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ON CIRIC TYPE MAPS

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1. Introduction and Definitions

Recently, Ciric [2] proved some non-unique fixed point theorems for orbitally continuous self mapping T on an orbitally complete metric space M which satisfy a condition of the type

$$\min \{d(Tx, Ty), d(x, Tx), d(y, Ty)\}$$

$$- \min \{ d(x, Ty), d(y, Tx) \} \leqslant q d(x, y)$$

for all $x, y \in M$ and for some $q \in (0,1)$.

Achari [1] and Pachpatte [3] also proved some non-unique fixed point theorem on Ciric type maps. The purpose of the present paper is to establish a fixed point theorem in metric space for a Ciric type map which is not necessarily continuous.

Definition 1. Let (M,d) be a metric space. A mapping T on M is said to be orbitally continuous if $\lim_i T^{n_i} x = u$ implies $\lim_i TT^{n_i} x = Tu$ for each $x \in M$.

Definition 2. A metric M is said to be orbitally complete if every Cauchy sequence of the from $\left\{ T^{n_i} x \right\}_{i=1}^{\infty}$, $x \in M$ converges in M.

2. The Main Result

Now we state and prove our main result contained is the following

Theorem. *Let $T : M \to M$ be an orbitally continous mapping on M and let M be T orbitally complete metric space. If T satisfies the following condition:*

$$\min \{d(x, Tx)\, d(Tx, Ty), [\, d(x, y)\,]^2, d(x, Tx)\, d(y, Ty)\,\}$$

$$-\min \{\, d(x, Tx)\, d(x, y), d(x, Ty)\, d(y, Tx)\,\} \leqslant q d(x, Tx)\, d(x, Ty) \qquad \ldots(1)$$

for all $x, y \in M$ and $q \in (0,1)$, then, for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Proof. Let $x \in M$ be arbitrary. We define a sequence

$$x_0 = x,\ x_1 = Tx_0,\ x_2 = Tx_1, \ldots, x_n = Tx_{n-1}.$$

If for some n, $x_n = x_{n+1}$, then $\{x_n\}$ is a Cauchy sequence and the limit of $\{x_n\}$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$, for each $n = 0, 1, 2, \ldots$ By (1) for $x = x_{n-1}$ and $y = x_n$, we have

$$\min \{d(x_{n-1}, x_n)\, d(x_n, x_{n-1}), [\, d(x_{n-1}; x_n)\,]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1})\,\}$$

$$-\ \min \{\, [\, d(x_{n-1}, x_n)\,]^2, d(x_{n-1}, x_{n+1})\, d(x_n, x_n)\,\} \leqslant q\, [\, d(x_{n-1}, x_n)\,]^2$$

or

$$\min \{\, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [\, d(x_{n-1}, x_n) \quad]$$

$$-\ \min \{\, [\, d(x_{n-1}, x_n)\,]^2, 0\,\} \leqslant q[\, d(x_{n-1}, x_n)\,]^2$$

or

$$\min \{\, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [\, d(x_{n-1}, x_n)\,]^2\,\} \leqslant q[\, d(x_{n-1}, x_n,)\,]^2.$$

Since

$$[\, d(x_{n-1}, x_n)\,]^2 \leqslant q[\, d(x_{n-1}, x_n)\,]^2 \text{ is impossible (as } q < 1,$$

we have

$$d(x_{n-1}, x_n)\, d(x_n, x_{n+1}) \leqslant q[\, d(x_{n-1}, x_n)\,]^2$$

or

$$d(x_n, x_{n+1}) \leqslant qd(x_{n-1}, x_n).$$

Proceeding in the same manner, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant qd(x_{n-1}, x_n) \\ &\leqslant q^2 d(x_{n-2}, x_{n-1}) \\ &\leqslant q^3 d(x_{n-3}, x_{n-2}) \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\leqslant q^n d(x_0, x_1). \end{aligned}$$

By the triangular inequality, we have for $m > n$

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m), \\ &\leqslant (q^n + q^{n+1} + \ldots q^{m-1})\, d(x_0, x_1), \\ &\leqslant \frac{q^n}{1-q}\, d(x_0, x_1). \end{aligned}$$

Since $q < 1$, the right hand side of the above inequality tends to zero as $m, n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence. M being T–orbitally complete, there is some $u \in M$ such that $u = \lim_n T^n x$.

By orbital continuity of

$$Tu = \lim_n TT^n x = u$$

i. e. u is a fixed point of T. This completes the proof of the theorem.

3. An Illustrative Example

The following example illustrates the generality of the theorem

Example. Let $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = 0, \textit{ for } x \neq \tfrac{2}{3}$$

$$Tx = 1, \textit{ for } x = \tfrac{2}{3}$$

Clearly, T is discontinuous at $x = \frac{2}{3}$, and therefore it is not a contraction map, but it satisfies the inequality (1) for all $x, y \in [0, 1]$ and it has fixed point $x = 0$.

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[1] B. G. Pachpatte, On Ciric type mape with a non-unique fixed point, *Indian J. Pure Appl. Math.* **10** (1979), 1039–1043.

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[3] Lj. B. Ciric, On some maps with a non-uniq ie fixed point, *Publ. Inst. Math.* (*Beograd*) (*N. S.*) **17** (**31**) (1974), 52–58.