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PRIMITIVE ACCESSIBLE RINGS AND SEMI-SIMPLICITY

By

Yash Paul

Department of Mathematics, Indian Institute of Technology,

New Delhi-110016, India

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ABSTRACT

In this paper we show that a primitive accessible ring is either associative or simple with an identity element. Another result is that a semisimple accessible ring is a subdirect sum of simple rings with an identity element and associative primitive rings.

1 INTRODUCTION

Accessible rings were introduced in [1] where it was shown that simple accessible rings are either associative or commutative. In that paper primitive and semisimple accessible rings were studied under the assumption that the rings are without nilpotent elements in their centre. In this paper we dispense with this assumption and investigate the properties and the structures of these rings. As a by-product we prove that a simple, not associative accessible ring has no proper one-sided ideals. The notion of primitive ring lends itself to a structure theory for semisimple rings. Prime accessible rings are characterized in [3].

2. Preliminaries and Notation

A nonassociatve ring R is called accessible in case the following two identities hold :

(1) $(x, y, z) + (z, x, y) - (x, z, y) = 0$

(2) $([w, x,], y, z) = 0$

for all $w, x. y, z \in R$,where the associator (x, y, z) is defined by $(x,y,z) = (xy)z - x(yz)$ and the commutator $[x, y]$ is defined by $[x, y] = xy-yx$. Substituting $z = x$ in (1) we obtain the flexible law,$(x,y,z)=0$. A linearization of this identity yields

(3) $(x, y, z) = -(z, y, x)$

In any nonassociative ring, the following identity holds:

(4) $w(x, y, z) + (w, x, y) z = (wx, y, z) - (w, xy, z) + (w, x, yz)$

In what follows, any expression of the form (R, a, b) means the set of all finite sums of (x, a, b) for $x \in R$; analogous arguments are meant for all forms of similar expressions.

3. Construction of Ideals

Lemma 1. Let A be a right ideal of R. Then,

(i) $S = \{ s \in A : Rs \subseteq A\}$.

is a two-sided ideal of R;

(ii) $(A, R, R) \subseteq S$.

Proof. (i) For any $s \in S$, $x \in R$, consider sx and xs.

Let $y \in R$. Then using (1),

$$y(sx) = -(y, s, x) + (ys)x$$

$$= -(y, x, s) - (s, y, x) + (ys)x$$

$$= (s, x, y) - (s, y, x) + (ys)x$$

by (3). So $y(sx) \in A$, and hence $sx \in S$. Also,

$$y(xs) = -(y, x, s) + (yx)s$$
$$= (s, x, y) + (yx)s$$

by (3). So $y(xs) \in A$, and hence $xs \in S$. It follows that S is a two-sided ideal of R.

(ii) clearly

$$(A, R, R) \subseteq A.$$

Let $z \in R$ be arbitrary. Then by (4)

$$z(a, x, y) = (za, x, y) - (z, ax, y) + (z, a, xy) - (z, a, x)y$$

By (2), $(za, x, y) = (az, x, y)$ and by (1)

$$(z, ax, y) = -(ax, y, z) + (ax, z, y)$$
$$(z, a, xy) = -(a, xy, z) + (a, z, xy)$$
$$(z, a, x) = -(a, x, z) + (a, z, x)$$

Thus,

$$z(a, x, y) = (az, x, y) + (ax, y, z) - (ax, z, y) - (a, xy, z) + (a,z,xy)$$
$$+ (a, x, z)y - (a, z, x)y$$

and hence

$$z(a, x, y) \in A$$

This implies that

$$(A, R, R) \subseteq S,$$

Theorem 2. If R has a maximal right ideal $A \neq (0)$, which contains no two-sided ideals of R other than (0), then R is associative.

Proof. By lemma 1, S is a two-sided ideal of R contained in A. Therefore $S=(0)$, and hence $(A, R, R) = (0)$.

On the other hand, it is easy to verify that $A+RA$ is a two-sided ideal of R. Since $A \underset{\neq}{\subset} A+RA$, we must have $A+RA = R$. Thus, considering

$$(R, R, R) = (A+RA, R, R) = (RA, R, R).$$

But

$$(RA, R, R) = (AR, R, R) \subseteq (A, R, R) = (0)$$

Therefore,

$$(R, R, R,) = (0),$$

that is, R is an associative ring.

Lemma 3. If R is simple and not associative, then R has no proper one-sided ideals.

Proof. Suppose for example, that I is a non-zero right ideal of R. Then $(I, R, R) \subseteq I$; hence from (3), we have $(R, R, I) \subseteq I$ and from (1), we have $(R, I, R) \subseteq I$. Now we show that $I + RI$ is a two-sided ideal of R;

$$(I + RI)R \subseteq IR + (RI)R \subseteq I + R\,(IR) + (R, I, R) \subseteq I + RI,$$

and

$$R(I+RI) \subseteq RI + R(RI) \subseteq RI + (RR)\,I + (R, R, I) \subseteq RI + I.$$

SInce $I \neq (0)$ and R is simple, $R = I + RI$ Then from (2), we have

$$(R, R, R) \subseteq (I + RI, R, R) \subseteq (I, R, R) + (RI, R, R)$$

$$\subseteq (I, R, R) + (I, R, R) \subseteq (I, R, R) \subseteq I.$$

Since $IR \subseteq I$, $(R, R, R) + (R, R, R)R \subseteq I$. But it is known [2] that $(R, R, R) + (R, R, R)R$ is a two-sided ideal of R,hence it is equal to R since R is not associatative. Therfore $I = R$. A similar argument shows that R has no proper left ideals as well.

4. Primitive Rings

Definition. A right ideal A of R is called regular if there exists an element $g \varepsilon R$, such that $x-gx \varepsilon A$ for $x \varepsilon R$.

Definition. R is called *primitive* if it contains a regular maximal right ideal, which contains no two–sided ideal of R other than the zero ideal (0).

Theorem 4. If R is a primitive ring, then either R is associative or it is simple with an identity element.

Proof. Let A be a regular maximal right ideal of R, with a modular element g. Either $A = (0)$ or $A \neq (0)$. If $A \neq (0)$, by Theorem 2, R is associative. Thus, assume that every regular maximal right ideal of R is (0).

In particular,there exists $g \varepsilon R$,such that $x-gx=0$ for all $x \varepsilon R$,Therefore every right ideal of R is regular.By Zorn's Lemma any regular right ideal of R can be imbedded in a regular maximal right ideal of R. Therefore R has no proper right ideals, and hence R is simple. The proof will be complete if we show that g is the identity element of R. By assumption g is a left identity element. Consider the set

$$L = \{x \varepsilon R;\ xg = x\}.$$

Let $y \varepsilon R$, and $x \varepsilon L$. Then

$$O = -(g,x,y) = (y,x,g) = = (yx)g - yx.$$

Therefore $yx \varepsilon L$ which implies L is a left ideal of R Since $g \varepsilon L$, $L \neq (O)$ Then Lemma 3 implies that $L = R$, and thus g is a right identity element of R, therefore it is the identity element of R.

5 SEMI–SIMPLE RINGS

Lemma 5 Let Q denote the intersection of all regular maximal right ideals in R. Then Q is a two sided ideal of R.

Proof. Define an ideal B of R to be a primitive ideal if the ring R/B is a primitive ring. We prove that the intersection of all the primitive ideals of R is Q.

Suppose that B is a primitive ideal of R. R/B is a primitive ring: therefore by Theorem 4, R/B is either a simple ring with an identity element or it is an associative ring. In either case the intersection of the regular maximal right ideal of R/B is zero; if R/B is simple then Lemma 3, R/B has no one sided proper ideals. If R/B is associative, Jacobson's density theorem implies that the intersection of all regular maximal right ideals of the associative primitive ring R/B is zero. Thus, the intersection of all regular maximal right ideals of R is contained in B for any primitive ideal B of R.

Conversely, intersection of all primitive ideals of R is contained in every regular maximal right ideal A of R: this is clear by the maximality of A, for every such A. Thus the intersection of all primitive ideals is Q and hence Q is an ideal of R.

Definition. Q is defined to be the *radial* of R. R is called semisimple of $Q = (O)$.

Theorem 6. Let R be a semisimple ring. Then, R is a subdirect sum of simple rings with an identity element and associative primitive rings.

Proof. R is semisimple ; therefore $Q=(O)$ which is the intersection of all primitive ideals $\{B\}$ of R. Then R is a subdirect sum of rings each of which is isomorphic to R/B for some primitive ideal of B of R. Since each R/B is a primitive ring, Theorem 4 implies the conclusion. Thus, R/Q is a subdirect sum of simple rings with an identity element and associative primitive rings.

REFERENCES

1. E. Kleinfeld, Standard and accessible rings, *Canad. J Math* **8** (1956), 335–340

2. E, Kleinfeld, Assosymmetric rings. *Proc. Amer. Math. Soc.* **8** (1957), 983–986.

3. Yash Paul, On prime accessible rings, *Univ. Nac. Tucumán Rev. Ser- A* (to appear).