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AN EXPONENTIAL FOURIER SERIES FOR MULTIVARIABLE $\bar{H}$-FUNCTION

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ABSTRACT

An exponential Fourier serier for the H-function of several complex variables introduced by H. M. Srivastava and R. Panda[7] is obtained by using a result given by Srivastava and Panda [8]. An exponential Fourier ser.es for the H-Function of two variables follows as the special case of the main result. A recent result given by S. D. Bajpai [1] also follows as a special case of our main result.

I. Introduction

Recently, Srivastava and Panda [7, *p*. 271, *eqn*. (4. 1)] defined the H-function of several complex variables by means of the Mellin-Barnes in egral (see also [6, p. 251 *et seq*.])

$$(1.1)\ H^{0,\,\varepsilon\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C:\,(B',\,D')\,;\ldots;\,(B^{(r)},\,D^{(r)})}\left(\begin{matrix}[\,(a):\,\theta',\ldots,\,\theta^{(r)}]\,:\\ [\,(c):\,\Psi',\ldots,\,\Psi^{(r)}]\,:\end{matrix}\right.$$

$$\left.\begin{matrix}[\,(b')\,:\,\phi'\ ;\ldots;\,[\,(b^{(r)})\,:\,\phi^{(r)}\,]\,;\\ [\,(d')\,:\,\delta'\,]\,;\ldots;\,[\,(d^{(r)})\,:\,\delta^{(r)}\,]\,;\end{matrix}\ z_1\,,\ldots\,z_r\right)$$

$$=(2\pi\omega)^{-r}\int_{L_1}\cdots\int_{L_r}U_1(s_1)\,\ldots\,U_r(s_r)\,V\,(s_1\,,\ldots,\,s_r)$$

$$\cdot\, z_1^{s_1} \ldots z_r^{s_r}\, ds_1 \ldots ds_r, \quad \omega = \sqrt{-1},$$

where

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma[d_j^{(i)} - \delta_j^{(i)} s_i] \prod_{j=1}^{v^{(i)}} \Gamma[1 - b_j^{(i)} + \phi_j^{(i)} s_i]}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma[1 - d_j^{(i)} + \delta_j^{(i)} s_i] \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma[b_j^{(i)} - \phi_j^{(i)} s_i]},$$

$$i = 1, \ldots, r,$$

$$V(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\varepsilon} [1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i]}{\prod_{j=\varepsilon+1}^{A} \Gamma[a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i] \prod_{j=1}^{C} \Gamma[1 - c_j + \sum_{i=1}^{r} \Psi_j^{(i)} s_i]},$$

and, for convergence,

(1.2) $\quad |\arg z_i| < 1/2\, T_i \pi, \; i = 1, \ldots, r,$

where

(1.3) $$T_i = -\sum_{j=\varepsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \Psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \; i = 1, \ldots, r.$$

The following formulae are required in the proof :

(1.4) $$\int_0^{\pi} e^{(2m+1)ix} \sin(2n+1)x \, dx = 0, \text{ if } m \neq n,$$

$$= \pi i/2, \text{ if } m = n,$$

which has been given in ([3], p. 490, 3. 891).

$$(1.5)\quad \int_0^{\pi} \sin (2n+1)x\,(\sin x)^{\sigma}\, H\begin{matrix} 0,\ \varepsilon: (u',v');\ldots;(u^{(r)},v^{(r)}) \\ A,\ C: (B',D');\ldots;(B^{(r)},D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\right.$$

$$\left. z_1(\sin x)^{2h_1},\quad \ldots, z_r(\sin x)^{2h_r}\right) dx$$

$$= (-1)^n \sqrt{\pi}\, H\begin{matrix} 0,\ \varepsilon+2: (u',v');\ldots;(u^{(r)},v^{(r)}) \\ A+2,\ C+2: (B',D');\ldots;(B^{(r)},D^{(r)}) \end{matrix}$$

$$\left(\frac{[-\frac{\sigma}{2}:h_1,\ldots,h_r],\ [\frac{1-\sigma}{2};h_1,\ldots,h_r],\ [(a):\theta',\ldots,\theta^{(r)}]:}{[n+(\frac{1-\sigma}{2}):h_1,\ldots,h_r],\ [-n-(\frac{\sigma+1}{2}):h_1,\ldots,h_r],[(c):\Psi',\ldots,\Psi^{(r)}]:}\right.$$

$$\left.\begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1,\ldots,z_r\right)$$

where n is an integer, $h_i > 0$, $1 < i < r$, (1.2) and (1.3) hold, and

$$Re\left(\sigma + 2\sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}\right) > -1;\ j=1,\ldots,u^{(i)},$$

which is a known formula ([8], p. 178, eqn. (3.24)).

2 EXPONENTIAL FOURIER SERIES

The Fourier series to be established is

$$(2.1)\quad (\sin x)^{\sigma}\, H\begin{matrix} 0,\ \varepsilon: (u',v');\ldots;(u^{(r)},v^{(r)}) \\ A,\ C: (B',D');\ldots;(B^{(r)},D^{(r)}) \end{matrix}\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1(\sin x)^{2h_1},\ldots, z_r(\sin x)^{2h_r}\right)$$

$$= \frac{2(-1)^m}{\sqrt{\pi i}} \sum_{m=-\infty}^{\infty} H^{0,\, \varepsilon+2 \,:\, (u', v')\,;\, \;\,;\, (u^{(r)}, v^{(r)})}_{A+2,\, C+2 \,:\, (B', D')\,;\ldots;\, (B^{(r)}, D^{(r)})}$$

$$\left(\begin{matrix} [-\frac{\sigma}{2} : h_1 ,\ldots, h_r],\ [\frac{1-\sigma}{2} : h_1 ,\ldots, h_r],\ [(a): \theta' ,\ldots, \theta^{(r)}]: \\ [m+ (\frac{1-\sigma}{2}: h_1,\ldots,h_r],[-m-(\frac{1+\sigma}{2}: h_1,\ldots,h_r],\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] \,;\ldots;\, [(b^{(r)}) : \phi^{(r)}]; \\ [(d') : \delta'] \,;\ \ ;\, [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \; z_1 ,\ \ , z_r \right) . e^{(2m+1)ix},$$

where $h_i > 0$, $1 < i < r$, $0 < x < \pi$; $Re(\sigma) \gg 0$, and (1.2) and (1.3) hold true.

Proof : Let

$$(2.2) \quad f(x) = (sin\ x)^{\sigma} H^{0,\, \varepsilon :\, (u', v')\,;\ldots;\, (u^{(r)}, v^{(r)})}_{A,\, C :\, (B', D')\,;\ \ ;\, (B^{(r)}, D^{(r)})}$$

$$\left(\begin{matrix} [(a): \theta\ ,\ldots, \theta^{(r)}] : [(b'): \phi'] \,;\ldots;\, [(b^{(r)}) : \phi^{(r)}]; \\ [(c): \Psi' ,\ldots, \Psi^{(r)}] : [(b'): \delta'] \,;\ldots;\, [(b^{(r)}): \delta^{(r)}]; \end{matrix} \right.$$

$$\left. z_1 (sin\ x)^{2h_1} ,\ldots, z_r (sin\ x)^{2h_r} \right) = \sum_{m=-\infty}^{\infty} E_m\, e^{(2m+1)ix}, (0<x<\pi).$$

Equation (2.2) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(0, \pi)$.

Multiplying both sides of (2.2) by $\sin(2n+1)x$ and integrating with respect to x from 0 to π and using (1.4) and (1.5) we obtain

$$(2.3) \quad E_m = \frac{2(-1)^m}{\sqrt{\pi i}} H^{0,\, \varepsilon+2 :\, (u', v')\,;\ldots;\, (u^{(r)}, v^{(r)})}_{A+2,\, C+2 \,:\, (B', D')\,;\ldots;\, (B^{(r)}, D^{(r)})}$$

$$\left(\begin{matrix}[-\frac{\sigma}{2}: h_1, \ldots, h_r], [\frac{1-\sigma}{2}: h_1, \ldots, h_r], [(a): \theta', \ldots, \theta^{(r)}]: \\ [m+\frac{1-\sigma}{2}: h_1, \ldots, h_r], [-m-\frac{1+\sigma}{2}: h_1, \ldots, h_r], [(c): \Psi', \ldots, \Psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'): \phi']; \ldots; [(b^{(r)}: \phi^{(r)}]; \\ [(d'): \delta']; \ldots; [(d^{(r)}): \delta^{(r)}];\end{matrix}\; z_1, \ldots, z_r\right),$$

From (2.2) and (2.3), the formula (2.1) follows.

3. Special Cases

(i) For $r = 2$, we get from (2.1) the corresponding Fourier series for the H-function of two variables defined by Mittal and Gupta [5].

$$(\sin x)^{\sigma}\, H^{0,\, \varepsilon\, :\, (u',\, v');\, (u'',\, v'')}_{A,\, C:\, (B',\, D');\, (B'',\, D'')}\left(\begin{matrix}[(a): \theta', \theta'']: [(b'): \phi']; [(b''): \phi'']; \\ [(c): \Psi', \Psi'']: [(d'): \delta']; [(d''): \delta''];\end{matrix}\; z_1(\sin x)^{2h_1}, \; z_2(\sin x)^{2h_2}\right)$$

$$= \frac{2(-1)^m}{\sqrt{\pi}\, i} \sum_{m=-\infty}^{\infty} H^{0,\, \varepsilon+2\, :\, (u',\, v');\, (u'',\, v'')}_{A+2,\, C+2\, :\, (B',\, D');\, B'',\, D'')}$$

$$\left(\begin{matrix}[-\frac{\sigma}{2}: h_1, h_2], [\frac{1-\sigma}{2}: h_1, h_2], [(a): \theta', \theta'']: \\ [m-(\frac{\sigma-1}{2}): h_1, h_2], [-m-\frac{\sigma+1}{2}: h_1, h_2], [(c); \Psi', \Psi'']:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'): \phi']; [(b''): \phi'']; \\ [(d'): \delta']; [(d''): \delta''];\end{matrix}\; z_1, z_2\right)$$

(ii) Letting $A=0=C=\varepsilon=v''=B''=d''$, $u''=D''=\delta''=1$, $\sigma=1-2\rho$, $h_1=-h$ where $h>0$ and $z_2\to 0$ in case (i) above, then in view of the following formulae ([2], *p.* 18, *eqn.* 1. 5. 4)

$$H_{0,\,0:\,(B',\,D');\,(0,\,1)}^{0,\,0:\,(u',\,v'):\,(1,\,0)}\left(\begin{matrix}---:\,(b_B,\,\phi_B);\,---;\\ ---:\,(d_D,\,\delta_D);\,(0,\,1);\end{matrix}\;z_1,\,z_2\right)$$

$$=e^{-z_2}\,H_{B',\,D'}^{u',\,v'}\left[z_1\left|\begin{matrix}(b_B,\,\phi_B)\\ (d_D,\,\delta_D)\end{matrix}\right.\right]$$

and ([4], *p.* 13, (iii))

$$H_{B+2,\,D+2}^{u,\,v+2}\left[z\left|\begin{matrix}(-\frac{\sigma}{2},\,h),\,(\frac{1-\sigma}{2},\,h),\,(b_B,\,\phi_B)\\ (d_D,\,\delta_D),\,(\frac{1}{2}+m-\frac{\sigma}{2},h),\,(-m-\frac{\sigma}{2}-\frac{1}{2},\,h)\end{matrix}\right.\right]$$

$$=(-1)^m\,H_{B+2,\,D+2}^{u+1,\,v+1}\left[z\left|\begin{matrix}(-\frac{\sigma}{2},h)\,(b_B,\,\phi_B),\,(b_B,\,\phi_B),\,(\frac{1-\sigma}{2},h)\\ (m-\frac{\sigma-1}{2},h),\,(d_D,\delta_D),\,(-m-\frac{\sigma+1}{2},h)\end{matrix}\right.\right],$$

we obtained the result given by Bajpai [1].

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