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A REMARK ON A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. R. M. Goel defined a class $S(k)$ of analytic functions in the unit disk U and proved some results for this class. Moreover, K. S. Padmanabhan and R. Parvatham studied the partial sums of analytic functions in the class $S(k)$. It is the purpose of this paper to prove some results for the class $S(k)$.

Let $S(k)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{ |z| < 1\}$ and which satisfy

$$|f'(z) - k| < k \qquad (z \in U),$$

where k is a fixed real number greater than one.

In 1967, R. M. Goel [1] showed the following lemma.

Lemma 1. Let a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $S(k)$. Then we have

$$|a_n| \leqq \frac{2k-1}{kn} \qquad (n \geqq 2),$$

$$|f'(z)| \geqq \frac{k(1-|z|)}{k+(k-1)|z|}$$

and

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{(2k-1)|z|}{\{k+(k-1)|z|\}(1-|z|)}.$$

Furthermore, K. S. Padmanabhan and R. Parvatham [2] gave some results for the class $S(k)$.

Theorem 1. Let a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U and

$$\sum_{n=2}^{\infty} n\,|a_n| \leqq 1.$$

Then the function $f(z)$ is in the class $S(k)$.

Proof. By using the hypothesis of the theorem, we have

$$|f'(z)-k|-k = |(1-k)+\sum_{n=2}^{\infty} na_n z^{n-1}|-k$$

$$< \sum_{n=2}^{\infty} n\,|a_n| - 1$$

$$\leqq 0$$

for $z \in U$. Hence, by the maximum modulus theorem, the function $f(z)$ is in the class $S(k)$. Furthermore, the function

$$f(z) = z - \frac{1}{n} z^n$$

is an extremal function for the theorem.

Theorem 2. A function

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

is in the class $S(k)$ if, and only if,

$$\sum_{n=2}^{\infty} n a_n \leqq 1.$$

Proof. Assume that

$$\sum_{n=2}^{\infty} n a_n \leqq 1$$

and let $|z| = 1$. Then we have

$$|f'(z) - k| - k = |(1-k) - \sum_{n=2}^{\infty} n a_n z^{n-1}| - k$$

$$\leqq \sum_{n=2}^{\infty} n a_n - 1$$

$$\leqq 0.$$

Consequently, by the maximum modulus theorem, the function $f(z)$ belongs to the class $S(k)$.

For the convrse, assume that

$$|f'(z) - k| = |(1-k) - \sum_{n=2}^{\infty} n a_n z^{n-1}|$$

$$< k.$$

Since $|\text{Re}(z)| \leqq |z|$ for any z, we have

$$\text{Re}\{(k-1)+\sum_{n=2}^{\infty} na_n z^{n-1}\} < k.$$

Choose values of z on the real axis so that $f'(z)$ is real. And letting $z \to 1$ through real values in (1), we obtain

$$\sum_{n=2}^{\infty} na_n \leqq 1.$$

Furthermore, the function

$$f(z) = z - \frac{1}{n} z^n$$

is an extremal function for the theorem.

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