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A FIXED POINT THEOREM INVOLVING FOUR POINTS

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Recently, Pittnauer [1] obtained the following fixed point theorem by taking three point x_1, x_2, x_3 in the contractive definition:

Theorem A. *Let f be a maping from a complete metric space* (X, d) *into itself such that*

$$d(f(x_1), f(x_2)) \leqslant \delta\,[d(x_1, f^k(x_3)) + d(x_2, f^k(x_3))]$$

for arbitrary poinis $x_1, x_2, x_3 \in X$, *a fixed integer* $k \geqslant 1$, *and a fixed* δ, $0 < \delta < 1$. *Then f has a unique fixed point in X.*

We extend this result by taking a more general mapping and also four points $x_1, x_2, x_3, x_4 \in X$. We prove the following

Theorem. *Let* f_1 *and* f_2 *be two mappings of a complete metric space* (X, d) *tnto itself such that*

$$(1)\quad d(f_1(x_1), f_2(x_2)) \leqslant h.\ max\ \{d(x_1, x_2), d(x_1, f_1^k(x_3)), d(x_1, f_2^k(x_4)).$$
$$d(x_2, f_1^k(x_3)), d(x_2, f_2^k(x_4)), d(f_1^k(x_3), f_2^k(x_4))\}.$$

for arbitrary $x_1, x_2, x_3, x_4 \in X$, *a fixed integer* $k \geqslant 1.$ $0 < h < 1$, *and let* f_1, f_2 *be commutative. Then* f_1 *and* f_2 *have a unique common fixed point.*

Proof. Let $x, y \in X$ be arbitrary. Setting $x_1 = f_2^k(x)$,

$x_2 = f_1^k(y)$, $x_3 = y$, $x_4 = x$ in (1) yields

(2) $d(f_1(f_2^k(x)), f_2(f_1^k(y))) \leqslant h\, d((f_2^k(x), f_1^k(y))$.

For $x_0 \in X$, let $\{x_n\}$ be defined as follows:

$$x_n = \begin{cases} f_1(x_{n-1}), \text{ if } n \text{ is odd.} \\ f_2(x_{n-1}), \text{ if } n \text{ is even.} \end{cases}$$

In view of the fact that $f_1 \, o \, f_2 = f_2 \, o \, f_1$ we observe that

$x_{2n} = f_1^n f_2^n(x_0)$ and $x_{2n+1} = f_1^{n+1} f_2^n(x_0)$.

Let $n > k$, which gives $n = k + i$, for some integer $i \geqslant 1$. We have, on an easy computation, with the help of (2),

$d(x_{2n}, x_{2n+1}) \leqslant h^{2n-2k} d(x_{2k}, x_{2k+1})$.

For $m > n > k$, we have

$$d(x_{2n}, x_{2m}) \leqslant d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \dots + $$
$$+ d(x_{2m-1}, x_{2m})$$

$$\leqslant h^{2n-2k} - (1 + h + \dots + h^{2m-2n-1})\, d(x_{2k}, x_{2k+1})$$

$$< \frac{h^{2n-2k}}{1-h} d(x_{2k}, x_{2k+1}), \text{ so that}$$

$\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = u$, $u \in X$.

Now, let, if possible, $f_2(u) = u$. We have the closed ball (*cf.* [2]):

$H = \{ x : x \in X, d(x, u) \leqslant \pi\, d(u, f_2(u)) \}$,

(3) π being given by $0 < \pi < \frac{1}{1+2h}$

It is easy to see that for each $x \in H$,

(4) $d(x, u) \geqslant (1 - \pi)\, d(u, f_2(u))$.

Further, as $\lim_{n\to\infty} x_n = u$, a positive integer N such that $x_n \in H \ \forall\ n \geqslant N$. Setting $x_1 = x_{2n}$, $x_2 = u$, $x_3 = f_2^k(x_{2n})$, and $x_4 = f_1^k(x_{2n})$ in (1) yields

$$d(x_{2n+1}, f_2(u)) \leqslant h.\ max\ \{ d(x_{2n}, x_{2(n+k)}), d(u, x_{2(n+k)}) \}.$$

Taking the limit as $n \to \infty$ one obtains $u = f_2(u)$. Similarly, it can be shown that $u = f_1(u)$, so u is a common fixed point of f_1 and f_2.

Let v be another fixed point f_1 and f_2. By taking $x_1 = u = x_4$ and $x_2 = v = x_3$ in (1), uniqueness of the fixed point can be easily proved.

Remark. If f_1 and f_2 are each surjective, then (1) reduces to the Banach contraction principle, and hence a unique fixed point exists, without the assumption that f_1 and f_2 commute.

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