Jñānābha, Vol. 14, 1984

# SOME POLYNOMIALS OF BERNSTEIN TYPE

*By*

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(*Received: December 22, 1982; Revised: March 5, 1984*)

## ABSTRACT

The object of the present note is to given an estimate on the derivatives for function of two variables by the derivatives of some polynomials of Bernstein type in the uniform norm.

## 1. Introduction

The linear positive operators $(B_n^2 f)$ defined as $(B_n^2 f)(x)$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}(x, y) f(k/n, l/n), \tag{1.1}$$

where

$$p_{n,k,l}(x, y) = \binom{n}{k} \binom{n-k}{l} x^k l^l (1 - x - y)^{n-k-l},$$

converge uniformly to a continuous function $f(X)$ on the simplex $\triangle = \{x = (x, y) \mid x \geqslant 0, y \geqslant 0, x + y \leqslant 1\}$. The order of convergence for the polynomials (1.1) on $\triangle$ was given by Stancu [3].

**Theorem 1** ([3]) *If* $\dfrac{\partial^{r+s} f(x)}{\partial x^r \partial y^s}$ *is continuous with modulus of continuity* $\omega_{r,s}$ *and the maximum absolute value* $M_{r,s}$ $(r, s = 0, 1, 2, \ldots)$ *then for* $n \geqslant r + s$,

$$\left\| \frac{\partial^{r+s} B_n^2 f}{\partial x^r \partial y^s} - \frac{\partial^{r+s} f}{\partial x^r \partial y^s} \right\|$$

$$\leqslant \left[ 1 + \sqrt{1 + 2r + 2s}\; w \right] \left( \frac{1}{\sqrt{n-r-s}} \right) + \frac{(r+s)\,(r+s-1)}{2n} M_{r,s}. \tag{1.2}$$

Recently, Papanicolau [1] Considered Some Barnstein type operators which were originally proposed by A. Lupas. We recall here the sequence of linear positive operators $(p_n^2 f)$ as defined in [1]:

$$(p_n^2 f)\,(X) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}\,(a, b)\, f(x + k/n,\, y + l/n) \tag{1.3}$$

for all $(a, b) \in \triangle$, fixed $X \in \triangle$, and which maps the space of bounded continuous functions $C(\triangle)$ into itself.

The aim of this note is to present the degree of approximation for polynomials (1.3). We prove the following theorem.

**Theorem 2** *If $f \in C^{r+s}(\triangle)$, with its modulus of continuity $w_{r,s}$ and the maximum absolute value $M_{r,s}$ $(r, s = 0, 1, 2, \ldots)$, then for $n \geqslant r+s$,*

$$\left\| \frac{\partial^{r+s} p_n^2 f}{\partial a^r \partial b^s} - \frac{\partial^{r+s} f}{\partial a^r \partial b^s} \right\|$$

$$\leqslant (1 + \sqrt{\lambda^*_{n,r,s}})\, w\left(\frac{1}{\sqrt{n}}\right) + \frac{(r+s)\,(r+s-1)}{2n} M_{r,s}, \tag{1.4}$$

where

$$\lambda^*_{n,r,s} = \begin{cases} \frac{1}{2} + 2s - s/n \text{ if } r = s, & n > (r+s)(r+s-1) \\ 2\,(i + j^2/n). & \text{otherwise;} \end{cases}$$

$$\max\{r, s\} = i$$

$$\min\{r, s\} = j \tag{1.5}$$

We note that the estimate (1.4) is sharper then the estimate (1.2).

## 2. Proof of Theorem 2

We use the following results [2]:

$$\left.\begin{aligned}
&\sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}(a, b) .\, k = na \\
&\sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}(a, b) .\, k^2 = n\,(n-1)a^2 + na \\
&\sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}(a, b) .\, l = nb \\
&\sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{n,k,l}(a, b) .\, l^2 = n\,(n-1)\,b^2 + nb
\end{aligned}\right\} \tag{2.1}$$

It follows easily from (1.3) that

$$\left| \frac{\partial^{r+s} p_n{}^2 f(x)}{\partial a^r\, \partial b^s} - \frac{\partial^{r+s} f(a+x, b+y)}{\partial a^r\, \partial b^s} \right|$$

$$\leqslant \left| \frac{\partial^{r+s} f(a+x, b+y)}{\partial a^r\, \partial b^s} \right| \cdot \left| 1 - d_{n,r,s} \right| \qquad +$$

$$+ \left| d_{n,r,s} \right| \cdot \sum_{k=0}^{n-r-s} \sum_{l=0}^{n-r-s-k} p_{n-r-s-k-l}(a, b) \ldots$$

$$\ldots \left| f^{r+s}\left(x + \frac{k + r\theta_k}{n}, y + \frac{l + s\theta_l}{n}\right) - f^{r+s}(a+x, b+y) \right|,$$

where

$$d_{n,r,s} = \frac{n(n-1) \ldots (n-s+1)(n-s) \ldots (n-s-r+1)}{n^{r+s}}$$

and

$$\theta_k, \theta_l \in (0, 1).$$

Using (2.1) and the inequality

$$\left| f^{r+s}\left(x + \frac{k + r\theta_k}{n}, y + \frac{l + s\theta_l}{n}\right) - f^{r+s}(a + x, b + y) \right|$$

$$\leqslant \left[1 + 1/\delta \left\{(k/n-a)^2 + (l/n-b)^2 + \frac{r^2+s^2}{n} + \frac{2(rk+sl)}{n^2}\right\}^{\frac{1}{2}} \right] w(\delta)$$

for $\delta > 0$, we get after a little calculation that

$$\left| \frac{\partial^{r+s} p^2{}_n f(X)}{\partial a^r \, \partial b^s} - \frac{\partial^{r+s} f(a + x, b + y)}{\partial a^r \, \partial b^s} \right|$$

$$\leqslant S_1 + S_2, \ (say).$$

Clearly

$$S_1 = \left| \frac{\partial^{r+s} f(a + x, b + y)}{\partial a^r \, \partial b^s} \right| \cdot \left| 1 - d_{n,r,s} \right|$$

$$\leqslant \frac{(r + s)(r + s - 1)}{2n} M_{r,s}, \qquad (cf. \ [3])$$

and

$$S_2 = w(\delta) \left[1 + \frac{T^{1/2}}{n\delta}\right],$$

where

$$T = \{ (r + s)^2 - (n - r - s) \} (a^2 + b^2)$$

$$+ (n - r - s) \{ (2r + 1) a + (2s + 1) b \} + r^2 + s^2.$$

The main proof of this theorem is based on the optimization of expression $T$ over $\triangle$.

Putting $a + b = k, k \in [0, 1]$, we get that

$$T = 2 (\lambda a^2 - k\lambda a + \mu a) + k^2\lambda + k (2s+1) (n - r - s)+r^2+s^2, \tag{2.2}$$

where

$$\lambda = (r + s)^2 - (n - r - s) \textit{ and } \mu = (r - s) (n - r - s).$$

**Case 1**. Let $\lambda \geqslant 0$ in (2.2).

($A_1$). Further if $r \geqslant s$, then the maximum of (2.2) is

$$k^2\lambda + (2r + 1) (n - r - s) k + r^2 + s^2, k \in [0, 1].$$

So the maximum of (2.3) occurs at $k = 1$ and it is

$$\lambda + (2r + 1) (n - r - s) + r^2 + s^2$$

$$= 2rn + 2s^2.$$

($A_2$). Again, when $r \leqslant s$, then the required maximum of (2.2) is

$$2s_n + 2r^2.$$

**Case 2**. Let $\lambda < 0$ in (2.2).

($A_1$). When $r > s$, we get similarly

$$T_{max} = 2rn + 2s^2$$

$(A_2)$. When $r < s$, we get

$$T_{max} = 2s_n + 2r^2.$$

$(A_3)$. When $r = s$, then the maximum of (2.2) is

$$\frac{(4s^2 + 2s - n)}{2} k^2 + (2s + 1)(n - 2s) k + 2s^2, k \in [0, 1].$$

Now the maximum of this expression occurs at $k = 1$ and it is

$$\frac{8s^2 + (4s + 1)(n - 2s)}{2}$$

$$= (\tfrac{1}{2} + 2s) n - s.$$

Finally, we conclude that

$$T_{max} = n. \lambda^*_{n,r,s} ,$$

where

$\lambda^*_{n,r,s}$ is given by (1. 5).

By choosing $\delta = 1/\sqrt{n}$, we get

$$S_2 \leqslant (1 + \sqrt{\lambda^*_{n,r,s}}) w (1/\sqrt{n}).$$

This complete the proof of theorem 2.

**Acknowledgements**

The authors are thankful to Prof. O. P. Varshney, University of Roorkee, U. P., for his valuable help and suggestion.

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