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SOME RESULTS ON FIXED POINTS

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1. Introduction

Recently, B. Fisher [2] proved the following theorem for a pair of mappings :

Theorem 1 *Let S and T be a pair of self–mapping of a complete metric space (X, d) satisfying*

$$[d(Sx, Ty)]^2 \leqslant bd(x, Sx)\, d(y, Ty) + cd(x, Ty)\, d(y, Sx)$$

for all x, y in X, where $0 \leqslant b < 1$, $c \geqslant 0$. *Then S and T have a common fixed point. Further, if* $0 \leqslant b; c < 1$, *then each of S and T has a unique fixed point and these two points coincide.*

Here we present a theorem which is a generalization of the above result.

Theorem 2. *Let S and T be the self-mappings of a complete metric space (X, d) satisfying*

$$[\, d(Sx, Ty)\,]^2 \leqslant a \max [d\,(x, Sx)\, d(y, Ty),\, d(x, y)\, d(x, Sx),$$
$$d(x, y)\, d(y, Ty),\, cd(x, Ty)\, d(y, Sx)\,] \qquad (1)$$

for all x, y in X, where $0 < a < 1$, *and* $0 \leqslant c < 1$ *then S and T have a unique common fixed point.*

Proof : Let x_0 be any arbitrary point in X and let the sequence $\{x_n\}$ of elements in X be defined by

$$x_{2n+1} = Sx_{2n}\ ,\ x_{2n+2} = Tx_{2n+1}\ ,\ for\ n=0,\ 1,\ 2 \qquad (2)$$

If $x_n = x_{n+1}$ for $n = 0, 1, 2, \ldots$ then, it immediately follows that $\{x_n\}$ is a Cauchy sequence. So take up the case when $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$. Then, by (1) and (2), we have

$$[\ d(x_{2n+1}\ ,\ x_{2n+2})\ ]^2 = [\ d(Sx_{2n}\ ,\ Tx_{2n+1})\ ]^2$$

$$\leqslant a\ max\ [\ d(x_{2n},\ x_{2n+1})\ d(x_{2n+1},\ x_{2n+2}),\ d(x_{2n},\ x_{2n+1})\ d(x_{2n},\ x_{2n+1}),$$

$$d(x_{2n},\ x_{2n+1})\ d(x_{2n+1},\ x_{2n+2}),\ c\ d(x_{2n},\ x_{2n+2})\ d(x_{2n+1},\ x_{2n+1})\ ],$$

$$\leqslant a\ max\ \{\ d(x_{2n},\ x_{2n+1})\ d(x_{2n+1},\ x_{2n+2}),\ [\ d(x_{2n},\ x_{2n+1})\ ]^2$$

$$d(x_{2n},\ x_{2n+1})\ d(x_{2n+1},\ x_{2n+2}),\ 0\ \}$$

then,

either,

$$[\ d(x_{2n+1},\ x_{2n+2})\ ]^2 \leqslant a\ d(x_{2n},\ x_{2n+1})\ d(x_{2n+1},\ x_{2n+2})$$

which implies

$$d(x_{2n+1},\ x_{2n+2}) \leqslant a\ d(x_{2n},\ x_{2n+1})$$

or

$$[\ d(x_{2n+1},\ x_{2n+2})\ ]^2 \leqslant a\ [\ d(x_{2n},\ x_{2n+1})\ ]^2$$

which implies

$$d(x_{2n+1},\ x_{2n+2}) \leqslant a^{\frac{1}{2}}\ d(x_{2n},\ x_{2n+1})$$

Since $a < a^{\frac{1}{2}} < 1$, as $0 < a < 1$, it follows that

$$d(x_{2n+1}, x_{2n+2}) \leqslant a^{\frac{1}{2}} d(x_{2n}, x_{2n+1}) \tag{3}$$

for all $n=0, 1, 2,...$

Similarly, we can prove

$$d(x_{2n}, x_{2n+1}) \leqslant a^{\frac{1}{2}} d(x_{2n-1}, x_{2n}) \tag{4}$$

for all $n = 1, 2,...$

Thus, by (3) and (4), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leqslant a^{\frac{1}{2}} d(x_{2n}, x_{2n+1}) \leqslant (a^{\frac{1}{2}})^2 d(x_{2n-1}, x_{2n}) \\ &\leqslant (a^{\frac{1}{2}})^{2n+1} d(x_0, x_1) \end{aligned} \tag{5}$$

Since $a^{\frac{1}{2}} < 1$, the right-hand side of (5) tends to zero as $n \to \infty$. Hence it follows that $\{x_n\}$ is a Cauchy sequence and it will converge to a point z (say) as X is complete metric space.

Now, by triangular inequality, we have

$$\begin{aligned} [d(z, Sz)]^2 &\leqslant [d(z, x_{2n+2}) + d(Sz, Tx_{2n+1})]^2, \\ &\leqslant [d(z, x_{2n+1})]^2 + [d(Sz, Tx_{2n+1})]^2 + 2d(z, x_{2n+2}) \\ &\quad d(Sz, Tx_{2n+1}). \end{aligned} \tag{6}$$

Since

$$\begin{aligned} [d(Sz, Tx_{2n+1})]^2 \leqslant a \max\ [&d(z, Sz)\, d(x_{2n+1}, x_{2n+2}),\ d(z, x_{2n+1})\, d(z, Sz), \\ &d(z, x_{2n+1})\, d(x_{2n+1}, x_{2n+2}),\ cd(z, x_{2n+2})\, d(x_{2n+1}, Sz)] \end{aligned}$$

on letting $n \to \infty$, we have

$$[d(Sz, Tx_{2n+2})]^2 = 0.$$

Therefore, as $n \to \infty$, we have by (6)

$$[\, d(z, Sz)\,]^2 = 0$$

which implies $z = Sz$.

Similarly, we can prove that $z = Tz$.

Thus, we have

$$z = Tz = Sz.$$

i. e. z is the common fixed point of S and T.

Now to show the uniqueness of z, let us consider that $w \neq z$ is another common fixed point of S and T. Then

$$\begin{aligned}[\, d(z, w)\,]^2 &= [\, d(Sz, Tw)\,]^2 \\ &\leqslant a \max [\, d(z, Sz)\, d(w, Tw),\, d(z, w)\, d(z, Sz) \\ &\qquad d(z, w)\, d(w, Tw),\, c\, d(z, Tw)\, d(w, Sz)\,] \\ &\leqslant ac\, [\, d(z, w)\,]^2.\end{aligned}$$

But $ac < 1$ as $0 < a < 1$ and $0 \leqslant c < 1$, threfore, it follows that $[\, d(z, w)\,]^2 = 0$ and so $z = w$. This completes the proof.

3. Another Fixed Point Theorem

In this section we prove

Theorem 3. *Let X be a Hausdorff space, and let T_1 and T_2 be two continuous self–mappings of X. Let F be a symmetric continuous mapping of $X \times X$ into the set of non–negative reals satisfying*

$$[\, F(T_1x, T_2y)\,]^2 \leqslant a \max [\, F(x, T_1x),\, F(y, T_2y),\, F(x, y)\, F(x, T_1x),$$
$$F(x, y)\, F(y, T_2y),\, c\, F(x, T_2y)\, F(y, T_1x)\,] \quad (7)$$

for all distinct x, y in X, where $0 < a < 1$ *and* $0 \leqslant c < 1$. *Also let* $F(x, x) = 0$ *for all* x *in X.*

If, for some x_0 *in X, the sequence* $\{x_n\}$, *defined by*

$$x_{2n+1} = T_1 x_{2n}\ ,\ x_{2n+2} = T_2 x_{2n+1}\ ,\ \text{for } n=0, 1, 2, \ldots, \qquad (8)$$

has a convergent subsequence $\{x_{n_k}\}$ *in X, then* T_1 *and* T_2 *have a unique common fixed point.*

Proof : Assume $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \ldots$;

then, by (7) and (8),

$$[\ F(x_1, x_2)\ ]^2 \leqslant [\ F(T_1 x_0, T_2 x_1)\ ]^2$$

$$\leqslant a \max [\ F(x_0, x_1)\ F(x_1, x_2),\ F(x_0, x_1)\ F(x_0, x_1),$$

$$F(x_0, x_1)\ F(x_1, x_2),\ cF(x_0, x_2)\ F(x_1, x_1)\ ].$$

Thus, either

$$[\ F(x_1, x_2)\ ]^2 \leqslant a\ F(x_0, x_1)\ F(x_1, x_2)$$

which implies

$$F(x_1, x_2) \leqslant a^{\frac{1}{2}}\ F(x_0, x_1)$$

or

$$[\ F(x_1, x_2)\ ]^2 \leqslant a\ [\ F(x_0, x_1)\ ]^2$$

which implies

$$F(x_1, x_2) \leqslant a^{\frac{1}{2}}\ F(x_0, x_1).$$

Since $a < a^{\frac{1}{2}} < 1$, it follows that

$$F(x_1, x_2) \leqslant a^{\frac{1}{2}} F(x_0, x_1) < F(x_0, x_1).$$

Similarly,

$$[F(x_2, x_3)]^2 \leqslant [F(T_2x_1, T_1x_2)]^2 = [F(T_1x_2, T_2x_1)]^2$$

$$\leqslant a \max [F(x_2, T_1x_2) F(x_1, T_2x_1), F(x_2, x_1) F(x_2, T_1x_2),$$

$$F(x_2, x_1)F(x_1, T_2x_1), cF(x_2, T_2x_1)F(x_1, T_1x_2)]$$

$$\leqslant a \max [F(x_2, x_3) F(x_1, x_2), F(x_2, x_1) F(x_2, x_3),$$

$$F(x_2, x_1) F(x_1, x_2), cF(x_2, x_2) F(x_1, x_3)],$$

which implies

$$F(x_2, x_3) \leqslant a^{\frac{1}{2}} F(x_1, x_2) < F(x_1, x_2).$$

By repeating the same process in a similar manner we get a monotone sequence of non-negative real numbers

$$F(x_0, x_1) > F(x_1, x_2) > F(x_2, x_3) > \ldots > F(x_n, x_{n+1}) > \ldots$$

which must converge along with all its subsequences to some real number q (say). By hypothesis, we have a convergent subsequence of $\{x_n\}$ in X which converges to some z in X i. e.

$$z = \lim_{k \to \infty} x_{2n_k} .$$

Since T_1 and T_2 are continuous

$$T_1z = T_1 \lim_{k \to \infty} (x_{2n_k}) = \lim_{k \to \infty} (x_{2n_k} + 1).$$

$$T_2T_1z = T_2 \lim_{k\to\infty} (z_{2n_k+1}) = \lim_{k\to\infty} (x_{2n_k+2})$$

Now, from the continuity of F, we have

$$F(z, T_1z) = F(\lim_{k\to\infty} x_{2n_k}, \lim_{k\to\infty} x_{2n_k+1})$$

$$= \lim_{k\to\infty} F(x_{2n_k}, x_{2n_k+1})$$

$$= q$$

$$= \lim_{k\to\infty} F(x_{2n_k+1}, x_{2n_k+1})$$

$$= F(T_1z, T_2T_1z),$$

If $z \neq T_1z$, then by the inequality (7), we have

$$[F(T_1z, T_2T_1(z)]^2 \leqslant a \max [F(z, T_1z)\, F(T_1z, T_2T_1z), \; F(z, T_1z)\, F(z, T_1z), \; F(z, T_1z)\, F(T_1z, T_2T_1z), \; cF(z, T_2T_1z)\, F(T_1z, T_1z)],$$

which implies

$$F(T_1z, T_2T_1z) \leqslant a^{\frac{1}{2}} F(z, T_1z) < F(z, T_1z)$$

and so we have

$$F(z, T_1z) = F(T_1z, T_2T_1z) < F(z, T_1z)$$

a contradiction. Hence $z = T_1z$.

Similaly, we may prove that $z = T_2z$. Hence it follows that z is the common fixed point of T_1 and T_2.

Now, to show the uniqueness of z, let us consider that w is the another common fixed point of T_1 and T_2.

Then

$$
\begin{aligned}
F(z, w)]^2 = [F(T_1 z, T_2 w)]^2 & \leqslant a \max [F(z, T_1 z) F(w, T_2 w), \\
& F(z, w) F(z, T_1 z), \\
& F(z, w) F(w, T_2 w), \\
& cF(z, T_2 w) F(w, T_1 z)] \\
& \leqslant a. c [F(z, w)]^2.
\end{aligned}
$$

Since ac < 1, it follows that $F(z, w) = 0 \Rightarrow z = w$. This completes the proof of **Theorem 3**.

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