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A MEIR-KEELER TYPE FIXED POINT THEOREM FOR THREE MAPPINGS

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ABSTRACT. In this note a theorem of Park and Rhoades [6], involving a pair of mappings satisfying a Meir–Keeler type contractive definition, is extended to three mappings.

In [2] and [7] extensions of Meir–Keeler type contractive definitions have been made for three mappings. The extension in this paper, however, is different, and is made in such a way as to generalize the sufficiency portion of the theorem of Jungck [3], and includes as special cases results in [1] and [4].

Let f be a continuous selfmap of a metric space (X, d). $C_f=\{g \mid g : X\rightarrow X$ such that $fg=gf$ and $gX\subset fX\}$. For $x_0 \in X, g, h \in C_f$ the sequence $\{fx_n\}$ is called the f-iteration of x_0 under g and h, and is defined by $gx_{2n+1} = fx_{2n}$, $hx_{2n+2} = fx_{2n+1}$, $n = 0, 1, 2, \ldots$.

THEOREM. Let f be a continuous selfmap of a complete metric space (x, d), $g, h \in C_f$ and continuous and satisfying the following condition :

For each $\varepsilon > 0$ there exists a there exists a $\delta > 0$ such that

$$(1)\ \varepsilon \leqslant \max \{d(gx, hy), d(fx, gx), d(fy, hy), [d(fx, hy) + d(fy, gx)]/2\} < \varepsilon + \delta$$

implies $d(fx, fy) < \varepsilon$.

Then either there exists a point of coincidence of f and g, or f and h, or f, g and h have a unique common fixed point.

The theorem is proved with the aid of the following lemmas.

LEMMA 1. Let f, g, h, satisfy the conditions of the Theorem. Then

$$inf\ \{d\ (fx_n, fx_{n+1})\ |\ n = 1, 2, \ldots\} = 0.$$

LEMMA 2, Let f, g, h be as in the Theorem. If there exists a point $\xi \varepsilon x$ such that $f\xi = g\xi = h\xi$, then $f\xi$ is the unique common fixed point of f, g, and h.

Proof of Lemma 1. Let $r = inf_n\ d(fx_n, fx_{n+1})$, and assume $r > 0$.

From (1), for cach x and y such that $gx \neq hy$,

(2) $d(fx, fy) < max\ \{d(gx, hy), d(fx, gx), d(fy, hy), [d(fx, hy) + d(fy, gx)]/2\}$.

Since $r > 0$, for n even we have, from (2),

$$d(fx_{2n-1}, fx_{2n}) < max\ \{d(gx_{2n-1}, hx_{2n}),\ d(fx_{2n-1}, gx_{2n-1})$$

$$d(fx_{2n}, hx_{2n}), [d(fx_{2n-1}, hx_{2n}) + d(fx_{2n}, gx_{2n-1}]/2\},$$

which implies $d(fx_{2n-1}, fx_{2n}) < d(fx_{2n-2}, fx_{2n-1},)$. Similarly,

$d(fx_{2n+2}, fx_{2n+1}) < d(fx_{2n+1}, fx_{2n})$. Thus, $d_n = d(fx_n, fx_{n+1})$

is monotone decreasing in n, and lim $d_n = r$.

From (1) there exits a $\delta > 0$ such that (1) is satisfied with $\varepsilon = r$. Pick N such that $n \geqslant N$ implies $r \leqslant d_n < r + \delta$.

Note that

$$max\ \{d(gx_{2n-1}, hx_{2n}), d(fx_{2n-1}, gx_{2n-1}), d(fx_{2n}, hx_{2n}), [d(fx_{2n-1}, hx_{2n}) + d(fx_{2n}, gx_{2n-1})]/2\}$$

$$= max\ \{d_{2n-2}, d_{2n-1}, d(fx_{2n}, fx_{2n-2})/2\} = d_{2n-2}.$$

Therefore, from (1), $2n - 2 \geqslant N$ implies $d_{2n-1} < r$, a contradiction. Therefore $r = 0$.

Proof of Lemma 2. Suppose $f\xi = g\xi = h\xi = \eta$ and $f\eta \neq \eta$.

Then $f^2\xi \neq f\xi$, so from (2), and the fact that f commutes with g and h

$$d(f\xi, f^2\xi) < max\ \{d(g\xi, hf\xi), d(f\xi, g\xi), d(f^2\xi, hf\xi),$$

$$\times\ [d(f\xi, hf\xi) + d(f^2\xi, g\xi)]/2\}$$

$$= d(f\xi, f^2\xi),$$

a contradiction. Therefore $f\eta = \eta$ which implies

$$g\eta = gf\xi = fg\xi = ff\xi = f\eta = \eta \text{ and } h\eta = \eta.$$

Suppose η' is another common fixed point of f, g and h, $\eta \neq \eta'$.

Thus, from (2) we obtain $d(\eta, \eta') < d(\eta, \eta')$ a contradiction.

Proof of the Theorem. If $fx_{2n} = fx_{2n+1}$ for some n, then g and f have a coincidence point. If $fx_{2n+1} = fx_{2n+2}$ for some n, then f and h have a coincidence point. Assume $fx_{2n} \neq fx_{2n+1}$ for all n, and assume $\{fx_n\}$ in not Cauchy. Then, for each $\varepsilon' > 0$ there exists an integer N such that for each $m > n > N$, $d(fx_m, fx_n) \geqslant \varepsilon'$ Define $\varepsilon' = 2\varepsilon$ and choose δ, $0 < \delta < \varepsilon$ such that (1) is satisfied. Since $r = 0$ there exists an integer N such that $d_i < \delta/6$ for $i \geqslant N$. With this choice of N, pick $m > n > N$ such that

(3) $$d(fx_m, fx_n) \geqslant 2\varepsilon > \varepsilon + \delta.$$

From (3) it is clear that $m - n > 6$. For, otherwise, $d(fx_m, fx_n) \leqslant \sum_{i=0}^{5} d_{i+m} < \delta < \delta + \varepsilon$, a contradiction. Without loss of generality we may pick n even. From (3) there exists a smallest integer j with $2j - 1 > 2n$ such that

(4) $$d(fx_{2n}, fx_{2j-1}) \geqslant \varepsilon + \delta/3.$$

Hence $d(fx_{2n}, fx_{2j-3}) < \varepsilon + \delta/3$.

Thus

$$\begin{aligned}
\varepsilon < \varepsilon + \delta/3 \leqslant d(fx_{2n}, fx_{2j-1}) &= d(gx_{2n+1}, hx_{2j}) \\
&\leqslant \max \{d(gx_{2n+1}, hx_{2j}), d(fx_{2j}, hx_{2j}), d(fx_{2n+1}, gx_{2n+1}), \\
&\qquad [d(fx_{2n+1}, hx_{2j}) + d(fx_{2j}, gx_{2n+1})]/2\} \\
&= \max d(fx_{2n}, fx_{2j-1}), d_{2j-1}, d_{2n}, [d(fx_{2n+1}, fx_{2j-1}) \\
&\qquad + d(fx_{2j}, fx_{2n})]/2\} \\
&\leqslant \max \{\delta/3 + \varepsilon + \delta/3, \delta/6, \delta/6, [d_{2n} + d_{2j-3} + d_{2j-2} \\
&\qquad + \varepsilon + \delta/3 + \delta/2 + \varepsilon + \delta/3]/2\} \\
&< \varepsilon + \delta,
\end{aligned}$$

so that, from (1), $d(fx_{2n+1}, fx_{2j}) < \varepsilon$.

But

$$\begin{aligned}
d(fx_{2n}, fx_{2j-1}) &\leqslant d_{2n} + d(fx_{2n+1}, fx_{2j}) + d_{2j-1} \\
&< \delta/6 + \varepsilon + \delta/6 = \varepsilon + \delta/3,
\end{aligned}$$

a contradiction. Therefore $\{fx_n\}$ is Cauchy and hence converges, from the completeness of X, to a point $\xi \in X$.

Since $ffx_{2n} = fgx_{2n+1} = gfx_{2n+1}$, and $ffx_{2n+1} = fhx_{2n+2} = hfx_{2n+2}$, the continuity of f, g, and h implies $f\xi = g\xi = h\xi$. Now apply Lemma 2.

REMARKS 1. As noted in [4], Theorem 1 of [4] is a generalization of the result in [1]. From [5] it is clear that the contracitive definition of this paper is more general than that of [4].

2. Setting $f = g$ and $g = h = f$ in the Theorem of this paper yields Theorem 4 of [6].

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