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SEICHES ASSOCIATED WITH A CERTAIN CLASS OF LAKES

by

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ABSTRACT

Longitudinal seiches of small amplitude are investigated in the case of a type of lake with a single basin. The results depend upon the properties of the hypergeometric differential equation and its solutions.

1. Introduction

Systems of standing waves in lakes, known as seiches, are almost always present to some extent and are occasionally observed to have amplitudes of several feet. For a review and discussion of certain aspects of the associated theory (see Chrystal [1] and Hutchinson [4, *p*. 299 *et seq*.]).

In many instances, although the amplitude of the vertical oscillations may be quite small, the amplitude of the corresponding oscillations in the horizontal current is significent, and as such is of interest when the question of the deposition of sediment is involved. The rerder is also referred to various papers by Hidaka (see Hidaka [3] for example).

For one-dimensional problems of the type under consideration, seiches may either longitudinal or transverse, althogh the letter tend to be comparatively rare unless the breadth of the lake in question is appreciable compared with its length.

2. General Theory

Longitudinal seiches of small amplitude, where the wavelength is much greater then the depth of water, such that the vertical acceleration is negligable, are governed by the partial differential equation

$$\frac{\partial^2 \xi}{\partial t^2} = g \frac{\partial}{\partial x} \left\{ \frac{1}{B(x)} \frac{\partial}{\partial x} \left\{ A(x)\xi \right\} \right\} \tag{1}$$

$$\text{and } \zeta = - \frac{1}{B(x)} \frac{\partial}{\partial x} \left\{ A(x)\xi \right\}, \tag{2}$$

where x is the longitudinal coordinate, $A(x)$ and $B(x)$ are respectively the cross–sectional area and breadth of the free surface of the lake, t the time, g the accelearation due to gravity and ξ and ζ denote respectively the horizontal and vertical displacement of the water of the lake. See Chrystal [1] for example.

For convenience, let

$$u = A(x)\ \xi, \tag{3}$$

$$\text{so that } \frac{\partial^2 u}{\partial t^2} = g\ A(x) \frac{\partial}{\partial x} \left\{ \frac{1}{B(x)} \frac{\partial u}{\partial x} \right\}. \tag{4}$$

For normal modes of oscillation put

$$u = P(x)\ cos\ (wt + \varepsilon), \tag{5}$$

with it is found that the function $P(x)$ is determined by the ordinary differential equation

$$P'' - \frac{B'}{B} P' + \frac{\omega^2}{g H} P = 0. \tag{6}$$

where $H \equiv H(x)$ is the depth of the lake.

For the present purposes, we take $H(x)$ and $B(x)$ to be of the form

$$H(x) = h \; x^{2-\mu} \; (1 - x^{\mu}/l^{\mu}) \tag{7}$$

and

$$B(x) = W \; x^{\nu} (1 - x^{\mu}/l^{\mu})^{\tau} , \tag{8}$$

where l is the length of the lake, the constants ν and τ are positive and $1 \leqq \mu \leqq 2$ depending upon the configuration of the lake in question.

Under these circumstances, it is evident that the boundary conditions are that ξ and ζ should be finite for $0 \leqq x \leqq l$, and, because the cross–sectional area of the lake is zero at both ends, we have

$$A(0) \;\; \xi(0) = u(0) \tag{9}$$

and

$$A(l) \;\; \xi(l) = u(l), \tag{10}$$

which, in turn, imply that

$$P(0) = P(l) = 0. \tag{11}$$

if the expressions for $H(x)$ and $B(x)$ are substituted into (6), we have

$$z(1 - z) \, P'' + \left[1 - \frac{\nu + 1}{\mu} + \left(\frac{\nu + 1}{\mu} + \tau - 1 \right) z \right] P'$$

$$+ \frac{\omega^2 l^{\mu}}{g h \mu^2} \, P = 0, \tag{12}$$

where $z = (x/l)^{\mu}$.

The differential equation (12) is of hypergeometric form, so that a fundamental system of solutions consists of.

$$P_1(x) = {}_2F_1(p, q; r; z) \tag{13}$$

and $$P_2(x) = z^{1-r}\, {}_2F_1\,(1 + p - r, 1 + q - r; 2 - r; z), \tag{14}$$

where $r = 1 - (\nu + 1)/\mu$, $p + q = -\tau - (\nu + 1)/\mu$ and

$$pq = -\frac{\omega^2 l^{\mu}}{gh\mu^2}\ . \tag{15}$$

See Slater [5, *p.* 9].

The solution P_1 is discarded because it does not vanish at the origin, and we consider the behaviour of P_2 as $x = l$, or as $z = 1$. Provided that

$$p + q - r < 0, \tag{16}$$

the hypergeometric function on the right of (14) converges with $z = 1$, and it may then be summed by Gauss's summation theorem. See Exton [2, *p.* 19]. Now $p + q - r = -1 - \tau$, so that the condition (16) is satisfied on account of the fact that, by hypothesis, τ is positive, Hence

$$P_2(l) = \frac{\Gamma(2 - r)\,\Gamma(r - p - q)}{\Gamma(1 - p)\,\Gamma(1 - q)}\ . \tag{17}$$

We note that $2 - r = 1 + (\nu + 1)/\mu$ and $r - p - q = 1 + \tau$, so that niether of the gamma functions of (17) can possibly have an argument which is a negative integer. It thus follows that $P_2(l)$ is finite for all admissable values of its parar eters.

The eigenvalue equation may be obtained from the relation (17), and after a little reduction, we have

$$\omega^2 = \frac{hg\mu^2}{l^\mu} [N^2 + N(\tau + \frac{\nu+1}{\mu})], N = 1, 2, 3, \ldots, \qquad (18)$$

and where we have put $p = N$, so that $1/\Gamma(1-p) = 0$.

Apart from a multiplictative constant, we then have

$$P(x) = (x/l)^{\nu+1} {}_2F_1 \left(\frac{\nu+2}{\mu} + N, -\tau - N; 1 + \frac{\nu+1}{\mu}; \frac{x^\mu}{l^\mu} \right). \qquad (19)$$

This expression may be reduced to a more convenient form by the application of Euler's transformation (see Exton [2, p. 20]). We then obtain the result

$$P(x) = (x/l)^{\nu+1} (1 - x^\mu/l^\mu)^{\tau+1} {}_2F_1 ((1-N, 1 + \frac{\nu+1}{\mu} + N; 1 + \frac{\nu+1}{\mu} ; \frac{x^\mu}{l^\mu}), \qquad (20)$$

where the associated hypergeometric function is seen to be a polynomial in $(x/l)^\mu$ of degree N-1. This function does not converge at $x=l$ unless N is a positive integer, which is consistent with the above analysis.

In order to proceed further, we must discuss the from of $P'(x)$. By straightforward term by term differentiation, which is admissible in the case of a convergent power series, it follows that

$$d/dz [z^{c-1} {}_2F_1 (a, b; c, z)] = (c-1) z^{c-2} {}_2F_1 (a, b; c-1; z). \qquad (21)$$

From (19) we have

$$P'(x) = (\nu + 1 - \mu) \frac{x^\nu}{l^{\nu+1}} {}_2F_1 (\frac{\nu+1}{\mu} + N, -\tau - N; \frac{\nu+1}{\mu} ; \frac{x^\mu}{l^\mu}),$$

$$\mu/l\ (x/l)^{\nu}\left(1-\frac{x^{\mu}}{l^{\mu}}\right)^{\tau}\ {}_2F_1\left(-N,\left(\frac{\nu+1}{\mu}+\tau+N;\frac{\nu+1}{l^{\mu}};\frac{x^{\mu}}{\mu}\right)\right), \tag{22}$$

by futher application of Euler's transformation. We see that the hypergeometric function on the the right of (22) is a polynomial in x^{μ}/l^{μ} of degree N.

We then obtain the following explicit expressions respectively for the horizontal and vertical displacements:

$$\frac{\xi(x,\ t)}{cos\ (\omega t+\varepsilon)}=\frac{Kx^{\mu-1}}{Wh}\left(1-x^{\mu}/l^{\mu}\right)\ {}_2F_1\ (1-N,\ 1+\frac{\nu+1}{\mu}+\tau+N;$$

$$1+\frac{\nu+1}{\mu};\ x^{\mu}/l^{\mu}\Big) \tag{23}$$

and

$$\frac{\zeta(x,\ t)}{cos(\omega t+\varepsilon)}=-\frac{K(\nu+1)l^{\mu-\nu-1}}{W}\ {}_2F_1\left(-N,\ \frac{\nu+1}{\mu}++N;\frac{\nu+1}{\mu};x^{\mu}/l^{\mu}\right). \tag{24}$$

K is an arbitrary constant depending upon the initial conditions. For $\xi(x,\ t)$ to be finite in the closed interval of x $[0,\ l]$, it will be seen that the parameter μ must not be less then unity.

3. Conclusion

The model employed in this study conforms quite closely to many naturally occurring lakes with one single basin, and it is of interest to observe that the periods of the normal modes are given by the equation

$$T_n = 1/\surd\ \{a\ (n^2+bn)\ \}, \tag{2.5}$$

where a and b are suitable constants, and $n = 1,\ 2,\ 3,\ \ldots$.

This expression tends asymptotically to the case of a canal with uniform breadth and depth. It must be stressed, however, that for the higher modes, the conditions of the original approximation leading to equations (1) and (2) no longer apply.

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