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ON GENERALISED FRACTIONAL INTEGRAL AND INTEGRAL TRANSFORMS

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ABSTRACT

We first obtain the generalised fractional integrals of the H–function of one and several variables. Next we obtain four theorems showing relations between certain generalised fractional integrals and certain generalised integral transforms of one and several variables.

1. Introduction

Here we study the following generalised fractional integral operators of several variables :

$$R\left[\begin{matrix}(\alpha_r), (\gamma_r), (\mu_r) \\ (\beta_r), (m_r'), (\eta_r)\end{matrix}; f; z_1, \ldots, z_r\right]$$

$$= \prod_{i=1}^{r}\left\{\frac{\mu_i\, z_i^{-\eta_i-1}}{\Gamma(1-\alpha_i)}\right\}\int_0^{z_1}\cdots\int_0^{z_r}\prod_{i=1}^{r}\left\{t_i^{\eta_i}\; {}_2F_1\left(\begin{matrix}\alpha_i, \beta_i+m_i' \\ \gamma_i\end{matrix}; \frac{t_i^{\mu_i}}{z_i^{\mu_i}}\right)\right\}$$

$$\cdot f(t_1, \ldots, t_r)\, dt_1 \ldots dt_r \tag{1.1}$$

and

$$K\left[\begin{matrix}(\alpha_r), (\gamma_r), (\mu_r) \\ (\beta_r), (m_r'), (\delta_r)\end{matrix}; f; z_1, \ldots, z_r\right]$$

$$= \prod_{i=1}^{r} \left[\frac{\mu_i \, z_i^{\delta_i}}{\Gamma(1-a_i)} \right] \int_{z_1}^{\infty} \cdots \int_{z_r}^{\infty} \prod_{i=1}^{r} \left\{ t_i^{-\delta_i - 1} \; {}_2F_1 \left(\begin{matrix} a_i, \beta_i + m_i \\ \gamma_i \end{matrix} ; \frac{z_i^{\mu_i}}{t_i^{\mu_i}} \right) \right\}$$

$$. f(t_1, \ldots, t_r) \, dt_1 \ldots dt_r. \qquad (1.2)$$

These operators exist provided that : for $i = 1, \ldots, r$

(i) $m_i' = 0, 1, \ldots$ and $\mu_i > 0$

(ii) $\gamma_i \neq 0, -1, -2, \ldots$, Re $(1-\alpha_i) > 0$, Re $(1 + \gamma_i - \alpha_i - \beta_i - m_i') > 0$

(iii) Re $(\eta_i + l_i + 1) > 0$

(iv) Re $(b_i) > 0$ or Re $(b_i) = 0$ and Re $(\delta_i - l_i') > 0$

where we assume that

$$f(t_1, \ldots, t_r) = o\,(t_1^{l_1} \cdots t_r^{l_r}\,.\,),\; max\,\{t_1, \ldots, t_r\} \to 0$$

$$= o\,(e^{-b_1 t_1 - \ldots - b_r t_r}\; t_1^{l_1'} \ldots t_r^{l_r'}),\; min\,\{t_1, \ldots, t_r\} \to \infty.$$

In case $r = 2$ these fractional operators reduce to the operators given earlier in [2, *p*. 233, *eqns*. (1.1) *and* (1.2)].

Here we also require the following generalised fractional integral operators involving Fox's H–function : [3, *p*. 39, *eqns*. (3 5 1) *and* (3. 5. 2)]

$$R_{\mu,\,\sigma}^{\eta,\,a,\,\lambda} \; f(t) \; ; x \,]$$

$$= \mu \, x^{-\eta - \mu a - 1} \int_0^x t^{\eta} \, (x^{\mu} - t^{\mu})^{\alpha} \, H_{P,\,Q}^{M,\,N} \left[kU \,\middle|\, \begin{matrix} (c_j, \gamma_j)_{1,P} \\ (d_j, \delta_j)_{1,Q} \end{matrix} \right] f(t)\,dt \qquad (1.3)$$

$$K^{\delta,\,\alpha,\,\lambda}_{\mu,\,\sigma}\,[\,f(t);\,x\,]$$

$$= \mu\, x^{\delta} \int_x^{\infty} t^{-\delta-\mu\alpha-1}\,(t^{\mu}-x^{\mu})^{\alpha}\, H^{M,\,N}_{P,\,Q}\left[kV \,\middle|\, \begin{matrix}(c_j,\,\gamma_j)_{1,P}\\(d_j,\,\delta_j)_{1,Q}\end{matrix}\right] f(t)\,dt \tag{1.4}$$

where

$$U = \left(\frac{t^{\mu}}{x^{\mu}}\right)^{\lambda}\left(1-\frac{t^{\mu}}{x^{\mu}}\right)^{\sigma} \tag{1.5}$$

$$V = \left(\frac{x^{\mu}}{t^{\mu}}\right)^{\lambda}\left(1-\frac{x^{\mu}}{t^{\mu}}\right)^{\sigma} \tag{1.6}$$

and μ, λ, σ are positive provided that the integrals involved exist.

The H–function transform of $f(t)$ is defined as : [3, *p*. 42, *eqn.* (4. 2. 1)]

$$\bar{f}\,(s) = \int_0^{\infty} H^{m,\,n}_{p,\,q}\left[(st)^{\nu} \,\middle|\, \begin{matrix}(a_j,\,\alpha_j)_{1,p}\\(b_j,\,\beta_j)_{1,q}\end{matrix}\right] f(t)\,dt,\ \nu > 0 \tag{1.7}$$

provided that the integral is absolutely convergent. We also study here the following multidimensional integral transform involving the H–function of several variables defined by Srivastava and Panda [4, *P*. 121, *eqn*. (1. 15)] :

$$\bar{f}\,[s_1,\dots,s_r] = s_1 \dots s_r \int_0^{\infty} \dots \int_0^{\infty} H_1\left[\begin{matrix}(s_1\,t_1)^{h_1}\\ \vdots \\ (s_r\,t_r)^{h_r}\end{matrix}\right] \cdot f(t_1,\dots,t_r)\,dt_1 \dots dt_r \tag{1.8}$$

provided that the multiple integral in (1.8) exists, where

$$H\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right] = H^{0,\,n:\,m_1,\,n_1\,;\dots;\,m_r,\,n_r}_{p,\,q:\,p_1,\,q_1\,;\dots;\,p_r,\,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\,\right|$$

$$\left.\begin{matrix}(a_j;\ \alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\ \gamma_j^{(r)})_{1,p_r}\\(b_j;\ \beta_j',\ldots,\beta_j^{(r)})_{1,q} : (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\ \delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

and $H_1\begin{bmatrix}z_1\\ \vdots\\ z_r\end{bmatrix}$ is its particular case $n=0$, (the H-function of several variables is defined in Srivastava, Gupta and Goyal [3, p. 251, eqn. (C. 1)]).

2. The Fractional Integrals

For $\nu, \lambda, \mu, \sigma > 0$, we have :

$$R^{\eta,\alpha,\lambda}_{\mu,\sigma}\left[H^{m,n}_{p,q}\left[(zt)^{\nu}\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};x\right]\right]$$

$$=\left(\sum_{h=1}^{M}\sum_{r=1}^{\infty}\frac{(-1)^r\,(k)^{\rho_r}}{r!\,\delta_h}\,g(\rho_r)\,\Gamma(\alpha+\sigma\rho_r+1)\right.$$

$$\left.\cdot\,H^{m,n+1}_{p+1,q+1}\left[(zx)^{\nu}\,\middle|\,\begin{matrix}(1-(\eta/\mu)-(1/\mu)-\lambda\,\rho_r,\ \nu/\mu),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(-(\eta/\mu)-(1/\mu)-\alpha-(\lambda+\mu)\rho_r),\nu/\mu)\end{matrix}\right]\right) \quad (2.1)$$

where

$$g(\rho_r)=\frac{\prod\limits_{j=1\ j\neq h}^{M}\Gamma(d_j-\delta_j\,\rho_r)\prod\limits_{j=1}^{N}\Gamma(1-c_j+\gamma_j\,\rho_r)}{\prod\limits_{j=M+1}^{Q}\Gamma(1-d_j+\delta_j\,\rho_r)\prod\limits_{j=N+1}^{P}\Gamma(c_j-\gamma_j\,\rho_r)} \quad (2.2)$$

and $\rho_r=(d_h+r)/\delta_h$, (2 3)

with $\min\operatorname{Re}[\eta+\nu\,b_i/\beta_i+\mu\lambda\,d_j/\delta_j+1]>0$,

$\min\operatorname{Re}[\alpha+\sigma(d_j/\delta_j)+1]>0,\ i=1,\ldots,m;\ j=1,\ldots,M,$

provided that the series in (2.1) is absolutely convergent;

$$K^{\delta, \alpha, \lambda}_{\mu, \sigma} [(H^{m, n}_{p, q} [(zt)^{V} | {}^{(a_j, \alpha_j)_{1,p}}_{(b_j, \beta_j)_{1,q}}] ; x]$$

$$= \sum_{h=1}^{M} \sum_{r=0}^{\infty} \frac{(-1)^r (k)^{\rho_r}}{r! \, \delta_h} g(\rho_r) \Gamma(\alpha + 1 + \sigma \rho_r)$$

$$. H^{m+1, n}_{p+1, q+1} \left[(zx)^{V} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p}, (\alpha + (\delta/\mu + (\sigma+\lambda)\rho_r + 1, v/\mu) \\ ((\delta/\mu) + \lambda\rho_r, v/\mu), (b_j, \beta_j)_{1,q} \end{matrix} \right] \tag{2.4}$$

($g(\rho_r)$ and ρ_r being given by (2.2) and (2.3))

with Re $[\delta - (v(a_i - 1)/a_i) + \mu\lambda (d_j/\delta_j)] > 0$,

min Re $[\alpha + \sigma (d_j/\delta_j) + 1] > 0$, $i = 1, \ldots, n$; $j = 1, \ldots, M$,

provided that the series in (2.4) is absolutely convergent;

$$R^{\eta, \alpha, \lambda}_{\mu, \sigma} \left[H \middle| \begin{matrix} (y_1 t)^{v_1} \\ \vdots \\ (y_r t)^{v_r} \end{matrix} \middle| ; x \right]$$

$$= H^{0, n+1 : * ; \ldots ; * ; M, N+1}_{p+1, q+1 : * ; \ldots ; * ; P+1, Q}$$

$$\left[\begin{matrix} (y_1 x)^{v_1} \\ \vdots \\ (y^r x)^{v_r} \\ k \end{matrix} \middle| \begin{matrix} P_1 : * ; \ldots ; * ; (-\alpha, \sigma), (c_j, \gamma_j)_{1,P} \\ Q_1 : * ; \ldots ; * ; (d_j, \delta_j)_{1,Q} \end{matrix} \right] \tag{2.5}$$

where

$P_1 : (1 - (\eta/\mu) - (1/\mu); v_1/\mu, \ldots, v_r/\mu, \lambda), (a_j; \alpha_j', \ldots, \alpha_j^{(r)}, 0)_{1,p}$

$Q_1 : (-(\eta/\mu) - (1/\mu) - \alpha; v_1/\mu, \ldots, v_r/\mu, (\sigma + \lambda)), (b_j; \beta_j', \ldots, \beta_j'^{(r)}, 0)_{1,q}$

provided that $\nu_i > 0$, $\min \operatorname{Re}\,[\eta+\mu\lambda(d_h/\delta_h) + \sum_{i=1}^{r} \nu_i\, d_j^{(i)}/\delta_j^{(i)}] > 0$,

$\min \operatorname{Re}\,(\alpha + \sigma\,(d_h/\delta_h) + 1) > 0$ $(j=1,\ldots,m_i;\ i=1,\ldots,r;$
$h = 1,\ldots,m)$;

$$K_{\mu,\sigma}^{\delta,\alpha,\lambda}\left[H\begin{bmatrix}(y_1/t)^{\nu_1}\\ \vdots\\ (y_r/t)^{\nu_r}\end{bmatrix};x\right]$$

$$= H^{0,n+1:*;\ldots;*;M,N+1}_{p+1,q+1:*;\ldots;*;P+1,Q}\left[\begin{matrix}(y_1/x)^{\nu_1}\\ \vdots\\ (y_r/x)^{\nu_r}\\ k\end{matrix}\left|\begin{matrix}P_2 : * ;\ldots; * ; (-\alpha,\sigma),\ (c_j,\gamma_j)_{1,P}\\ Q_2 : * ;\ldots; * ; (d_j,\delta_j)_{1,Q}\end{matrix}\right.\right] \tag{2.6}$$

where

$P_2 : (1-(\delta/\mu);\ \nu_1/\mu,\ldots,\nu_r/\mu,\ \lambda),\ (a_j;\ \alpha_j',\ldots,\alpha_j^{(r)},\ 0)_{1,p}$

$Q_2 : (-(\delta/\mu)-\alpha;\ \nu_1/\mu,\ldots,\nu_r/\mu,\ (\lambda+\sigma)),\ (b_j;\ \beta_j',\ldots,\beta_j^{(r)},\ 0)_{1,q}$

provided that $\nu_i > 0$. $\min \operatorname{Re}\,[\delta + \lambda\mu(d_h/\delta_h) + \sum_{i=1}^{r} \nu_i(d_j^{(i)}/\delta_j^{(i)})] > 0$,

$\min \operatorname{Re}\,(\alpha + \sigma(d_h/\delta_h) + 1) > 0$, $j = 1,\ldots,m_i$; $i = 1,\ldots,r$; $h=1,\ldots,M$;

for $Re\,(1-\alpha_i) > m_i'$, $h_i > 0$ *and* $m_i' = 0, 1, 2, \ldots$; $i=1,\ldots,r$, we have:

$$R\left[\begin{matrix}(\alpha_r),\ (\beta_r),\ (\mu_r)\\ (\beta_r),\ (m_r'),\ (\eta_r)\end{matrix} : H\begin{bmatrix}(u_1\ t_1)^{h_1}\\ \vdots\\ (u_r\ t_r)^{h_r}\end{bmatrix};z_1,\ldots,z_r\right]$$

$$= \prod_{i=1}^{r}\left\{\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+m_i')}\right\} H^{0,n:\,m_1+1,\,n_1+1;\ldots;\,m_r+1,\,n_r+1}_{p,q:\,p_1+2,\,q_1+2;\ldots;\,p_r+2,\,q_r+2}$$

$$\left[\begin{array}{c|c} (u_1\ z_1)^{h_1} & * : P_3 \\ \vdots & \\ (u_r\ \ z_r)^{h_r} & * : Q_3 \end{array}\right] \tag{2.7}$$

where

$P_3 : (1-(\eta_1/\mu_1)-(1/\mu_1), (h_1/\mu_1)), (c_j',\gamma_j')_{1,\ p_1}, (\beta_1-(\eta_1/\mu_1)-(1/\mu_1), h_1/\mu_1)$

$;...;(1-(\eta_r/\mu_r)-(1/\mu_r), (h_r/\mu_r)), (c_j^{(r)}, \gamma_j^{(r)}),_{1,\ p_r}, (\beta_r-\eta_r/\mu_r-1/\mu_r, h_r/\mu_r)$

$Q_3: (\beta_1+m_1'-(\eta_1/\mu_1)-(1/\mu_1), h_1/\mu_1), (d_j',\delta_j')_{1,q_1}, (\alpha_1-(\eta_1/\mu_1)-(1/\mu_1), h_1/\mu_1)$

$;...; (\beta_r+m_r'-(\eta_r/\mu_r)-(1/\mu_r), h_r/\mu_r), (d_j^{(r)},\delta_j^{(r)})_{1,\ q_r}, (\alpha_r-\eta_r/\mu_r-1/\mu_r, h_r/\mu_r)$

provided that min $Re\,(\eta_i+h_i\ \mu_i\ d_j^{(i)}/\delta_j^{(i)}) > 0,\ i=1,...,r;\ j=1,...,m_i;$

$$K\left[\begin{array}{c} (\alpha_r), (\beta_r), (\mu_r) \\ (\beta_r), (m_r'), (\delta_r) \end{array} ; H_1 \left[\begin{array}{c} (u_1\ t_1)^{h_1} \\ \vdots \\ (u_r\ t_r)^{h_r} \end{array}\right] ; z_1, ..., z_r\right]$$

$$= \prod_{i=1}^{r} \left\{\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+m_i')}\right\} H^{0,\ 0:\ m_1+1,\ n_1+1\ ;...;\ m_r+1,\ n_r+1}_{p,\ q:\ p_1+2,\ q_1+2\ ;...;\ p_r+2,\ q_r+2}$$

$$\left[\begin{array}{c|c} (u_1\ z_1)^{h_1} & * : P_4 \\ (u_r\ \ z_r)^{h_r} & * : Q_4 \end{array}\right] \tag{2.8}$$

where

$P_4: (1-\beta_1-m_1'+(\delta_1/\mu_1), h_1/\mu_1), (c_j', \gamma_j')_{1,p_1}, (1-\alpha_1 + (\delta_1/\mu_1), h_1/\mu_1)$

$;...;(1-\beta_r-m_r'+(\delta_r/\mu_r), h_r/\mu_r), (c_j^{(r)},\gamma_j^{(r)})_{1,\ p_r}, (1-\alpha_r+(\delta_r/\mu_r, h_r/\mu_r)$

$Q_4: (\delta_1/\mu_1, h_1/\mu_1), (d_j', \delta_j')_{1_1\ q_1}, (1-\beta_i+(\delta_1/\mu_1, h_1/\mu_1)\ ;.;$

$(\delta_r/\mu_r,\ h_r/\mu_r),\ (d_j^{(r)},\ \delta_j^{(r)})_{1,\ q_r},\ (1-\beta_r+(\delta_r/\mu_r),\ h_r/\mu_r)$

with max $Re\ [\ \delta_i - h_i\ (c_j^{(i)}-1)/\gamma_j^{(i)}\ ] > 0,\ j=1,\ldots,n_i\ ;\ i=1,\ldots,r.$

Method of proof : To establish (2.1), we apply the definition (1.3), make a change of variable and then evaluate the inner integral with the help of [3, *p* 64, *eqn*. (5. 2. 8)], to arrive at the R. H. S. of (2.1). (2. 4) — (2.8)) are similarly established with the help of [3, *p*. 12, *eqn*. (2. 2. 4) ; *p*. 10, (2. 1. 1); *p*. 251, *eqn* (C. 1) and *p*. 242, *eqn*. (A.4)] and [1, *p*. 398, *eqn*. (2)] respectively.

3. The Theorems

With $g(\rho_r)$ and ρ_r given by (2.2) and (2.3) and for $\nu,\ \lambda,\ \mu,\ \sigma > 0$, we have :

Theorem I: *If* $R_{\mu,\ \sigma}^{\eta,\ \alpha,\ \lambda}\ [\ f(t);\ x]$ *and* $\bar{f}(x)$ *be given by* (1.3) *and* (1.7), *respectively, then* :

$$R_{\mu,\ \sigma}^{\eta,\ \alpha,\ \lambda}\ [\ \bar{f}(t);\ x\ ] = \sum_{h=1}^{M}\ \sum_{r=1}^{\infty}\ \frac{(-1)^r\ (k)^{\rho_r}}{r\ \delta_h}\, g(\rho_r)\ \Gamma(1+\alpha+\sigma\rho_r)$$

$$\times \int_0^\infty f(u)\ \phi_1\ (xu)\ du \tag{3.1}$$

where

$$\phi_1(x) = H_{p+1,\ q+1}^{m,\ n+1}\left[x^\nu \left| \begin{matrix} (1-(\eta/\mu)-(1/\mu)-\lambda\rho_r,\ \nu/\mu),\ (a_j,\ \alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q},(-\alpha-(\eta/\mu)-(1/\mu)-(\lambda+\sigma)\rho_r,\ \nu/\mu) \end{matrix} \right. \right] \tag{3.2}$$

provided that min $Re\ [\eta+\nu\ (b_i/\beta_i)+\mu\lambda\ (d_j/\delta_j)+1\ ] > 0,$

min Re $[a + \sigma\,(d_j/\delta_j) + 1] > 0$ $(i = 1,\ldots, m;\ j = 1,\ldots, M)$, *and the series in* (3.1) *is absolutely convergent.*

Theorem II: *If* $K^{\delta,\,\alpha,\,\lambda}_{\mu,\,\sigma}\,[f(t);x]$ *and* $f(x)$ *be given by* (1.4) *and* (1.7), *respectively, then* :

$$K^{\delta,\,\alpha,\,\lambda}_{\mu,\,\sigma}\,[\bar{f}(t);x] = \sum_{h=1}^{M}\sum_{r=0}^{\infty}\frac{(-1)^r\ (k)^{\rho_r}}{r!\ \delta_h}\,g(\rho_r)\,\Gamma(\alpha+1+\sigma\rho_r)$$

$$\int_0^{\infty} f(u)\ \phi_2\,(xu)\ du \tag{3.3}$$

where

$$\phi_2(x) = H^{m+1,\,n}_{p+1.q+1}\left[x^{v}\,\middle|\,\begin{matrix}(a_j,a_j)_{1,p},\ (1-a+(\delta/\mu)+(\lambda+\sigma)\rho_r,\ v/\mu)\\ ((\delta/\mu)+\lambda\rho_r,\ v/\mu),\ (b_j,\beta_j)_{1,q}\end{matrix}\right] \tag{3.4}$$

provided that $Re\ [\ \delta + v\,(a_i - 1)/a_i + \mu\lambda\,(d_j/\delta_j)\ > 0,$

min Re $[\alpha + \sigma\,(d_j/\delta_j) + 1] > 0$ $(i = 1,\ldots, n;\ j = 1,\ldots, M)$,

and the series in (3.3) *is absolutely convergent.*

Theorem III : *If* $R\left[\begin{matrix}(a_r),\ (\beta_r),\ (\mu_r)\\ (\beta_r),\ (m_r'),\ (\eta_r)\end{matrix}\ ;f\ ;z_1,\ldots,z_r\right]$

and $\bar{f}(z_1,\ldots,z_r)$ *be given by* (1.1) *and* (1.8) *respectively, then* :

$$R\left[\begin{matrix}(a_r),\ (\beta_r),\ (\mu_r)\\ (\beta_r),\ (m_r'),\ (\gamma_r)\end{matrix}\ ;\bar{f}\ ;z_1,\ldots,z_r\right]\prod_{i=1}^{r}\left\{\frac{\Gamma(\beta_i)\ z_i}{\Gamma(\beta_i+m_i')}\right\}$$

$$\cdot\int_0^{\infty}\cdots\int_0^{\infty}\Psi_1(z_1 u_1,\ldots,z_r u_r)\ f(u_1,\ldots,u_r)\ du_r \ldots du_r \tag{3.5}$$

where

$$\Psi_1(u_1,\ldots,u_r) = H^{0,0:m_1+1,n_1+1;\ldots;m_r+1,n_r+1}_{p,q:p_1+2,q_1+2;\ldots;p_r+2,q_r+2}$$

$$\left[\begin{array}{c|c} u_1^{h_1} & *:A \\ \vdots \\ u_r^{h_r} & *:B \end{array}\right], \tag{3.6}$$

$A:(1-(\eta_1/\mu_1)-(2/\mu_1),(h_1/\mu_1),(c_j',\gamma_j')_{1,p_1},(\beta_1-(\eta_1/\mu_1)-(2/\mu_1),h_1/\mu_1)$

$;\ldots;(1-\eta_r/\mu_r-2/\mu_r,h_r/\mu_r),(c_j^{(r)},\gamma_j^{(r)})_{1,p_r},(\beta_r-\eta_r/\mu_r-2/\mu_r,h_r/\mu_r)$

$B:(\beta_1-(\eta_1/\mu_1)-(2/\mu_1)+m_1',h_1/\mu_1).(d_j',\delta_j')_{1,q_1},(\alpha_1-(\eta_1/\mu_1)-2/\mu_1,h_1/\mu_1)$

$;\ldots;(\beta_r-\eta_r/\mu_r-2/\mu_r+m_r',h_r/\mu_r),(d^{(r)},\delta_j^{(r)})_{1,q_r},(\alpha_r-\eta_r/\mu_r-2/\mu_r,h_r/\mu_r)$

provided that $Re\,[\eta_i+h_i\,(d_j^{(i)},/\delta_j^{(i)})+1]>0$ $i=1,\ldots,r;$

$j=1,\ldots,m_i$ and $m_i'=0,1,2,\ldots$, $Re\,(1-\alpha_i)>m_i'$ $(i=1,\ldots,r$ and

the multiple integral in (3.5) is absolutely convergent.

Theorem IV : *If* $K\left[\begin{array}{c}(\alpha_r),(\beta_r),(\mu_r)\\(\beta_r),(m_r'),(\delta_r)\end{array};f;z_1,\ldots,z_r\right]$

and $\bar{f}(z_1,\ldots,z_r)$ are given by (1.2) and (1.8) respectively, then

for $m_i'=0,1,2,\ldots$ and $Re\,(1-\alpha_i)>m_i'$ $(i=1,\ldots,r)$:

$$K\left[\begin{array}{c}(\alpha_r),(\beta_r),(\mu_r)\\(\beta_r),(m_r'),(\delta_r)\end{array};\bar{f};z_1,\ldots,z_r\right]\prod_{i=1}^{r}\left\{\frac{z_i\,\Gamma(\beta_i)}{\Gamma(\beta_i+m_i')}\right\}$$

$$\cdot\int_0^\infty\cdots\int_0^\infty \Psi_2(z_1u_1,\ldots,z_ru_r)\,f(u_1,\ldots,u_r)\,du_1\ldots du_r \tag{3.7}$$

where

$$\Psi_2 (u_1 ,..., u_r) = H \begin{matrix} 0, 0: m_1 + 1, n_1 + 1 ;...; m_r + 1, n_r + 1 \\ p, q: p_1 + 2, q_1 + 2 ;...; p_r + 2, q_r + 2 \end{matrix}$$

$$\left[\begin{matrix} u_1^{h_1} \\ \vdots \\ u_r^{h_r} \end{matrix} \middle| \begin{matrix} * : E \\ \\ * : F \end{matrix} \right], \tag{3.8}$$

$$E: (1-\beta_1-\delta_1/\mu_1-1/\mu_1-m_1', h_1/\mu_1), (c_j', \gamma_j')_{1,p_1}, (1-\alpha_1+\delta_1/\mu_1-1/\mu_1, h_1/\mu_1)$$

$$;...;(1-\beta_r-\delta_r/\mu_r-1/\mu_r-m_r', h_r/\mu_r), (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}, (1-\alpha_r+\delta_r/\mu_r-1/\mu_r, h_r/\mu_r)$$

$$F: (\delta_1/\mu_1-1/\mu_1, h_1/\mu_1), (d_j', \delta_j')_{1,q_1}, (1-\beta_1+\delta_1/\mu_1-1/\mu_1, h_1/\mu_1);...;$$

$$(\delta_r/\mu_r)-(1/\mu_r), h_r/\mu_r), (d_j^{(r)}, \delta_j^{(r)})_{1,q_r}, (1-\beta_r+(\delta_r/\mu_r)-1/\mu_r), h_r/\mu_r)$$

provided that $\max Re [\delta - h_i (c_j^{(i)} - 1)/\gamma_j^{(i)}] > 0$, $i=1 ,..., r$;

$j=1 ,..., n_i$) *and the multiple integral in* (3.7) *exists.*

Method of proof. To establish theorem I, we use (1.3) and (1.7) in (3.1), change the order of integration and then use (2.1) to arrive at the desired result. Theorems II,III, and IV are established in a similar manner on using (2.4), (2.7) and (2.8) respectively.

Remark The fractional integrals and the theorems established here are capable of yielding a number of particular cases, both known and new ones, but these are not recorded here for lack of space.

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