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A NOTE ON THE ANALYTIC UNIVALENT FUNCTIONS

By

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Abstract. A class $S(\alpha, \beta)$ of univalent functions of the from

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

with negative coefficients was studied by Gupta and Ahmad.
In this paper, we consider a subclass $S_c(\alpha,\beta)$ of $S(\alpha,\beta)$. It is the purpose of this paper to prove a result for convex linear combinations and a distortion theorem for $f(z)$ in $S_c(\alpha, \beta)$.

1. Introduction

Let T denote the class of functions of the from

$$(1.1) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

which are analytic and univalent in the unit disk $U = \{z: |z| < 1\}$.
A function $f(z) \in T$ is said to be in the class $S\{\alpha, \beta)$ if and only if

$$(1.2) \quad \left| \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} \Big/ \left\{ \frac{zf'(z)}{f(z)} + (1-2\alpha) \right\} \right| < \beta \quad (z \in U)$$

for some α $(0 \leqq \alpha < 1)$ and β $(0 < \beta \leqq 1)$.

For this class $S(\alpha, \beta)$, Gupta and Ahmad [3] showed the following lemma.

Lemma. A necessary and sufficient condition for a function $f(z)$ defined by (1.1) to be in the class $S(\alpha,\beta)$ is that

$$(1.3)\quad \sum_{n=2}^{\infty} \{ (n-1) + \beta(n+1-2\alpha) \} a_n \leqq 2\beta(1-\alpha).$$

In view of Lemma, we can see that the function $f(z)$ defined by (1.1) in $S(\alpha,\beta)$ satisfies

$$(1.4)\quad a_2 \leqq \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)}.$$

Let $S_c(\alpha, \beta)$ denote the class of functions $f(z) \in S(\alpha, \beta)$ of the form

$$(1.5)\quad f(z) = z - \frac{2c\beta(1-\alpha)}{1+\beta(3-2\alpha)} z^2 - \sum_{n=3}^{\infty} a_n z^n \qquad (a_n \geqq 0),$$

where $0 \leqq c \leqq 1$.

Silverman [5], Silverman and Silvia [6], Al-Amiri [1], Finkelstein [2], Netanyahu [4], Suffridge [7] and Tepper [8] showed many interesting results for certain subclasses of univalent functions with a fixed second coefficient.

2. Convex linear combinations

Theorem 1. Let

$$(2.1)\quad f_2(z) = z - \frac{2c\beta(1-\alpha)}{1+\beta(3-2\alpha)} z^2$$

and

$$(2.2)\quad f_n(z) = z - \frac{2c\beta(1-\alpha)}{1+\beta(3-2\alpha)} z^2 - \frac{2(1-c)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} z^n$$

for $n = 3, 4, \ldots$. Then $f(z)$ is in the class $S_c(\alpha, \beta)$ if and only if it can be expressed in the form

$$(2.3)\quad f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z),$$

wehre $\lambda_n \geqq 0$ and

$$(2.4)\quad \sum_{n=2}^{\infty} \lambda_n = 1.$$

Proof. Assume that $f(z)$ can be expressed in the form (2.3). Then

$$(2.5)\quad f(z) = z - \frac{2c\beta\,(1-\alpha)}{1+\beta(3-2\alpha)}\, z^2$$

$$- \sum_{n=3}^{\infty} \frac{2\lambda_n(1-c)\,\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}\, z^n$$

$$= z - \sum_{n=2}^{\infty} b_n z^n ,$$

where

$$(2.6)\quad b_2 = \frac{2c\beta\,(1-\alpha)}{1+\beta(3-2\alpha)}$$

and

$$(2.7)\quad b_n = \frac{2\lambda_n(1-c)\,\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \qquad (n = 3, 4, \ldots).$$

Consequently we can see that

$$(2.8)\quad \sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\}b_n$$

$$= 2\beta(1-\alpha)\{c + (1-c)\sum_{n=3}^{\infty} \lambda_n\}$$

$$= 2\beta(1-\alpha)\{1 - \lambda_2(1-c)\}$$

$$\leqq 2\beta(1-\alpha).$$

This shows that $f(z) \in S_c(\alpha, \beta)$ by Lemma.

Conversely, suppose that $f(z)$ defined by (1.5) is in the class $S_c(\alpha, \beta)$. By using Lemma, we have

(2.9) $$2c\beta(1-\alpha) + \sum_{n=3}^{\infty} \{(n-1) + \beta(n+1-2\alpha)\} a_n \leqq 2\beta(1-\alpha),$$

that is,

(2.10) $$a_n \leqq \frac{2(1-c)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \qquad (n=3, 4, \ldots).$$

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(2.11) $$\lambda_n = \frac{(n-1)+\beta(n+1-2\alpha)}{2(1-c)\beta(1-\alpha)} a_n \qquad (n=3, 4, \ldots)$$

and

(2.12) $$\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n,$$

we get (2. 3). Thus we have the theorem.

3. Distortion theorem

Theorem 2. Let $f_n(z)$ be defined by (2.2) and $n \geqq 4$. Then

(3.1) $$|f_n(re^{i\theta})| \leqq |f_4(-r)|.$$

Proof. Since $r^n/\{(n-1)+\beta(n+1-2\alpha)\}$ is a decreasing function of n, we obtain

$$(3.2)\quad |f_n(re^{i\theta})| \leqq r + \frac{2c\beta(1-\alpha)}{1+\beta(3-2\alpha)}r^2 + \frac{2(1-c)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}r^r$$

$$\leqq r + \frac{2c\beta(1-\alpha)}{1+\beta(3-2\alpha)}r^2 + \frac{2(1-c)\beta(1-\alpha)}{3+\beta(5-2\alpha)}r^4$$

$$= -f_4(-r)$$

which shows (3.1). Hence we have the theorem.

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