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A STUDY OF SOME MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

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ABSTRACT

We study here certain multidimensional fractional integral operators involving a general multivariable function in their kernel. We give five basic properties of these operators, and then establish two theorems and two corollaries, which are believed to be new. These basic theorems exhibit structural relationships between the multidimensional integral transforms. The one-and two-dimensional analogues of these results, which are new and of interest in themselves, can easily be deduced. Special cases of these latter theorems will give rise to certain known results obtained from time to time by several earlier authors.

I. Introduction and Definitions

Fractional integral operators have been defined and studied by various, authors notably by Riemann and Liouville [4], Weyl [4], Erdélyi [1,2], Kober [5], Sneddon [11], Kalla [18] and Saxena [13] (See also Srivastava and Buschman [6]). These oberators play an important role in the theory of integral equations and problems concerning mathematical physics. In this paper, we shall study the following multidimensional fractional integral operators having the general function ϕ as their kernel. Also, for the sake of brevity, we shall use the symbol $f(\mathbf{x})$ to represent $f(x_1, \ldots, x_r)$.

$$R\{f(\mathbf{x})\} = R\{f(\mathbf{x}); t_1, \ldots, t_r; \gamma_1, \ldots, \gamma_r\}$$

$$= \prod_{i=1}^{r} (t_i^{-\gamma_i - 1}) \int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r} (x_i^{\gamma_i}) \, \phi(x_1/t_1, \ldots, x_r/t_r) \times$$

$$\times f(\mathbf{x}) \, dx_1 \ldots dx_r \quad (1.2)$$

and

$$W\{f(\mathbf{x})\} = W\{f(\mathbf{x}); t_1 \ldots t_r; \delta_1, \ldots, \delta_r\}$$

$$= \prod_{i=1}^{r} (t_i^{\delta_i}) \int_{t_1}^{\infty} \cdots \int_{t_r}^{\infty} \prod_{i=1}^{r} (x_i^{-\delta_i - 1}) \, \phi(t_1/x_1, \ldots, t_r/x_r) \times$$

$$\times f(\mathbf{x}) \, dx_1 \ldots dx_r$$

where the kernel ϕ is such that the above integrals make sense. The above operators exist under the following conditions.

(i) $p_i \leqslant 1, q_i < \infty, p_i^{-1} + q_i^{-1} = 1, i = 1, \ldots, r$

(ii) $Re(\gamma_i) > -q_i^{-1}, Re(\delta_i) > -p_i^{-1}, i = 1, \ldots, r$

(iii) $f(\mathbf{x}) \in L_{p_i}((0, \infty) \times \ldots \times (0, \infty)), i = 1, \ldots, r$

The following special case of the multidimensional fractional integral operators involviug product of Gauss's hypergeometric functions [14, p. 153, eqn (i) and (2)] will be used in Section 3.

$$I\{f(\mathbf{x})\} = \prod_{i=i}^{r} \left\{ \frac{t_i^{-\gamma_i - 1}}{\Gamma(1-a_i)} \right\} \int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r}$$

$$\left\{ x_i^{\gamma_i} \, {}_2F_1\left(a_i, \beta_i + m; \beta_i; \frac{x_i}{t_i}\right) \right\} \times f(\mathbf{x}) \, dx_1 \ldots dx_r \quad (1.3)$$

and

$$K\{f(\mathbf{x})\} = \prod_{i=1}^{r} \left\{ \frac{t_i^{\delta_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \cdots \int_{t_r}^{\infty} \prod_{i=1}^{r}$$

$$\left\{x_i^{-\delta_i-1}\ {}_2F_1\left(\alpha_1, \beta_i + m; \beta_i; \frac{t_i}{x_i}\right)\right\} \times f(\mathbf{x})\, dx_1 \ldots dx_r \qquad (1\ 4).$$

The conditions of existence of these operators follow easily from the conditions given in the paper referred to above.

The generalized multidimensional integral transform T, defined below, will also be required during the course of our further study.

$$T\{f(\mathbf{x})\ ; s_1, \ldots, s_r\} = \int_0^\infty \cdots \int_0^\infty k(s_1k_1, \ldots, s_rx_r)\ f(\mathbf{x})\ dx_1 \ldots dx_r \qquad (1.5)$$

where $k\ (s_1x_1, \ldots, s_rx_r)$ is the kernel of the transform T and the multiple integral occurring in the equation (1.5) is assumed

The following multivariable H-function transform which is a special case of the transform defined by Srivastava and Panda [9, *pt. I*, *p.* 121, eqn. (1.15)], will also be used in the sequel:

$$H\{f(\mathbf{x}); s_1, \quad , s_r\} = \int_0^\infty \cdots \int_0^\infty H \begin{matrix} 0, 0: (m', n')\ ; \ldots ; (m^{(r)}, n^{(r)}) \\ P, Q: [p', q'] , \ldots , [p^{(r)}, q^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [\ (a): A'\ , \ldots, A^{(r)}]\ :\ [\ (b'): B']\ ; \ldots ; [\ (b^{(r)}): B^{(r)}] \\ [\ (c): C'\ , \ldots, C^{(r)}]: [\ (d'): D']\ ; \ldots ; [\ (d^{(r)}: D^{(r)}] \end{matrix} ; s_1x_1, \ldots, s_rx_r \right)$$

$$\times f(\mathbf{x})\ dx_1 , \ldots, dx_r \qquad (1.6)$$

The transform defined above will be denoted symbolically as follows :

$$H\left\{ f(\mathbf{x}); \begin{matrix} [\ (a): A'\ , \ldots, A^{(r)}]\ :\ [\ (b'): B'], \ldots, [\ (b^{(r)}): B^{(r)}] \\ [\ (c): C'\ , \ldots, C^{(r)}]: [\ (d'): D'], \ldots, [(d^{(r)}): D^{(r)}] \end{matrix} ; s_1, \ldots, s_r \right\} \qquad (1.7)$$

The kernel of the transform given by (1.6) is a special case of the multivariable H-function defined by Srivastava and Panda in another paper [7, p. 271, eqn. (4.1) to (4.3)]. For explanation of the various symbols, notations and conditions of existence of multivariable H-function we refer to the paper mentioned above (see also Srivastava, Gupta and Goyal [10], p. 251, eqn. (C.1) et seq). Also, for the sake of brevity, three dots (...) occurring in a multivariable H-function at a particular place will indicate that the parameters at those places are same as those of multivariable H-function occurring in (1.7).

2. SOME BASIC PROPERTIES

Property I. If $f(x) \varepsilon L_{p_i}((0, \infty) \times \ldots \times (0, \infty))$, $1 \leqslant p_i \leqslant 2$ (or $f(\mathbf{x}) \varepsilon M_{p_i}((0, \infty) \times \ldots \times (0, \infty))$, $p_i > 2$, where $M_{p_i}((0,\infty) \times \ldots \times (0, \infty))$ denotes the class of all functions $f(x) \varepsilon L_{p_i}((0, \infty) \times \ldots \times (0, \infty)$ with $p_i > 2$, which are the inverse Mellin transforms of functions belonging to $L_{q_i}((-\infty, \infty) \times \quad \times (-\infty, \infty))$, Re $(\gamma_i) > -q_i^{-1}$, Re $(\delta_i) > -p_i^{-1}$, $p_i^{-1} + q_1^{-1} = 1$ $(i=1, \quad ,r)$ and the multidimensional Mellin transform of the function $f(\mathbf{x})$ exists, then

(a) $M[R\{f(\mathbf{x}); t_1, \ldots, t_r; \gamma_1, \ldots, \gamma_r\}; s_1, \ldots, s_r]$

$$= M[f(\mathbf{x}); s_1, \ldots s_r] W\{1; t_1, \ldots, t_r; \gamma_1 - s_1 + 1, \ldots, \gamma_r - s_r + 1\} \qquad (2.1)$$

(b) $M[W\{f(\mathbf{x}); t_1, \ldots, t_r; \delta_1, \ldots, \delta_r\}; s_1, \ldots, s_r]$

$$= M[f(\mathbf{x}); s_1, \ldots, s_r] R\{1; t_1, \ldots, t_r; \delta_1 + s_1 + 1, \quad, \delta_r + s_r + 1\} \qquad (2.2)$$

where the symbol M occurring in (2.1) and (2.2) stands for the multi dimensional Mellin transform defined in the following way:

$$M[f(\mathbf{x}); s_1, \quad, s_r] = \int_0^\infty \cdots \int_0^\infty f(\mathbf{x}) \prod_{i=1}^{r} (x_i^{s_i - 1}) dx_1 \ldots dx_r \quad (2.3)$$

provided that the multiple integral involved in (2.3) exists,

Proof :- To prove (2.1), we use (2.3) and (1.1) to obtain

$$M[R\{f(\mathbf{x})\}] = \int_0^\infty \cdots \int_0^\infty (t_i^{s_i-1})$$

$$\Bigg\{ \prod_{i=1}^{r} (t_i^{-\gamma_i - 1}) \int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r} (x_i^{\gamma_i}) f(\mathbf{x}) \times$$

$$\times \phi(x_1/t_1, \ldots, x_r/t_r)\, dx_1 \quad dx_r \Bigg\} dt_1 \ldots dt_r \quad (2.4)$$

Now interchanging the orders of t_i and x_i $(i=1,\ldots,r)$ integrals in (2.4), which is easily seen to be permissible under the conditions stated with (2.1) and interpreting the result thus obtained with the help of (1.2), we arrive at the required result (2.1).

The result (2.2) can be established in a similar manner.

Property II If $f(x) \varepsilon L_{p_i}((0,\infty) \times \ldots \times (0,\infty))$,

$g(x) \varepsilon L_{q_i}((0,\infty) \times \ldots \times (0,\infty))$, $p_i^{-1} + q_i^{-1} = 1$,

$\text{Re}(\gamma_i) > \max.(-p_i^{-1}, -q_i^{-1})$, $(i=1,\ldots,r)$ then

$$\int_0^\infty \cdots \int_0^\infty f(\mathbf{x})\, R\{g(\mathbf{x})\}\, dx_1 \ldots dx_r$$

$$= \int_0^\infty \cdots \int_0^\infty g(\mathbf{x})\, W\{f(\mathbf{x})\}\, dx_1 \ldots dx_r \quad (2.5)$$

provided that the multiple integrals involved in (2.5) are absolutely convergent.

Property III.

(a) $R\{f(\mathbf{wx});\ t_1,\dots,t_r;\ \gamma_1,\dots,\gamma_r\} = R\{f(\mathbf{x});\ t_1w_1,\dots,t_rw_r;\ \gamma_1,\dots\gamma_r\}$ (2.6)

(b) $W\{f(\mathbf{wx});\ t_1,\dots,t_r;\ \delta_1,\dots,\delta_r\} = W\{f(\mathbf{x});\ t_1w_1,\dots,t_rw_r;\ \delta_1,\dots,\delta_r\}$ (2.7)

provided that the multiple integrals involved in (2.6) and (2.7) are absolutely convergent.

Property IV.

(a) $R\{f(\mathbf{1}/\mathbf{x});\ t_1,\dots,t_r;\ \gamma_1,\dots,\gamma_r\} = R\{f(\mathbf{x});\ 1/t_1,\dots,(1/t_r);$
$\gamma_1+1,\dots,\gamma_r+1\}$ (2 8)

(b) $W\{f(\mathbf{1}/\mathbf{x});\ t_1,\dots,t_r;\ \delta_1,\dots,\delta_r\} = W\{f(\mathbf{x});\ (1/t_1),\ \ ,(1/t_r);$
$\delta_1-1,\dots,\delta_r-1\}$ (2.9)

provided that the multiple integrals involved in (2.8) and (2 9) are absolutely convergent.

Property V.

(a) $R\left\{\prod_{i=1}^{r}(x_i^{\beta_i})\,f(\mathbf{x});\ t_1,\dots,t_r;\ \gamma_1,\dots,\gamma_r\right\}$

$$= \prod_{i=1}^{r}(t_i^{\beta_i})\,R\{f(\mathbf{x});\ t_1,\dots,t_r;\ \gamma_1+\beta_1,\dots,\gamma_r+\beta_r\} \qquad (2.10)$$

(b) $W\left\{\prod_{i=1}^{r}(x_i^{\beta_i})\,f(\mathbf{x});\ t_1,\dots,t_r;\ \delta_1,\dots,\delta_r\right\}$

$$= \prod_{i=1}^{r} (t_i^{\beta_i}) \, W \{ f(\mathbf{x}); t_1 ,..., t_r; \delta_1 - \beta_1 ,..., \delta_r - \beta_r \} \qquad (2.11)$$

provided that the multiple integrals involved in (2.10) and (2.11) are absolutely convergent.

The proofs of the Properties II, to V are straight forward and follow from the definitions (1.1) and (1.2) of the multidimensional fractional integral operators involved therein.

3. Relationship Between Multidimensional Fractional Integral Operators And Multidimensional Integral Transforms

In this section, first we shall establish two most general theorems exhibiting interconnections between the fractional integral operators R and W defined by (1.1) and (1.2) respectively and the integral transform T defined by (1.5). Next, we give two interesting corollaries interconnecting the multidimensional fractional integral operators defined by (1.3) and (1.4) and the multidimensional H-function transform defined by (1.6).

Theorem I. If

$$\tau (s_1 ,..., s_r) = T \{ \Psi (\mathbf{u}^{\rho}) \, g(\mathbf{u}); s_1 ,..., s_r \}$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} k \, (\mathbf{su}) \, \Psi \, (\mathbf{u}^{\rho}) \, g \, (\mathbf{u}) \, du_1 ,..., du_r \qquad (3.1)$$

and

$$\Psi (t_1 ,..., t_r) = R \{ f(\mathbf{x}^{\sigma}); t_1 ,..., t_r \}$$

$$= \prod_{i=1}^{r} (t_1^{-\gamma_i - 1}) \int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r} (x_i^{\gamma_i}) \, f(\mathbf{x}^{\sigma})$$

$$\phi\,(x_1/t_1\,,\ldots,\,x_r/t_r)\times dx_1\ldots dx_r \tag{3.2}$$

then

$$\tau\,(s_1\,,\ldots,\,s_r) = \frac{1}{\rho_1\ldots\rho_r}\int_0^\infty\ldots\int_0^\infty f(\mathbf{x}^\sigma)$$

$$\theta\,(s_1\,,\ldots,\,s_r,\,x_1\,,\ldots,\,x_r,\,\rho_1\,,\ldots,\,\rho_r)\,dx_1\ldots dx_r \tag{3.3}$$

where

$$\theta\,(s_1\,,\ldots,\,s_r,\,t_1\,,\ldots,\,t_r,\,\rho_1\,,\ldots,\,\rho_r)$$

$$= W\left\{\prod_{i=1}^{r}\,(x_i^{1/\rho_i-1})\,g\,(\mathbf{x}^{1/\rho})\,k\,(\mathbf{sx}^{1/\rho});\,t_1\,,\;\;,\,t_r\right\}$$

$$= \prod_{i=1}^{r}\,(t_i^{\gamma_i})\int_{t_1}^\infty\ldots\int_{t_r}^\infty\prod_{i=1}^{r}\,(x_i^{1/\rho_i-\gamma_i-2})\,g\,(\mathbf{x}^{1/\rho})\,k\,(\mathbf{sx}^{1/\rho})\times$$

$$\times\,\phi\,(t_1/x_1\,,\ldots,\,t_r/x_r)\,dx_1\ldots dx_r \tag{3.4}$$

ρ_i and σ_i $(i = 1,\ldots,r)$ are non zero real numbers of the same sign and all the integrals involved in equations (3.1) to (3.4) are assumed to be absolutely convergent. Also in (3.6) $k\,(\mathbf{sx}^{1/\sigma})$ stands for $k\,(s_1x_1^{1/\rho_1}\,,\ldots,\,s_rx_r^{1/\rho_r})$ and so on.

Proof. Applying the formula (2.5) to the pair of equations (3.2) and (3. 4), we get:

$$\int_0^\infty\ldots\int_0^\infty f\,(\mathbf{x}^\sigma)\,\theta\,(s_1\,,\ldots,\,s_r,\,x_1\,,\ldots,\,x_r,\,\rho_1\,,\ldots,\,\rho_r)\,dx_1\ldots dx_r$$

$$\int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{r}\,(x_i^{1/\rho_i-1})\,g\,(\mathbf{x}^{1/\rho})\,k\,(\mathbf{sx}^{1/\rho})\,\Psi\,(\mathbf{x})dx_1\ldots dx_r$$

(3.5)

Now changing the variables of integrations on the right hand side of (3.5) slightly and interpreting the result thus obtained with the help of (3.1) we easily arrive at (3.3) after little simplification.

If in the above theorem, we replace ρ_i by h_i $(i=1 ,\ldots, r)$ and take

$$g(\mathbf{x}) = \prod_{i=1}^{r} \left\{ x_i^{h_i(c_i+1)-1} \right\}, \quad \phi(\mathbf{x}) = \prod_{i=1}^{r} \left\{ \frac{1}{\Gamma(1-a_i)} {}_2F_1(a_i, \beta_i+m; \beta; \mathbf{x}_i) \right\}$$

and T to be multidimensional H-function transform defined by (1.6), the right hand side of equation (3.4) assumes the following form:

$$\prod_{i=1}^{r} \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-a_i)} \right\} \int_{t_1}^{\infty} \cdots \int_{t_r}^{\infty} \prod_{i=1}^{r} \left\{ x_i^{c_i-\gamma_i-1} \; {}_2F_1(\alpha_i, \; \beta_i+m; \; t_i/x_i) \right\}$$

$$H \begin{pmatrix} s_1 x_1^{i/h_i} \\ \vdots \\ s_r x_r^{1/h_r} \end{pmatrix} dx_1 \ldots dx_r \qquad (3.6)$$

On evaluating the above integral with the help of known results [(4, p 398, eqn. (2)] and [3, p. 105, eqn. (1)]) and the definition of multivariable H-function [7, p. 271, eqn (4.1)] and substituting the value of θ $(s_1 ,\ldots, s_r, t_1 ,\ldots, t_r, \rho_r ,\ldots, \rho_r)$ thus obtained, in the right hand side of equation (3.3), we obtain the following interesting corollary with the help of equations (3.1) and (3.3).

Corollary 1. If $h_i > 0$, Re $(1-a_i) > m$, $m \varepsilon W$, (the set of whole number) σ_i $(i = 1 ,\ldots, r)$ are non zero real numbers ofthe same sign.

$\beta_i \neq 0, -1, -2 ,\ldots,$ Re $(C_i) \geqslant 0$

$f(x) = O\left(x_i^{A_i}\right)$ for small values of x_i

$$= O\left(\left(x_i^{B_i} \; e^{-C_i x_i} \right) \right) \text{ for large values of } x_i,\ i=1,\dots,r,$$

the multivariable H-function satisfy the conditions corresponding appr-opriately to those given by Srivastava and Panda [8, p. 130, equations (1.6) to (1.10)]

then

$$H\left\{ \prod_{i=1}^{r} \left(x_i^{h_i(c_i+1)-1} \, \Psi(\mathbf{x}^h) ; \begin{array}{l} \dots : [(b') : B']_{1,p'} \\ \dots : (1-\alpha_1+\gamma_1-c_1,\ 1/h_1), \end{array} \right.\right.$$

$$\left. \begin{array}{l} ;\dots; [(b^{(r)}) : B^{(r)}]_{1,p^{(r)}} \\ [(d'):D']_{2,q'-1}, (1-\beta_1+\gamma_1-m-c_1, 1/h_1); \dots ;(1-\alpha_r+\gamma_r-c_r, \\ 1/h_r), [(d^{(r)}) : D^{(r)}]_{2,q^{(r)}-1}, (1-\beta_r+\gamma_r-m-c_r, 1/h_r) \end{array} ;s_1,\dots,s_r \right\}$$

$$= \frac{1}{\prod_{i=1}^{r} \{ (\beta_i)_m \}} H\left\{ \prod_{i=1}^{r} \left(x_i^{h_i(c_i+1)-1} \right) f(\mathbf{x}^{h\sigma}); \right.$$

$$\begin{array}{l} \dots : [(b') : B']_{1,p'} \\ \dots : (\gamma_1-c_1, \frac{1}{h_1}), [(d') : D']_{2,q'-1}, (1-\beta_1+\gamma_1-c_1, 1/h_1), ;\dots; \end{array}$$

$$\left. \begin{array}{l} ;\dots; [(b^{(r)}) : B^{(r)}]_{1,p^{(r)}} \\ (\gamma_r-c_r, 1/h_r), [(d^{(r)}) : D^{(r)}]_{2,q^{(r)}-1}, (1-\beta_r+\gamma_r-c_r, 1/h_r) \end{array} ; s_1,\dots,s_r \right\}$$

(3.7)

where

$$\Psi(t_1, \quad, t_r) = I\{f(\mathbf{x}^\sigma); t_1, \ldots, t_r\}$$

$$= \prod_{i=1}^{r} \left(\frac{t_i^{-\gamma_i - 1}}{\Gamma(1-\alpha_i)} \right) \int_0^{t_1} \cdots \int_0^{t_r} \prod_{i=1}^{r} \left\{ x_i^{\gamma_i} {}_2F_1(\alpha_i, \beta_i + m; \beta_i; \frac{x_i}{t_i}) \right\}$$

$$f(\mathbf{x}^\sigma)\, dx_1 \ldots dx_r \qquad (3.8)$$

provided that

$$\mathrm{Re}\,\{ c_i - \gamma_i + \frac{1}{h_i} \frac{(b_j^{(i)} - 1)}{B_j^{(i)}} \} < 0, 1 \leqslant j \geqslant n^{(i)}$$

$$\mathrm{Re}\,\{ \sigma_i B_i + c_i + 1 + \frac{1}{h_i} (\frac{b_j^{(i)} - 1}{B_j^{(i)}}) \} < 0, 1 \leqslant j \leqslant n^{(i)},$$

$$\mathrm{Re}\,\{\sigma_i A_i + \gamma_i + 1\} > 0, \mathrm{Re}\,\{1 + c_i + \sigma_i A_i + \frac{1}{h_i} (\frac{d_k^{(i)}}{D_k^{(i)}})\} > 0,$$

$2 \leqslant k \leqslant m^{(i)}, i = 1, \ldots, r.$

Theorem II. If

$$\tau(s_1, \quad, s_r) = T\{\Psi(\mathbf{u}^\rho)\, g(\mathbf{u}); s_1, \ldots, s_r\}$$

$$= \int_0^\infty \cdots \int_0^\infty k(\mathbf{su})\, \Psi(\mathbf{u}^\rho)\, g(\mathbf{u})\, du_1 \ldots du_r \qquad (3.9)$$

and

$$\Psi(t_1, \ldots, t_r) = W\{f(\mathbf{x}^\sigma); t_1, \quad, t_r\}$$

$$= \prod_{i=1}^{r} (t_i^{\gamma_i}) \int_{t_1}^\infty \cdots \int_{t_r}^\infty \prod_{i=1}^{r} (x_i^{-\gamma_i - 1})\, \phi(t_1/x_1, \ldots, t_r/x_r)\, f(\mathbf{x}^\sigma)$$

$$dx_1 \ldots dx_r \qquad (3.10)$$

then

$$\tau(s_1,\ldots,s_r) = \frac{1}{\rho_1\cdots\rho_r}\int_0^\infty \cdots \int_0^\infty f(\mathbf{x}^\sigma)\,\theta(s_1,\ldots,s_r,$$

$$x_1,\ldots,x_r,\rho_1,\ldots,\rho_r)\,dx_1\ldots dx_r \tag{3.11}$$

where

$$\theta(s_1,\ldots,s_r,t_1,\ldots,t_r,\rho_1,\ldots,\rho_r)$$

$$= R\prod_{i=1}^{r}\left(x_i^{1/\rho_i-1}\right) g(\mathbf{x}^{1/\rho})\,k(\mathbf{s}\mathbf{x}^{1/\rho});\,t_1,\ldots,t_r\}$$

$$= \prod_{i=1}^{r}\left(t_i^{-\gamma_i-1}\right)\int_0^{t_1}\cdots\int_0^{t_r}\prod_{i=1}^{r}\left(x_i^{(1/\rho_i+\gamma_i-1)}\right) g(\mathbf{x}^{1/\rho})\,k(\mathbf{s}\mathbf{x}^{1/\rho})$$

$$\phi(x_1/t_1,\ldots,x_r/t_r)\,dx_1\ldots dx_r \tag{3.12}$$

ρ_i and σ_i $(i=1,\ldots,r)$ are nonzero real numbers of the same sign and the integrals involved in the equations (3.9) to (3.12) are assumed to be absolutely convergent.

Proof. The proof of the above theorem can be easily developed on the lines similar to those of theorem I.

Again, if in theorem II, we replace ρ_i by h_i $(i=1,\ldots,r)$ and take

$$g(\mathbf{x}) = \prod_{i=1}^{r}\left\{x_i^{h_i(c_i+1)-1}\right\},$$

$$\phi(\mathbf{x}) = \prod_{i=1}^{r}\left\{\frac{1}{\Gamma(1-a_i)}\,{}_2F_1(\alpha_i,\beta_i+m;\beta_i;x_i)\right\}$$

and T to be multidimensional H-function transform defined by (1.6), then proceeding on the lines similar to those of corollary I, we obtain the following interesting corollary:

Corollary II. If $h_i > 0$, Re $(1-a_i) > m$, $m \varepsilon W$ (The set of the whole numbers), σ_i $(i=1, \ldots, r)$ are non zero real numbers of the same sign, $\beta_i \neq 0, -1, -2, \ldots$, Re $(C_i) \geqslant 0$,

$$f(\mathbf{x}) = O(x_i^{A_i}) \text{ for small values of } \mathbf{x}_i$$

$$= O(x_i^{B_i} e^{-C_i x_i}) \text{ for large values of } x_i,\ i=1, \ldots, r,$$

the multivariable H-function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda [8, p. 130, equations (1.6) to (1.10)].

Then

$$H \left\{ \prod_{i=1}^{r} (x_i^{h_i(c_i+1)-1}) \Psi(\mathbf{x}^h); \begin{array}{l} \ldots : [(b'): B']_{1,p'} \\ \ldots : (\beta_1-\gamma_1-c_1-1, \frac{1}{h_1}), [(d'):D']_{2,q'-1}, (-\gamma_1-c_1, \frac{1}{h_1}) \\ ; \ldots ; [(b^r): B^{(r)}]_{1,p^{(r)}} \\ ; \ldots ; (\beta_r-\gamma_r-c_r-1, \frac{1}{h_r}), [(d^{(r)}): D^{(r)}]_{2,q^{(r)}-1}, (-\gamma_r-c_r, \frac{1}{h_r}) \end{array} ; s_1, \ldots, s_r \right\}$$

$$= \frac{1}{\prod_{i=1}^{r} \{(\beta_i)_m\}} H \left\{ \prod_{i=1}^{r} (x_i^{h_i(c_i+1)-1}) f(\mathbf{x}^{h\sigma}); \right.$$

$$\cdots : [\,(b') : B'\,]_{1,p'}$$

$$\ldots : (\beta_1-\gamma_1-c_1+m-1,\ 1/h_1),\ [(d') : D']_{2,\ q'-1},(a_1-\gamma_1-c_1-1,\ 1/h_1)$$

$$;\ldots;\ [\,(b^{(r)}) : B^{(r)}\,]_{1,\ p^{(r)}}$$

$$;\ldots;(\beta_r-\gamma_r-c_r+m-1,1/h_r),\ [(d^{(r)}):D^{(r)}]_{2,\ q^{(r)}-1},(a_r-\gamma_r-c_r-1,\ 1/h_r)\ ;$$

$$\left. s_1 ,\ldots, s_r \right\} \qquad (3.13)$$

where

$$\Psi\,(t_1 , \ , t_r) = K\,\{\,f\,(\mathbf{x}^{\sigma})\ ;\ t_1 ,\ldots, t_r\}$$

$$= \prod_{i=1}^{r} \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \cdots \int_{t_r}^{\infty} \prod_{i=1}^{r} \left\{ x_i^{-\gamma_i-1}\,{}_2F_1(\alpha_i,\beta_i+m;\beta_i;t_i/x_i) \right\}$$

$$f\,(\mathbf{x}^{\sigma})\ dx_1 \ldots dx_r \qquad (3.14)$$

provided that

$$\mathrm{Re}\,\{\gamma_i + c_i + 1/h_i\,(\frac{d_i}{D_j^{(i)}}) + 1\} > 0,\ 1 \leqslant j \leqslant m^{(i)},$$

$$\mathrm{Re}\,\{\sigma A_i+\beta_i- \gamma_i+m\} >0,\ \mathrm{Re}\,\{c_i+1+\sigma_i A_i+ 1/h_i\,(\frac{d_j^{(i)}}{D_j^{(i)}})\}> 0,$$

$$2 \leqslant j \leqslant m^{(i)},$$

$$\mathrm{Re}\,\{\,c_i+1+\sigma_i B_i +1/h_i\ (\frac{b_k^{(i)}-1}{B_k^{(i)}})\,\} < 0,\ 1 \leqslant k \leqslant n^{(i)},$$

$$i = 1 ,\ldots, r$$

The one-and two-dimensional analogues of Theorems I and II can easily be deduced but since the theorems contain a reasonably detailed analysis of the multidimensional case we prefer to omit their details. The corollaries I and II given earlier are also new. They give rise to interesting theorems involving multidimensional analogues of fractional integral operators defined by Kober [5], Riemann Liouville [4] and Weyl [4] and simpler multidimensional integral transforms, on suitably specializing the fractional integral operators and multidimensional H-function transform involved therein. Again, the one-and two-dimensional analogues of corollaries I and II, yield theorems esentially similar to those given earlier by Gupta, Goyal and Handa [12, pp 165-170]. Handa [15, pp. 200-204], Kalla [16, p. 1008, 1010; 17, pp. 54-56] and Mathur [19, pp. 108-121].

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