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CHARACTERIZATIONS OF SEMI-CHEBYSHEV SUBSPACES AND ELEMENTS OF ε-APPROXIMATION IN LOCALLY CONVEX SPACES

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ABSTRACT

In this paper we characterize semi-Chebyshev subspaces and elements of ε-approximation in locally convex spaces with a family of semi-norms.

1. Introduction

Some characterizations of semi-Chebyshev subspaces of normed linear spaces were obtained by Singer in [4]. In Section 2 we give a characterization of semi-Chebyshev subspaces of locally convex spaces with a family of semi-norms by using a characterization of elements of best approximation in locally convex spaces with a family of semi-norms (the study initiated by R. A. Hirschfeld [3]) given in [2]. In the third section we characterize elements of ε-approximation in locally convex spaces with a family of semi-norms. In normed linear spaces the elements of ε–approximation have been introduced and characterized by R. C. Buck [1] who used the term 'element of good approximation' instead of ε–approximation. A characterization of these elements in normed linear spaces has also been given by Singer in 4] (Theorem 6. 12).

2. A Characterization of Semi-Chebyshev Subspaces

Let E be a locally convex space with a family P of semi–norms, G a linear subspace of E, $x \in E/G$ and $g_0 \in G$. Then g_0 is called an element of *best approximation* of x by means of elements of G if for every $p \in P$

$$p(x-g_0) = \inf_{g \varepsilon G} p(x-g)$$

and we denote by $L_G(x)$, the set of all such $g_0 \varepsilon G$.

The subspace G is called *semi-Chebyshev* if for every $x \in E$ the set $L_G(x)$ contains at most one element.

The following characterization of elements of best approximation was given by Ellumalai in [2].

Theorem 2.1 *Let* E *be a locally convex space with a family* P *of semi–norms,* G *a linear subspace of* E, $x \in E/G$ *and* $g_0 \in G$. *Then* $g_0 \in L_G(x)$ *if and only if for every* $p \in P$ *there exists* $f^p \in E^*$ *(the dual of* E*) such that*

$$f^p(g) = 0 \tag{2.1}$$

$$| f^p(x-g_0) | = p(x-g_0) \tag{2.2}$$

$$| f^p(x-g) | \leqslant p(x-g) \tag{2.3}$$

for every $g \in G$.

Using Theorrm 2.1, we give the following characterization of semi–Chebyshev subspaces.

Theorem 2.2. *Let E be a locally convex space with a family P of semi-norms, G a linear subspace of E,* $x \in E/\overline{G}$ *and* $g_0 \in G$.

Then the following condition are equivalent :

(a) $L_G(x) = \{g_0\}$

(b) $g_0 \in L_G(x)$ *and there do not exist* $g \in G/\{g_0\}$ *and* $f^p \in E^*$ *such that*

$$f^p(g_0 - g) = 0 \tag{2.4}$$

$$| f^p(x - g) | = p(x - g) \tag{2.5}$$

$$| f^p(x - g_0) | \leqslant p(x - g_0) \tag{2.6}$$

Proof (a) ⇒ (b). Suppose there exists $g \in G/\{g_0\}$ and $f^p \in E^*$ such that (2.4), (2.5) and (2.6) are satisfied. Then

$$p(x - g) = | f^p(x - g) | \leqslant | f^p(x - g_0) | \leqslant p(x - g_0)$$

and therefore $g \neq g_0 \in L_G(x)$, contrary to the assumption.

(b) ⇒ (a). Let us suppose that (a) does not hold good. i. e. there exists some $g_k \in L_G(x)$ such that $g_k \neq g_0$.

Therefore (by Theorem 2.1) there exists $f^p \in E^*$ such that (2.4), (2.5) and (2.6) are satisfied for $g = g_k \in G/\{g_0\}$. This contradicts (b).

3. Characterizations of Elements of ε-Approximation

Let E be a locally convex space with a family P of semi-norms, G a linear subspace of E, $x \in E$ and $\varepsilon > 0$. An element $g_0 \in G$ is said to be an *element of ε-approximation* of x by means of elements of G if we have

$$p(x - g_0) \leqslant \inf_{g \in G} p(x - g) + \varepsilon \text{ for all } p \in P.$$

We shall denote by $L_G(x, \varepsilon)$ the set of all elements of ε-approximation of x. In particular, for $\varepsilon = 0$ we find the elements of best approximation of x and the sets $L_G(x)$.

Now we give some characterizations of elements of ε-approximation.

Theorem 3.1. *Let E be a locally convex space with a family P of semi-norms, G a linear subspace of E, $x \in E$, $g_0 \in G$ and $\varepsilon > 0$. Then $g_0 \in L_G(x, \varepsilon)$ if and only if for every $p \in P$ there exists $f^p \in E^*$ such that*

$$f^p(g) = 0 \tag{3 1}$$

$$f^p(x - g_0) = p(x - g_0) - \varepsilon \tag{3 2}$$

$$f^p(x - g) \leqslant p(x - g) \tag{3.3}$$

for every $g \in G$.

Proof. Let $g_0 \in L_G(x, \varepsilon)$. Then for all $g \in G$ and $p \in P$, we have

$$p(x - g_0) - \varepsilon \leqslant \inf_{g \in G} p(x - g)$$

In particular, for $\alpha \neq 0$,

$$p(x - g_0) - \varepsilon \leqslant p(x - g_0 + g/\alpha). \tag{3.4}$$

For each $g \in G$, let

$$M = \{g + \alpha(x - g_0) : \alpha \text{ a scalar}\}.$$

Define f_0^p on M as

$$f_0^p\,[g + \alpha\,(x - g_0)\,] = \alpha\,[p\,(x - g_0) - \varepsilon\,] \tag{3.5}$$

for every $g \in G$. Therefore $f_0^p\,(g) = 0$ and $f_0^p\,(x - g_0) = p(x - g_0) - \varepsilon$.

Now for $\alpha \neq 0$

$$f_0^p\,[g + \alpha\,(x - g_0)\,] = \alpha\,[p(x - g_0) - \varepsilon]$$

$$\leqslant \alpha p\,(x - g_0 + g/\alpha),\ \text{by (3 4)}$$

$$= p[g + \alpha\,(x - g_0)]\ \text{for each } g \in G.$$

For $\alpha = 0$ and $g \in G$ the inequality

$$f_0^p\,[g + \alpha(x - g_0)\,] \leqslant p[g + \alpha(x - g_0)\,]$$

is trivially satisfied. Thus for every $z \in M$ and $p \in P$, we have

$$f_0^p\,(x) \leqslant p(z). \tag{3.6}$$

Clearly, f_0^p is a continuous linear functional on M and so by Hahn–Banach theorem, f_0^p can be extended to a continuous linear functional f^p on the whole space E such that

$$f^p\,(x) \leqslant p(x) \text{ for all } x \in E$$

and

$f^p\,(z) = f_0^p$ for all $z \in M$. These conditions imply that f^p satisfies the relations (3.1), (3.2) and (3.3).

Conversely, let the given conditions be satisfied. Then by (3.2)

$$p(x-g_0)-\varepsilon = f^p(x-g_0) \leqslant f^p(x-g) \leqslant p(x-g)$$

for all $g \in G$ and $p \in P$. This implies that $p(x-g_0)-\varepsilon \leqslant \inf_{g \in G} p(x-g)$

i. e. $g_0 \in L_G(x, \varepsilon)$.

Theorem 3 2. *Let E be a locally convex space with a family P of semi-norms, M a linear manifold in E, $x \in E$ and $m_0 \in M$. Then $m_0 \in L_M(x, \varepsilon)$ if and only if for each $p \in P$ there exists a continuous linear functional $f^p \in E^*$ such that*

$$f^p(m - m_0) = 0 \tag{3 7}$$

$$f^p(x - m_0) = p(x - m_0)-\varepsilon \tag{3.8}$$

$$f^p(x - m) \leqslant p(x - m) \tag{3.9}$$

for every $m \in M$.

Proof. Since M is a linear manifold in E and $m_0 \in M$, $M-m_0$ is a linear subspace of E. It is easy to see that $m_0 \in L_M(x, \varepsilon)$ if and only if $0 \in L_{M-m_0}[(x-m_0), \varepsilon]$. So applying theorem 3.1 to the subspace $M-m_0$, there exists $f^p \in E^*$ such that (3.7), (3.8) and (3.9) are satisfied.

Conversely, let the given conditions be satisfied.

Then, by (3.8)

$$p(x - m_0)-\varepsilon = f^p(x - m_0) \leqslant f^p(x - m) \leqslant p(x - m)$$

for every $p \in P$ and every $m \in M$. This implies that $m_0 \in L_M(x, \varepsilon)$.

Theorem 3 3. *Let E be a locally convex space with a family P of semi-norms, G a linear subspace of E, $x \in E$ and $g_0 \in G$.*

Then $g_0 \subset L_G(x, \varepsilon)$ *if and only if* $g_0 \subset L_G[tx + (1 - t)\, g_0, \varepsilon]$ *for all scalars* t *with* $|t| \leqslant 1$.

Poof. Let $g_0 \subset L_G(x - \varepsilon)$. The result is trivially true for $t=0$. Suppose $t \neq 0$. Consider

$$p[tx + (1-t)\, g_0 - g] = |t|\, p(x - g_0 + \frac{g_0 - g}{t})$$

$$\geqslant |t|\, [\, p(x - g_0) - \varepsilon]$$

$$= p[tx + (1-t)\, g_0 - g_0] - |t|\, \varepsilon$$

$$\geqslant p\; tx + (1-t)\, g_0 - g_0 - \varepsilon \text{ as } |t| \leqslant 1.$$

This implies that $\inf\limits_{g \subset G} p\,(tx + (1-t)\, g_0 - g)$

$$\geqslant p\,(tx + (1-t)\, g_0 - g_0) - \varepsilon.$$

Consequently, $g_0 \subset L_G\,(tx + (1-t)\, g_0, \varepsilon)$.

The converse part follwos by taking $t = 1$.

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