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SOME RESULTS ON FIXED POINTS FOR ORBITALLY CONTINUOUS FUNCTIONS

by

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ABSTRACT

In this paper some fixed point theorems in orbitally complete metric spaces for orbitally continuous functions are obtained; these theorems generalize some results of D. S. Jaggi [2].

1. Introduction

Ćirić [1] has introduced the concept of orbitally continuous mappings and orbitally complete spaces.

Let (x, d) denote a metric space and f a self-map defined on it. An orbit of f at a point $x \in X$ is the set

$$O(x, f) = \{x, f(x), \ldots, f^n(x), \ldots\}.$$

Definition 1. X is said to be (x, f)-orbitally complete if $\bar{O}(x, f)$, the closure of $O(x, f)$, is complete. Further, X is said be f-orbitally complete if it is (x, f)-orbitally complete for all $x \in X$.

Definition 2. A self-map f difined on X is said to be x-orbitally continuous if $f/\bar{O}(x, f)$ is continuous and f is said to be orbitally continuous if it is x-orbitally continuous for all $x \in X$.

It is known that every complete metric space is orbitally complete and every continuous function of X into itself is orbitally continuous, but the converse of these statements are not true [1].

2. Main Results

Theorem 1. *Let f be a self-map defined on a metric space (X, d) satisfying*

$$(*)\quad d(f(x), f(y)) \leqslant \frac{\alpha_1\, d(x, f(x))\, d(y, f(y))}{d(x, y)}$$

$$+ \frac{\alpha_2\, d(x, f(y))\, d(y, f(x))}{d(x, y)} + \frac{\alpha_3\, d(y, f(x))\, d(y, f(y))}{d(x, y)}$$

$$+ \frac{\alpha_4\, d(x, f(x))\, d(y, f(x))}{d(f(x), f(y))}$$

$$+ \beta_1\, d(x, y) + \beta_2\, d(x, f(x)) + \beta_3 d(y, f(y))$$
$$+ \beta_4 d(x, f'y)) + \beta_5 d(y, f(x))$$

for all $x, y \in X$, $x \neq y$, $f(x) \neq f(y)$ and for some $\alpha_i, \beta_j \in [0, 1)$, where $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$ with $\alpha_1 + 2\alpha_3 + 2\alpha_4 + \beta_1 + \beta_2 + \beta_3$ $2\beta_5 < 1$ and $\alpha_2 + \beta_1 + \beta_4 + \beta_5 < 1$. It there exists sone $x_0 \in X$ such that X is (x_0, f)-orbitally complete and f is x_0-orbitally continuous, then f has a unique fixed point.

Proof. Let $\{x_n\}$ be a sequence of iterates of f at x.

If $x_n = x_{n+1}$ for some n, then the result is immediate.

So let $x_n \neq x_{n+1}$ for all n. Now

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$$

$$\leqslant \frac{\alpha_1 d(x_n, f(x_n))\, d(x_{n-1}, f(x_{n-1}))}{d(x_n, x_{n-1})} + \frac{\alpha_2 d(x_n, f(x_{n-1}))\, d(x_{n-1}, f(x_n))}{d(x_n, x_{n-1})}$$

$$+ \frac{\alpha_3 d(x_{n-1}, f(x_n))\, d(x_{n-1}, f(x_{n-1}))}{d(x_n, x_{n-1})} + \frac{\alpha_4 d(x_n, f(x_n))\, d(x_{n-1}, f(x_n))}{d(f(x_n), f(x_{n-1}))}$$

$$+ \beta_1 d(x_n, x_{n-1}) + \beta_2 d(x_n, f(x_n)) + \beta_3 d(x_{n-1}, f(x_{n-1}))$$
$$+ \beta_4 d(x_n, f(x_{n-1})) + \beta_5 d(x_{n-1}, f(x_n))$$

which implies that

$$d(x_{n+1}, x_n) \leqslant K d(x_n, x_{n-1}) \leqslant \ldots \leqslant K^n d(x_1, x_0),$$

where $K = \left(\dfrac{\alpha_3 + \alpha_4 + \beta_1 + \beta_3 + \beta_5}{1 - \alpha_1 - \alpha_3 - \alpha_4 - \beta_2 - \beta_5} \right) < 1.$

Using triangle inequality, we can easily check that $d(x_n, x_m) \to 0$ as $m, n \to \infty$ independently. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is (x_0, f)-orbitally complete, there exists a $u \in \bar{O}(x_0, f)$ such that $\{x_n\} \to u$ as $n \to \infty$. As f is x_0-orbitally continuous, we have

$$f(u) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = u.$$

This shows that u is a fixed point of f. The uniqueness of u follows immediately from the condition (*) and the fact $\alpha_2 + \beta_1 + \beta_4 + \beta_5 < 1$.

Corollary. *Let f be a continuous self-map difined on a complete metric space (X, d). If f satisfies the condition (*), then f has a unique fixed point.*

It is interesting to note that there exist functions which may not be x-orbitally continuous for any $x \in X$, yet some of their iterates may be x-orbitally continuous, in fact orbitally continuous. This inspires the following result.

Theorem 2. *Let f be a self-map defined on a metric space (X, d) such that the condition (*) holds. If, for some x_0 and a positive integer m, f^m is x_0-orbitally continuous and X is (x_0, f^m)-orbitally complete, then f has a unique fixed point.*

Proof. It immediately follows that the sequence $\{x_n\}$ of iterates of f at x_0 is Cauchy sequence. Therefore, its subsequence $\{x_{n_k}$ $(n_k = km)$ is also Cauchy. Further the subsequence is in $O(x_0, f^m)$ which is complete, so there exists $u \in O\ (x_0, f^m)$ such that $\{x_{n_k} \to u$ as $k \to \infty$.

Since f^m is x_0-orbitally continuovs, we have

$$f^m(u) = f^m\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} f^m(x_{n_k})$$

$$= \lim_{k \to \infty} x_{n_{k+1}} = u.$$

Therefore u is a fixed point of f^m. We now show that $f(u) = u$.

If $f(u) \neq u$, then

$$d(f(u), u) = d(f(u), f^m(u))$$

$$\leqslant \frac{\alpha_1\, d(u, f(u))\, d(f^{m-1}(u), f^m(u))}{d(u, f^{m-1}(u))} + \frac{\alpha_2\, d(u, f^m(u))\, d(f^{m-1}(u), f(u))}{d(u, f^{m-1}(u))}$$

$$+ \frac{\alpha_3 d(f^{m-1}(u), f(u))\, d(f^{m-1}(u), f^m(u))}{d(u, f^{m-1}(u))}$$

$$+ \frac{\alpha_4\, d(u. f(u))\, d(f^{m-1}(u), f(u))}{d(f(u), f^m(u))}$$

$$+ \beta_1\, d(u, f^{m-1}(u)) + \beta_2\, d(u, f(u))\ \beta_3\, d(f^{m-1}(u), f^m(u))$$

$$+ \beta_4\, d(u, f^m(u)) + \beta_5\, d(f^{m-1}(u), f(u))$$

which implies that

$$d(f(u), u) \leqslant Kd(u, f^{m-1}(u)),$$

$$\text{where } K = \left(\frac{\alpha_3 + \alpha_4 + \beta_1 + \beta_3 + \beta_5}{1 - \alpha_1 - \alpha_3 - \alpha_4 - \beta_2 - \beta_5}\right).$$

It is easy to see that

$$d(u, f^{m-1}(u)) = d(f^m(u), f^{m-1}(u))$$

$$\leqslant K\, d(f^{m-1}(u), f^{m-2}(u)) \leqslant \ldots \leqslant K^{m-1}\, d(f(u), u))$$

Therefore, $d(f(u), u) \leqslant K^m (d(f(u), u)$, a contradiction since $K < 1$.

Hence $f(u) = u$. The uniqueness of u follows easily.

We observe that Theorem 1 can be further generalized to

Theorem 3. *Let f be self-map defined on a metric space (X, d) such that, for some positive integer m, f satisfies the condition*

$$d(f^m(x), f^m(y)) \leqslant \frac{\alpha_1\, d(x, f^m(x))\, d(y, f^m(y))}{d(x, y)}$$

$$+ \frac{\alpha_2\, d(x) f^m(y))\, d\, y, f^m(x))}{d(x, y)}$$

$$+ \frac{\alpha_3\, d(y, f^m(x))\, d(y, f^m(y))}{d(x, y)} + \frac{\alpha_4\, d(x, f^m(x))\, d(y, f^m(x))}{d(f(x), f(y))}$$

$$+ \beta_1\, d(x, y) + \beta_2\, d(x, f^m(x)) + \beta_3\, d(y, f^m(y))$$

$$+ \beta_4\, d(x, f^m(y)) + \beta_5\, d(y, f^m(x))$$

for all $x, y \in X$, $x \neq y$, $f^m(x) \neq f^m(y)$ and for some $\alpha_i, \beta_j \in [0, 1]$, where $i = 1, 2, 3, 4$, and $j = 1, 2, 3, 4, 5$ with $\alpha_1 + 2\alpha_3 + 2\alpha_4 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1$ and $\alpha_2 + \beta_1 + \beta_4 + \beta_5 < 1$. If there exists some x_0

such that f^m is x_0-orbitally continuous and X is (x_0, f^m)-orbitally complete, then f has a unique fixed point.

Proof. The existence of a unique fixed point u (*say*) of f^m follows from Theorem 1. Also,

$$f(u) = f(f^m(u)) = f^m(u)).$$

This implies that $f(u) = u$. Further, since a fixed point of f is also a fixed point of f^m and u is a unique fixed point of f^m, it follows that u is a unique fixed point of f.

Remark. If we take $\alpha_i = o (i = 2, 3, 4)$ and $\beta_j = o$ $(j=2, 3, 4, 5)$ in the above results, we obtain some results, due to Jaggi ([2], Theorem 1 (including Corollary), Theorem 2 and Theorem 3).

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