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ON AN EXTENSION OF A FIXED POINT THEOREM OF MEIR AND KEELER FOR THREE MAPPINGS

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Following Park and Bae [3], in this note we generalize the result of Meir and keeler [2] to three mapping under Ćirić type [1] condition.

1. Singh and Singh [6] have given the following mapping condition for three mappings:

(1) $d(fx, gy) \leqslant \alpha\ max.\ \{ d(hx, hy), d(fx, hx), d(gy, hy),$

$\frac{1}{2} [d(fx, hy) + d(gy, hx)] \}$, where

$\alpha \in (0, 1)$, f, g, h are mappings from X into itself, h is continuous and $fh = hf$, $gh = hg$ and $f(X) \cup g(X) \subseteq h(X)$.

Borrowing the idea of three maps from the above, we define the Meir-Keeler type mapping condition for three maps as follows:

The self-maps f and g of X are called an (ε, δ)-h contraction if for $\varepsilon > 0$, three exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$,

(2) $\varepsilon \leqslant max\ \{ d(hx, hy), d(fx, hx), d(gy, hy), [d(fx, hy)$

$+ d(gy, hx)]/2 \} < \varepsilon + \delta(\varepsilon)$ implies

$d(fx, gy) < \varepsilon$.

We take h continuous, $fh = hf$, $gh = hg$, and $f(X) \cup g(X) \subseteq h(X)$. Let C_h denote the class of self maps f and g satisfying above.

Given a point x_0 in X, we consider a sequence $\{hx_n\}_{n=1}^{\infty}$ recursively, given by the rule $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \ldots$; with the understanding that, if $hx_{2n+1} = hx_{2n+2}$ for some n, then $hx_{2n+1} = hx_{2(n+j)}$ for each $j \geqslant 0$. Such a sequence $\{hx_n\}_{n=1}^{\infty}$ is called an h-iteration of x_0 under f and g respectively. For an (ε, δ)-h contraction f, g we have from (2):

$$(3) \quad d(fx, gy) < max \{ d(hx, hy), d(fx, hx), d(gy, hy), [d(fx, hy) + d(gy, hx)]/2 \}, \text{ for}$$

three pairs (x, y) for which the right hand side of (3) is positive.

2. We require the following two lemmas:

Lemma 1. *Let h be a self-map of metric space X and f, g be (ε,δ)-h contraction. If there exists an $x_0 \in X$ and h-iteration $\{hx_n\}_{n=1}^{\infty}$ of x_0 under f and g, then*

$$inf. \{ d(hx_n, hx_{n+1}) \mid n = 1, 2, \quad \} = 0.$$

The proof runs parallel to that for lemma 2.1 in [3].

Lemma 2. *Let f, g be (ε, δ)-contraction commuting with h. If there exists a $z \in X$ such that $hz = fz = gz$, then hz is the unique common fixed point of f, g and h.*

Proof. Let $fz = hz = gz = v$. Suppose that $hv \neq v$.

Then $d(v, hv) = d(fz, hgz) = d(fz, ghz) < max,$

$$\{ d(hz, h^2z), d(hz, fz), d(h^2z, ghz), [d(fz, h^2z) + d(ghz, hz)]/2 \},$$

where $h^2z = h(hz)$. Therefore, $d(v, hv) < d(v, hv)$, a contradiction. Hence $hv = v$ and $fv = fhz = hfz = hv = v$ and also $gv = v$. Therefore, hz is a common fixed point of f, g and h. Uniqueness easily follows.

3. Now we state our main result:

Theorem 1. *Let h be a self-map of a metric space X and f, g be (ε, δ)–h contraction commuting with h. If a point $x_0 \in X$ has any h–iteration of x_0 under f and g with a cluster point $z \in X$ at which f, g, h are continuous, then $\{hx_n\}$ converges to z and hz is the common fixed point of f and g.*

Proof. By Lemma 2, it is sufficient to find a point $z \in X$ such that $fz = hz = gz$.

Let $x_0 \in X$ and let $\{hx_n\}_{n=1}^{\infty}$ be an h–iteration of x_0 under f and g. By Lemma 1, $\inf_n d(hx_{2n+1}, hx_{2n+2}) = 0$. If there exists an n such that $d(hx_{2n+1}, hx_{2n+2}) = 0$, then $hx_{2n+1} = fx_{2n} = gx_{2n+1}$, and we are finished. Let us assume that $d(hx_{2n+1}, hx_{2n+2}) \neq 0$ for every n. Following the proofs of Theorem 2.3 [3] and Theorem 4 [4], we can easily show that $\{hx_n\}$ is Cauchy, hence convergent to a point $z \in X$ Since $hhx_{2n} = hfx_{2n-1} = fhx_{2n-1}$ the continuity of h and f at z implies $hz = fz$. Similarly, $hz = gz$, as g is also continuous at z. This completes the proof.

If $f(x) \cup g(x) \subseteq h(x)$, then every $x_0 \in X$ has an h–iteration under f, g. Therefore, from Theorem 1 we have:

Theorem 2. *Let h be a continuous self-map of a complete metric* *space* (X, d), f, g *be* (ε, δ)-h *contraction in* C_h *and continuous. Then* f, g *and* h *have a common fixed point* w *in* X, *and, for any* $x_0 \in X$, *any* h-*iteration of* x_0 *under* f *and* g *converges to some* $z \in X$ *satisfying* $hz = w$.

Proof. It follows easily from Theorem 1 above.

Remark. 1. For $f = g$ in our Theorem 2, we have the Theorem **4** of Park and Rhoades [4] which, in turn, is an extension of the work of Ćirić [1] for a single mapping. In fact, Theorem 4 [4] is an extension of the earlier basic result of Park and *Bae* [3] for Meir and Keeler type [2] mapping. It will not be discourteous to mention that the author of the present paper obtained Theorem 4 of Park and Rhoades [4] independently, based upon the mapping condition of Ćirić [1] and the technique of Park and Bae [3] in mid. 1981, and thereby generalized it for three maps, as presented in the present note.

Remark 2. Note that (1) $\not\Rightarrow$ (2) $\Rightarrow$ (3). For two mappings, i. e., for $f = g$, Rhoades [5] has shown that (1) $\Rightarrow$ (3), and not conversely. In view of the fact that (3) $\not\Rightarrow$ (1), our Theorems 1 and 2 are extensions of the result of Singh and Singh [6].

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