Jñānābha, Vol. 14, 1984

# THE RAYLEIGH-TAYLOR INSTABILITY THROUGH POROUS MEDIUM OF VISCOELASTIC FLUID IN THE PRESENCE OF A HORIZONTAL MAGNETIC FIELD

*By*

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(*Received: November 1, 1982; Revised: August 3, 1983*)

## ABSTRACT

The character of an incompressible, viscous, finitely conducting, rotating viscoelastic fluid through porous medium in the presence of a horizontal magnetic field along the direction of gravitational field has been investigated. It has been shown that the solution is characterized by a variational principle. Based on the existence of variational principle, an approximate solution has been derived for the case of a fluid having exponentially varying density in the vertical direction. Due to permeability of the medium it is found that permeability has a stabilizing influence for small wave numbers of disturbance but it has a destabilizing effect for large wave numbers.

## 1. INTRODUCTION

Rayleigh [1] investigated the equilibrium of a stratified inviscid fluid and found that the equilibrium of a horizontal layer of an incompressible fluid variable density is stable or unstable according as the density decreases everywhere or increases anywhere. Chandrasekhar [1] considered the effect of viscosity on Rayleigh-Taylor instability and he found though that new modes are introduced by viscosity, the latter does not effect the criterion of stability.

Talwor [6] discussed the effect of a horizontal magnetic field on the character of equilibrium of an inviscid, incompressible, infinitely conducting, rotating fluid of variable density. Sharma and Ariel [3] further investigated the character of equilibrium of an inviscid, incompressible, finitely conducting, rotating fluid of variable density in the presence of a horizontal magnetic field and tried to find out to what extent the instability of the configuration is affected by changing the resistivity of the medium. Sharma [4] studied the Rayleigh-Taylor instability of the plane interface between two viscoelastic superposed conducting fluids through porous medium when the whole system is immersed in a uniform horizontal megnetic field.

In the present paper, we investigate the effect of permeability of the porous medium on the equilibrium of the viscoelastic, finitely conducting, rotating fluid of variable density in the presence of a horizontal magnetic field. The effect of permeabilty on the dispersion relation obtained for the ideal fluid is then discussed in both the long and short wavelength limits.

## 2. BASIC EQUATIONS

Cosider the motion of an incompressible, finitely conducting viscoelastic flu'd through porous medium in the presence of a uniform magnetic $\mathbf{H}$ $(= 0, 0, H)$ Let the configuration rotate uniformly with an angular velocity $\Omega$ about z axis. Let $\mathbf{u}$ $(=u,v,w)$, $\mathbf{h}$ $(=h_x,h_y,h_z)$, $\delta p$ and $\delta\rho$ denote the perturbations in velocity, magnetic field, pressure and density respectively. Then the linearized hydromagnetic perturbation equations of viscoelastic fluid throug porous medium are

$$(1+\lambda\frac{\partial}{\partial t})\,\rho\,\frac{\partial u}{\partial t} = (1+\lambda\frac{\partial}{\partial t})\,[-\ \nabla\delta p + \mathbf{g}\delta\rho$$

$$+ \frac{1}{4\pi} (\nabla \times \mathbf{h}) \times \mathbf{H} + 2\rho (\mathbf{u} \times \underline{\Omega})]$$

$$+ (1 + \lambda_0 \frac{\partial}{\partial t}) [\mu \nabla^2 \mathbf{u} - \frac{\mu}{k_1} + (\frac{\partial w}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial z}) \frac{d\mu}{dz}], \quad (1)$$

$$div\ \mathbf{u} = 0, \quad (2)$$

$$div\ \mathbf{h} = 0, \quad (3)$$

$$\frac{\partial h}{\partial t} = curl\ (\mathbf{u} \times \mathbf{H}) + \eta \nabla^2 \mathbf{H} \quad (4)$$

$$\frac{\partial}{\partial t} (\delta\rho) = -(\mathbf{u}. \nabla)\rho, \quad (5)$$

where, $\mu$, $\mathbf{g}$ $(=0, 0, -g)$, $\eta$, $\lambda$, $\lambda_0$ $(\lambda_0 < \lambda)$, $k_1$ and $\mathbf{x}$ denote the viscosity of the fluid, the acceleration due to gravity, resistivity of the medium, the stress relaxation time, the strain retardation time, permeability of the medium and the position vector respectively.

Analysing the disturbance into normal modes, we seek solutions whose dependence on $x$, $y$ and $t$ is given by

$$f(z)\ exp\ (ik_x x + ik_y y + nt), \quad (6)$$

where $k_x$ and $k_y$ are the horizontal components of the wave number vector $\mathbf{k}$ and $n$ is the growth rate of disturbance,we obtain the following set of equations in $w$, $h$, $\xi$ ond $\zeta$, after eliminating some of the quan tities from equations (1)-(5):

$$n [k^2 \rho w - D(\rho D w)] - \frac{gk^2}{n} (D\rho) w + ik_x \frac{H}{4\pi} (D^2 - k^2) h_z - D(2\rho\Omega\zeta)$$

$$+ \frac{1+\lambda_0 n}{1+\lambda n} [\mu (D^2 - k^2)^2 w + 2D\mu (D^2 - k^2) Dw + D^2\mu (D^2 + k^2) w$$

$$+\frac{1}{k_1}\,(k^2\mu w - D(\mu Dw)\,)\,] = 0, \tag{7}$$

$$[n\,\rho - \frac{1+\lambda_0 n}{1+\lambda n}\,\mu(D^2 - k^2 - \frac{1}{k_1}) - D\mu D_\rfloor \zeta - ik_x \frac{H}{4\pi}\xi = 2\rho\Omega\, Dw, \tag{8}$$

$$[n - \eta(L^2 - k^2)\,]\, h_z = ik_x\, Hw, \tag{9}$$

$$[\,n - \eta(\mathrm{D}^2 - k^2)\,]\,\xi = ik_x\, H\zeta, \tag{10}$$

where

$$\zeta = ik_x v - ik_y u, \tag{11}$$

and

$$\xi = ik_x h_y - ik_y h_x, \tag{12}$$

are the vertical components of the vectors of curl **u** and curl **h** respectively.

## 3. Boundary Conditions

We assume that the fluid is confined between two rigid planes $z = 0$ and $z = d$, since the normal velocity at the bonudary surface vanishes hence, we have

$$w = 0 \text{ at a rigid boundary}, \tag{13}$$

For electromagnetic boundary counditions, we see that if the fluid is bounded by an ideal conductor, no disturbance within the fluid can charge $E$ and $H$ outside the fluid. Since surface charges and surface currrents can allow discontinuities in $E_z$, $h_x$ and $h_y$, we must require that

$$h_z = E_x = E_y = 0 \tag{14}$$

which leads to

at a surface boundary by an ideal conductor.

We do not require a boundary condition on $\zeta$ for the value of $\zeta$ can be obtained by substituting the value of $\xi$ in eqn. (10)

## 4. A Variational Principle

Multiplying the equation (7) for the characteristic value $n_i$ by $w_j$ and integrating across the vertical extent of the fluid we obtain the following equations, after a series of integrations by parts

$$n_i\,(I_1 - I_9) - \frac{gk^2}{n_i}\,I_2 + n_j\,(I_7 - I_4 - I_5)$$

$$+ \frac{1+\lambda_0 n}{1+\lambda n}\,(I_3 + I_8) - \eta\, k^2\,(I_4 + 2I_5 + I_6 + I_9 + I_{10}) = 0\,, \tag{16}$$

where

$$I_1 = \int_0^d \rho(k^2\, w_i\, w_j + Dw_i\, Dw_j)\, dz, \tag{17}$$

$$I_2 = \int_0^d (D\rho)\, w_i\, w_j\, dz, \tag{18}$$

$$I_3 = \int_0^d \mu\,[(D^2 + k^2)\, w_i\, (D^2 + k^2) w_i + 4k^2\, Dw_i\, Dw_j$$

$$+ \frac{1}{k_1}\,(k^2\, w_i\, w_j + Dw_i\, Dw_j)\,]\, dz, \tag{19}$$

$$I_4 = \frac{k^2}{4\pi}\int_0^d h_i\, h_j\, dz, \tag{20}$$

$$I_5 = \frac{1}{4\pi}\int_0^d Dh_i\, Dh_j\, dz, \tag{21}$$

$$I_6 = \frac{1}{4\pi k^2}\int_0^d D^2 h_i\, D^2 h_j\, dz, \tag{22}$$

$$I_7 = \int_0^d \zeta_i \, \zeta_j \, dz, \tag{23}$$

$$I_8 = \int_0^d \mu \left[ \left( k^2 + \frac{1}{k_1} \right) \zeta_i \, \zeta_j + D\zeta_i \, D\zeta_j \right] dz, \tag{24}$$

$$I_9 = \frac{1}{4\pi} \int_0^d \xi_i \, \xi_j \, dz, \tag{25}$$

$$I_{10} = \frac{1}{4\pi k^2} \int_0^d D\xi_i \, D\xi_j \, dz. \tag{26}$$

If we write $i = j$ in equation (16), we obtain

$$n(I_1 + I_7 - I_4 - I_5 - I_9) - \frac{gk^2}{n} I_2 + \frac{1+\lambda_0 n}{1+\lambda n} (I_3 + I_8) -$$

$$- \eta \, k^2 \, (I_4 + 2I_5 + I_6 + I_9 + I_{10}) = 0 \tag{27}$$

Now by considering a small change $\delta n$ in $n$ consequent upon first arbitrary variations $\delta w$, $\delta h$, $\delta \zeta$ and $\delta \xi$ in $w$, $h$, $\zeta$ and $\xi$ respectively which the boundary conditions of the problem, we can show, by proc–eeding along standard lines, that a necessary and sufficient conditions for $\delta n$ to vanish is that $w$, $h$, $\zeta$ and $\xi$ be solutions of the characteristic value problem.

## 5. A continuously stratified fluid of finite depth

In this section, a use will be made of the existence of the variatio–nal principle to obtain the solution of the problem of a continuously stratified fluid layer of depth $d$ in which the undisturbed density distribution is

$$\rho(z) = \rho_0 \, exp \, \beta z, \tag{28}$$

where $\rho_0$ is the density at the lower boundary $\beta$ is constant.

A further assumption, namely that $|\beta d| << 1$ is made, implying that the density variation within fluid is a good deal less then the average density. We shall now consider the case of the fluid confined between two rigid boundaries $z=0$ and $z=d$ which are both ideally conducting. Therefore, the boundary conditions that are to be satisfied are

$$\left.\begin{aligned} w(0) &= w(d) = 0, \\ h(0) &= h(d) = 0, \\ D\xi(0) &= D\xi(d) = 0, \end{aligned}\right\} \tag{29}$$

Consistent with these boundary conditions, we assume the following trial functions for $w(z)$, $h(z)$ and $\xi(z)$ as

$$w(z) = W \sin lz,\ h(z) = K\cos lz,\ \xi = X \sin lz,$$

where $l(=\pi s/d)$, $s$ is an intger.

The value of trial function $\zeta(z)$ can be obtained by inserting the value of $\xi(z)$ from equation (10).

Evaluating the integrals defined in equetion (27) with assumed forms of $w$, etc., and eliminating the constants $W$, $K$ and $X$ with the help of equations (8)–(10), we obtain the following dispersion relation between $n$ and $k$:

$$[n_1 n_2+k_x{}^2V^2 (\lambda n+1)]\ [nn_1n_2 + nk_x{}^2V^2\ (\lambda n+1 - \frac{g\beta k^2}{l^2+k^2} n_2\ (\lambda n+1)]$$

$$+ \frac{4n\ \Omega^2\ l^2}{l^2+k^2}\ n^2{}_2\ (\lambda n + 1)^2 = 0, \tag{31}$$

where

$$n_1=\lambda n^2 + n\ [\lambda_0 \nu\ (l^2 + k^2 + 1/k_1) + 1] + \nu\ (l^2 + k^2 + 1/k_1), \tag{32}$$

$$n_2 = n + \eta\ (l^2 + k^2), \tag{33}$$

and $V^2 = (H^2/4\pi \rho_0)$ is the 'Alfven velocity' based on a magnetic field strength $H$.

Equation (31) forms the basis of the discussions. Its particular cases have been studied by various authors from time to time. Rayleigh [2] obtained the corresponding equation when $H = \Omega = \nu = \eta = \lambda = \lambda_0 = 1/k_1 = 0$. The case of inviscid, finitely conducting fluid ($1/k_1 = \nu = \lambda = \lambda_0 = 0$) has been discussed in detail by Sharma and Ariel [3], where the destabilizing nature of resistivity of the medium has been demonstrated. Sharma [5] also obtained the corresponding result when $1/k_1 = \lambda = \lambda_0 = 0$. In the present investigation we will discuss eqn. (31) in its general form.

It is convenient to discuss eqn. (31) in a non-dimensional form. Measuring $n$ and $k$ in terms or $(\pi s V/d)\text{sec}^{-1}$ and $(\pi s/d)\text{cm}^{-1}$, equation (31) takes the following dimensionless form

$$n^7\lambda^2 + 2n^6\lambda\,[2\lambda R(1+k^2) + L] + n^5\,[(L + \lambda R\,(1 + k^2)\,)^2$$
$$+ 2\lambda\,(2SM + 2LR\,(1 + k^2) + k^2\,) +$$

$$+ \frac{\lambda^2}{1 + k^2}\,(A - Bk^2) \quad + n^4\,[2(L + 2\lambda R\,(1 + k^2)\,)$$
$$(4RS\lambda_0\,(1 + k^2)\,M + 2R\,(1 + k^2) + 2SM) +$$

$$+ \frac{\lambda^2}{1+k^2}\,(A - Bk^2)\,] + n^4\,[2\,(L + 2\lambda R\,(1 + k^2)\,)$$
$$(4RS\lambda_0\,(1 + k^2)\,M + 2R\,(1 + k^2) + 2SM) +$$

$$+ 2\lambda k^2 + \frac{\lambda_0}{1+k^2}\,\{2A\,(1 + 2\lambda R\,(1 + k^2)\,)$$
$$- Bk^2\,(L + 1 + 4\lambda LR(1 + k^2)\,)\,\}\,] + n^3\,[4R^2\,(i + k^2)^2$$

$$(L^2 + 2L - 1) + 16RSLM\,(1 + k^2) + 4S^2M^2 + 2Lk^2$$
$$+ 4\lambda SMk^2 + 4\lambda LKk^2\,(1 + k^2) + 4\lambda Rk^2\,(1 + k^2) +$$

$$\lambda k^4 + \frac{A}{1+k^2}(4\lambda^2 R^2 (1 + k^2) + 8\lambda R (1 + k^2) + 1)$$

$$-\frac{Bk^2}{1+k^2}(L + 2S\lambda M + 8\lambda RL (1 + k^2) +$$

$$+ 4\lambda^2 R^2 (1 + k^2)^2 + \lambda k^2) ] + n^2 [2 (2R(L - 1) ( 1 + k^2) + 2SM)$$

$$(4RSM (1 + k^2) + k^2) +$$

$$8\lambda k^2 RSM (1 + k^2) + 2\lambda k^4 + \frac{2A}{1+k^2}(2R (1 + k^2)$$

$$+ 4\lambda R^2 (1 + k^2) + 1) - \frac{Bk^2}{1 + k^2}\{2SM + 4RL (1 + k^2)$$

$$+ 4\lambda RL (1 + k^2) + 4\lambda R^2 (L + 1) (1+k^2)^2 + 2\lambda k^2 (1+R(1+k^2))\}]$$

$$+ n [ (4RSM (1 + k^2) + k^2)^2 + 4AR^2 (1 + k^2)$$

$$- \frac{Bk^2}{1+k^2}\{4LR^2 (1 + k^2)^2 + 8RSM (1 + k^2) +$$

$$+ 8\lambda SR^2 (1 + k^2)^2 + k^2 (1 + 4\lambda R (1 + k^2) ) \} ]$$

$$- 2BRk^2 (4RSM (1 + k^2) + k^2) = 0, \qquad (34)$$

where

$$\left.\begin{array}{l} L=2S\lambda_0 M+1,\ M=(1+k^2+1/P),\ A=4\Omega^2/l^2V^2,\ B= g\beta/l^2V^2\} \\ S = \nu l/2V,\ R = \eta l/2V,\ 1/P = l^2/k_1,\ k_x = k\cos\theta\ (\theta = 0). \end{array}\right\} \qquad (35)$$

From the above equation, we see that, there are five parameters required specify $n$ for any given $k$. These number $A$, $B$, $S$, and R respectively represent measures of coriolis forces, buoyancy forces, viscous forces and finite resistivity in terms of magnetic field. $P$ is the measure of the permeability of the porous medium. $\theta$ being the inclination of the direction of wave-vector to that of the magnetic field.

## 6. Discussion

Equation (34) is a seventh degree in $n$, therefore, it must give seven roots. To obtain the explicit expression for each value of $n$ for general values of parameter is a task which is mathematically too involved. However, some general conclusions can be drawn. If $B < 0$, equation (34) dose not admit any positive root and the equilibrium is always stable. Since the magnetic in general has a stabilizing influence such results is to be expected. If $B > 0$, equation (34) will have at least one positive and the equilibrium will be unstable for all wave numbers. The behaviour of growth rates with respect to viscosity and resistivity of the medium have been already studied by various authors time to time. We now examine the behaviour of growth rate with respect to permeability of the medium. First we observe that

$$n \to \frac{2BS(1 + p^{-1})\, k^2}{A + 4S^2(1 + p^{-1})^2}, \quad (k \to 0) \tag{36}$$

and

$$n \to \frac{B}{2Sk^2}. \qquad (k \to \infty) \tag{37}$$

We shall now study the behaviour of $n$ on varying the value of $P$, the measure of porisity of the medium with respect to $k$. Clearly we must make distinction between the following cases : (i) $A < 4S^2$, (ii) $A > 4S^2$.

(i) $A < 4S^2$. In this case we find that with increasing $P$, the value of $n$ will also increase for all wave numbers. Thus, more the permeability of the medium, more it tends to destabilizes the configuration. It can be seen that in the absence of rotaion the behaviour of $P$ is reversed. Now an increase in the value of $P$, brings down the value of $n$ for all wave numbers.

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(ii) $A > 4S^2$. In this case a peculiar tendency is exhibited by $n$ as we vary the value of $P$. Two cases arise: (a) $P < P^*$, (b) $P > P^*$, where

$$P^* = \frac{2S}{\sqrt{A} - 2S}\,. \tag{38}$$

So long as $P < P^*$, an increase in the value of $P$ leads to the incre ase in the value of $n$ and therefore, again the system departs faster from one position of equilibrium as we increases the permeability of the medium. However, when $P$ becomes larger than $P^*$, this behaviour is reversed for small values of $k$. Now an increase in the value of $P$ leads to decrease in the value of $n$ for small values of $k$, but to an incre ase in the value of $n$ for large values of $k$. Thus, we can conclude that permeability of the medium has a stabilizing influence for small wave numbers of disturbance but it has a destabilizing effect for large wave numbers. It can be further seen that at higher rotation the effect of rotation the effect of increase in the value of $P$, for a fixed $R$ and $S$, is to inhibit the onset of instability and at lower rotation it favours the onset of instability.

It is the further noted that the absolute term in equation (34) does not contain any trem involving $\lambda$ and $\lambda_0$. It is clear that $n$ decre-ases with increasing $\lambda$ or $\lambda_0$ for every value of $k$. Thus, we can say that both stress relaxation time and strain retardation time parameters have a inhibiting influence on the configuration.

Finally, we have also observed in the present problem, that the inclusion of $\lambda$ alters the order of the dispersion relation, and that the boundary conditions are independent of the presence, or absence of either $\lambda$ or $\lambda_0$ or both.

**Acknowledgements**

The author is highly thankful to Professor H. M. Srivastava for suggesting certain improvements which led to the present form of the paper.

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