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NONARCHIMEDEAN BANACH ALMOST PERIODIC FUNCTIONS DEPENDING ON PARAMETERS

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Let Ω be a subset of the n-dimensional complex space and let $\mathcal{Z} = (z_1, \ldots, z_n)$ be the point of Ω with cordinates $z_1, \ldots, z_n$ and let X be a complex nonarchimedean Banach Space over nonarchimedean valued field F [2] with the nonarchimedean norm topology. The purpose of the present note is to define a new class of almost periodic functions depending on parameters called nonarchimedean Banach valued almost periodic functions depending on parameters and to study some of its properties.

Definition 1. A continuous function $f(\mathcal{Z}, x)$ on $\Omega \times R$, $f(\mathcal{Z}, x) : \Omega \times R \to X$ is called almost periodic in $x \in R$, uniformly with respect to $\mathcal{Z} \in \Omega$, if for any $\varepsilon > 0$, there exist a number $l(\varepsilon) > 0$ with the property that any interval of length $l(\varepsilon)$ the real line contains at least one point of abscissa τ, such that

$$\| f(\mathcal{Z}, x+\tau) - f(\mathcal{Z}, x) \| < \varepsilon, \ \mathcal{Z} \in \Omega, \ X \in R.$$

A number τ for which the above inequality holds is called an ε- translation number of the function $f(\mathcal{Z}, x)$. The uniform dependence on parameters follows from the fact that $l(\varepsilon)$ and τ are independent of $\mathcal{Z}$.

Theorem 1. *If a function $f(Z, x)$ is almost periodic with values in X and uniformly with respect to $Z \in \Omega$ and if c is a complex number and a is real, then* (i) $cf(Z, x)$, (ii) $f(Z, x + a)$, (iii) $\bar{f}(Z, x)$, (iv) $| f(Z, x) |$, (v) $1/f(Z, x)$, *where* $f(Z, x) > 0$, *are almost periodic functions, uniformly with respect to* $Z \in \Omega$.

The proof follows direectly from the definition of almost periodic functions.

Theorem 2. *An almost periodic function $f(Z, x)$ with values in X and uniformly with respect to $Z \in \Omega$, where Ω is a closed and bounded set, is bounded in the sense of nonarchimedean norm.*

Proof. Let $f(Z, x)$ be almost periodic function uniformly with respect to $Z \in \Omega$ and $l(1)$ the number corresponding to $\varepsilon = 1$, from the definition of almost periodic function on the set $\Omega \times [0,1]$, the function $f(Z, x)$ is continuous and thus bounded. Let $M > 0$ be such that

$$\| f(Z, x) \| \leqslant M, \quad (Z, x) \in \Omega \times [0, 1].$$

Consider a real number x and 1 — translation number τ of $f(Z, x)$ which belongs to the interval $(-x, -x + l)$. Then

$$\begin{aligned} \| f(Z, x) \| &= \| f(Z, x) - f(Z, x + \tau) + f(Z, x + \tau) \| \\ &\leqslant max \{ \| f(Z, x + \tau) \| , \| f(Z, x + \tau) - f(Z, x) \| \} \\ &\leqslant max \{M, 1\} \\ &= m, \end{aligned}$$

which shows that function $f(Z, x)$ is bounded.

Theorem 3. *A nonarchimedean Banach valued almost periodic function $f(Z, x)$ depending on parameter $Z \in \Omega$ (closed and bounded set) is* *uniformly continuous on* $\Omega \times R$.

Proof. Let ε be a positive number in the definition of almost periodic function and let $l = l(\varepsilon/3) > 0$ which corresponds to $\varepsilon/3$.

The function $f(Z, x)$ is continuous on the set $\Omega \times [-1, 1+l]$ and hence it is uniformly continuous.

Let $\delta = \delta(\varepsilon/3) > 0$, $\delta < 1$, we have

$$\| f(Z_2, Y_2) - f(Z_1, Y_1) \| < \varepsilon/3,$$

provided that

$$| Z_2 - Z_1 | < \delta, \; | Y_2 - Y_1 | < \delta, \; (Z_2, Y_2), (Z_1, Y_1) \in \Omega \times [-1, 1+l].$$

Finally, let (Z_2, x_2) and (Z_1, x_1) be any two points of $\Omega \times R$ such that $| Z_2 - Z_1 | < \delta$, $| x_2 - x_1 | < \delta$.

If τ is an $(\varepsilon/3)$ — translation number in the interval $[-x_1, -x_1 + l]$ then we have $0 \leqslant x_1 + \tau \leqslant l$, $-1 < x_2 + \tau \leqslant 1 + l$.

Therefore $(Z_2, x_2 + \tau), (Z_1, x_1 + \tau) \in \Omega \times [-1, 1 + l]$. Thus

$$\| f(Z_2, x_2) - f(Z_1, x_1) \| = \| f(Z_2, x_2) - f(Z_2, x_2 + \tau) + f(Z_2, x_2 + \tau) - f(Z_1, x_1 + \tau) + f(Z_1, x_1 + \tau) - f(Z_1, x_1) \|$$

$$\leqslant \max \{ \| f(Z_2, x_2) - f(Z_2, x_2 + \tau) \|, \| f(Z_2, x_2 + \tau) - f(Z_1, x_1 + \tau) \|, \| f(Z_1, x_1 + \tau) - f(Z_1, x_1) \| \}$$

$$< \epsilon,$$

which completes the proof.

Theorem 4 *If* $\{ f_n(Z, x) \}$ *be a sequence of almost periodic function with values in X and uniformly depending on parameters* $Z \in \Omega$ *and if*

$\lim_{n\to\infty} f_n(Z, x) = f(Z, x)$ *uniformly on* $\Omega \times R$ *in the sense of convergence in the nonarchimedean norm, then* $f(Z, x)$ *is almost periodic.*

Proof. For each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that

$$\| f_n(Z, x) - f(Z, x) \| < \varepsilon/3, \ (Z, x) \in \Omega \times R, \ n \geqslant N(\varepsilon) \ \ldots \qquad (1)$$

we fix n_0 for which (1) is true, and consider $(\varepsilon/3)$ determined from the almost periodicity of f_{n_0} and τ an $(\varepsilon/3)$ — translation number of f_{n_0} (Z, x).

For any $(Z, x) \in \Omega \times R$ we have

$$\| f(Z,x+\tau) - f(Z,x) \| = \| f(Z,x+\tau) - f_{n_0} f(Z,x+\tau) + f_{n_0} (Z,x+\tau)$$

$$- f_{n_0} (Z, x) + f_{n_0} (Z, x) - f(Z, x) \|$$

$$\leqslant \max \{ \| f(Z, x+\tau) - f_{n_0} (Z, x+\tau) \|, \| f_{n_0} (Z,x+\tau) - f_{n_0} (Z, x) \|$$

$$\| f_{n_0} (Z, x) - f(Z, x) \| \}$$

$$< \varepsilon,$$

which proves the almost periodicity of $f(Z, x)$.

Definition 2. A continuous function $f : \Omega \times R \to X$ is called normal if any set of translates of f has a subsequence, uniformly convergent in the sense of nonarchimedean norm.

Theorem 5 *The necessary and sufficient condition for a function* $f(Z, x)$ *to be almost periodic, with values in* X, *uniformly with respect to* $Z \in \Omega$, *where* Ω *is a closed and bounded set, is that it be normal on* $\Omega \times R$.

Proof. The necessary condition can be proved by the following well known method of diagonal extraction.

Let $\{f_{h_n}(Z, x)\}$ be a sequence of translates of $f(Z, x)$, and $S = \{S_n\}$ be a dense sequence in $\Omega \times R$, Let $\{f_{h_{1n}}(Z,x)\}$ be the subsequence of $\{f_{h_n}(Z, x)\}$ convergent in S_1. Applying the previous argument to $\{f_{h_{1n}}(Z, x)\}$, choose a subsequence $\{f_{h_{2n}}(Z, x)\}$ converging in S_2. Continuing in this way, we have the diagonal sequence $\{f_{h_{nn}}(Z, x)\}$ which converges pointwise in S. Let us denote it by $\{f_{k_n}(Z, x)\}$. Now we will show that this sequence converges uniformly on $\Omega \times R$ in the sense of nonarchimedean norm.

Consider a number $l = l(\varepsilon/5) > 0$ and $\delta = \delta(\varepsilon/5)$, We cover the set $\Omega \times [0, l]$ with a finite number of intervals of length less than δ, and in each of these intervals we choose one point of the set S. Thus we obtain a finite set $S_0 = \{r_1, r_2, \ldots, r_p\}$, and so the sequence $\{f_{k_n}(Z, x)\}$ is uniformly convergent in S. Therefore, there exists a natural number $N(\varepsilon/5)$, such that $m, n \geqslant N(\varepsilon/5)$ we have

$$\| f_{k_n}(Z, r_i) - f_{k_m}(Z, r_i) \| < \varepsilon/5, \ i = 1, 2, \ , p.$$

Let τ be an $(\varepsilon/5)$ — translation number in $[-t, -t + l]$ and let r_i be a point from S_0 for which $| t + \tau - r_i | < \delta$. For $n, m \geqslant N(\varepsilon/5)$, it follows that

$$\| f_{k_n}(Z, t) - f_{k_m}(Z, t) \| = \| f(Z, t + k_n) - f(Z, t + k_m) \|$$

$$\leqslant max \{ \| f(Z, t+k_n) - f(Z, t+k_n+\tau) \|, \| f(Z, t+k_n+\tau) - f(Z, r_i+k_n) \|,$$

$$\| f(Z, r_i+k_n) - f(Z, r_i+k_m) \|, \| f(Z, r_i+k_m) - f(Z, t+k_m+\tau) \|,$$

$$\| f(Z, t+k_m+\tau) - f(Z, t+k_m) \| \}$$

which shows that the sequence $\{ f_{k_n} (Z, x) \}$, satisfied the Cauchy uniform convergence condition.

The sufficient condition can be proved in the same way as Theorem 1–10 of [1].

Theorem 6. *If Ω is a closed and bounded set, the sum and product of two almost periodic functions with values in X depending on parameter are almost periodic functions with values in X, depending on parameter.*

Proof. Let $f(Z, x)$ and $g(Z, x)$ be almost periodic functions with values in nonarchimedean Banach space and depending on parameter and let $\{h_n\}$ be an arbitrary sequence of real numbers.

To prove the almost periodicity of the sum, it suffices to show that from the sequence of translates $\{ f_{h_n} (Z, x) + g_{h_n} (Z, x) \}$ one can extract a subsequence converging uniformly on $\Omega \times R$. From the sequence of translates $\{ f_{h_n} (Z, x) \}$, according to above theorem we choose a uniformly convergent subsequence on $\Omega \times R$, say $\{ f_{l_n} (Z, x)\}$. From the sequence of translates $\{g_{h_n} (Z, x) \}$ we choose a subsequence uniformy convergent on $\Omega \times R$, say $\{g_{l_n} (Z, x)$. Then the sequence $\{f_{l_n} (Z, x) + g_{l_n} (Z, x) \}$, which is a subsequence of the sequence $\{ f_{h_n} (Z, x) + g_{h_n} (Z, x) \}$ is uniformly convergent on $\Omega \times R$ and the theorem is proved.

It is easy to show that the product of two almost periodic functions is again almost periodic function using the identity

$$f(Z, x)\, g(Z, x) = \frac{\{ [f(Z, x) + g(Z, x)]^2 - [f(Z, x) - g(Z, x)]^2 \}}{4}$$

In view of Theorem 1 and Theorem 6, we have

Theorem 7. *The set of all nonarchimedean Banach valued almost periodic functions depending on parameters is a linear space.*

REFERENCES

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[2] L. Narici, E. Beckenstein and G. Bachman Functional Analysis and Valuation Theory, Marcel Dekker, New York, 1971.