)$ is invertible, hence

$$d(T^2(x'), T(x')) < (I - L)^{-1} Ld(Tx', x'). \tag{4}$$

Clearly we have that

$$d(T^{n+1}(x'), T^n(x')) < (I - L)^{-n} L^n d(T(x'), x'). \tag{5}$$

Indeed, for $n = 1$, it has been already proved.

Assume that (5) is true for $n = 1$, then

$$d(T^{n+1}(x'), T^n(x')) = d(T(T^n(x'), T(T^{n-1}(x')) <$$

$$< Ld(T^{n+1}(x'), T^n(x')) + Ld(T^n(x'), T^{n-1}(x')).$$

That is

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < Ld(T^n(x'), T^{n-1}(x)).$$

By the inductive assumption

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < L(I - L)^{-(n-1)} L^{n-1} d(T(x'), x').$$

Since

$$L(I - L)^{-(n-1)} = (I - L)^{-(n-1)} L,$$

we obtain

$$(I - L)\, d(T^{n+1}(x'), T^n(x')) < (I - L)^{-(n-1)} L^n d(T(x'), x').$$

Thus (5) is proved.

Furthermore, we have

$$d(T^{n+m}(x'), T^n(x')) < d(T^{n+m}(x'), T^{n+m-1}(x') + d(T^{n+m-1}(x'), T^{n+m-2}(x')) +$$

$$+ \dots + d(T^{n+1}(x'), T^n(x')) <$$

$$< \{(I - L)^{-(n+m-1)} L^{n+m-1} + (I - L)^{-(n+m-2)} L^{n+m-2} + \dots +$$

$$+ (I - L)^{-n} L^n\} d(T(x'), x') <$$

$$< (I-L)^{-n}L^n \left(\sum_{m=1}^{\infty} (I-L)^{-(m-1)} L^{m-1} (d(T(x'), x')) \right) = (I-L)^{-1}L)^n u'$$

where u' is the unique solution of

$$u = (I-L)^{-1}Lu + d(T(x'), x').$$

Indeed, from the spectral mapping theorem, we have that the spectral operator $(I-L)^{-1}L$ is such that $r((I-L)^{-1} L) < 1.$

Hence

$$d(T^{n+m}(x'), T^n(x')) < ((I-L)^{-1}L)^n u'$$

Being K *normal,* we have

$$\| d(T^{n+m}(x'), T^n(x') \| \leqslant \sigma \| ((I-L)^{-1} L)^n u' \| . \tag{6}$$

Since the right-hand side of (6) is going to zero when $n \to +\infty$, we obtain that $\{T^n(x')\}$ is a Cauchy-sequence with respect to the metric d^*. Being (X, d^*) complete, we denote by x^* the lemit of $\{T^n(x')\}$. Then the following inequalitieshold

$$d(T(x^*), x^*) < d(T(x^*), T^n(x') + d(T^n(x'), x^*) <$$

$$< (I-L)^{-1} Ld(x^*, T^{n-1} (x')) + d(T^n(x'), x^*).$$

Finally, using the normality of K,

$$\| d(T(x^*), x^*) \| \leqq \sigma (\| (I-L)^{-1}L \| . \| d(x^*, T^{n-1}(x') \| +$$

$$+ \| d(T^n(x'), x^*) \|). \tag{7}$$

Letting $n \to +\infty$ in (7), we obtain

$$T(x^*) = x^*.$$

At last, we would like to add in passing that the Theorem would be still true if the map T is a "*generalized contraction*" map (cfr. [2 Ch. 1] for an extensive bibliography).

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