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AN APPLICATION OF THE MULTIVARIABLE H-FUNCTION IN FREE OSCILLATIONS OF WATER IN A CIRCULAR LAKE

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ABSTRACT

The aim of this paper is to make use of the multivariable H-function of H. M. Srivastava and R. Panda (See [6] and [7]) in obtaining a solution of the partial differential equation of free oscillations of water in a circular lake.

1 Introduction

As an example of the application of the H-function of several complex variables introduced by Srivastava and Panda ([6]; see also [5], p. 251) in applied mathematics, we consider the problem of determining the free oscillations $\xi\,(y, \theta, t)$ of water in a circular lake.

The free oscillations of water in a circular lake is given by the following partial differential equation

$$y^2 \frac{\partial^2 \xi}{\partial y^2} + y \frac{\partial \xi}{\partial y} + \frac{\partial^2 \xi}{\partial \theta^2} + \mu^2 y^2 \xi = 0 \tag{1.1}$$

where ξ stands for the depth of water surface from its position of equilibrium and $\mu = w/\sqrt{(gd)}$

We shall assume that

(i) the lake is stationary in space;

(ii) in any vibrational mode ξ varies harmonically with time and ξ is small enough for its square to be neglected, and

(iii) there is no loss of energy.

The solution of (1.1) we take in the following form given by McLachlan ([3], p. 62)

$$\xi(y, \theta, t) = \sum_{\alpha=0}^{\infty} R_\alpha J_\alpha(\beta y) \cos(\alpha\theta - \alpha\phi) \cos(\alpha wt - \alpha\Psi) \quad (1.2)$$

When $\theta = 0 = t$, let $\xi(y, 0, 0) = f(y)$ (1.3)

Here we shall consider

$$f(y) = y^\lambda H_{p,q}^{m,n}\left[M y^{2k} \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right]$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,0:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix} [(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 y^{2h_1}, \ldots, z_r y^{2h_r} \right), \quad (1.4)$$

where the multivariable function is the H-function of several complex variables introduced by Srivastava and Panda ([6];see also [5],p. 251). We require the following series representation of the H-function, due to Skibiński [4] :

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] = \sum_{N=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s \; z^{\eta_s}}{f_N \; s!} \phi(\eta_s) \quad \ldots(1.5)$$

where

$$\phi(\eta_s) = \prod_{\substack{i=1 \\ i \neq N}}^{m} \Gamma(b_i - f_i \eta_s) \prod_{i=1}^{n} \Gamma(1 - a_i + e_i \eta_s)$$

$$\left\{ \prod_{i=m+1}^{q} \Gamma(1 - b_i + f_i \eta_s) \prod_{i=n+1}^{p} \Gamma(a_i - e_i \eta_s) \right\}^{-1}$$

and

$$\eta_s = \frac{b_N + s}{f_N}.$$

2. An Infinite Integral

The integral to be evaluated is

$$\int_0^\infty y^{\lambda-1} J_v(\beta y)\, H_{p,q}^{m,n}\left[M y^{2k} \middle| \begin{matrix} (a_p, e_p) \\ b_q, f_q) \end{matrix} \right]$$

$$. H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,0:(u',v');\ldots;(u^{(r)},v^{(r)})} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}),\delta^{(r)}]; \end{matrix}\; z_1 y^{2h_1},\ldots, z_r y^{2h_r} \right) dy$$

$$= \sum_{N=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s M^{\eta_s}\, 2^{2k\eta_s + \lambda - 1}}{\beta^{2k\eta_s+\lambda} f_N\, s!} \phi(\eta_s)$$

$$. H_{A+2,\, C+2:\,[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,1\qquad :(\ ',v);\ldots;(u^{(r)},v^{(r)})}$$

$$\left(\begin{matrix} [1 - \lambda/2 - v/2 - k\eta_s : h_1,\ldots,h_r], \\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \right.$$

$$[(a):\theta',\ldots,\theta^{(r)}],\ [1 - v/2 - \lambda/2 - k\eta_s : h_1,\ldots,h_r]:$$

$$\begin{matrix}[(b') : \phi'] ;\ldots; [(b^{(r)} : \phi^{(r)}]; \\ [(d') : \delta'] ;\ldots; [(d^{(r)} : \delta^{(r)}] ;\end{matrix} \quad (2/\beta)^{2h_1} z_1 ,\ldots, (2/\beta)^{2h_r} z_r \Bigg), \qquad (2.1)$$

provided that $k > 0$, $\beta > 0$, $h_i > 0$, $1 \leqslant i \leqslant r$,

$$\text{Re}\,(\lambda + v + K\, b_j'/f_j' + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}) > 0\,,$$

$$1 \leqslant j' \leqslant m;\ 1 \leqslant i \leqslant r;\ 1 \leqslant j \leqslant u^{(i)},$$

$$\text{Re}\,(\lambda - k(1-a_j')/e_j' - \sum_{i=1}^{r} h_i(1\text{-}b_j^{(i)}) \,/\, \phi_j^{(i)}) < 3/2,$$

$$|\arg z_i| < \tfrac{1}{2}\,\Delta_i\,\pi \text{ and } |\arg M| < \tfrac{1}{2}\,\nabla\pi$$

$$(\nabla = \sum_{\sigma=1}^{n} e_\sigma - \sum_{\sigma=n+1}^{p} e_\sigma + \sum_{\sigma=1}^{m} f_\sigma - \sum_{\sigma=m+1}^{q} f_\sigma).$$

Evaluation of (2.1). To establish (2.1), express the H-function in series with the help of (1.5), then interchanging the order of summation and integration (which is justified due to the absolute convergence of the integral involved in the process), evaluate the inner integral with the help of a result of Srivastava and Panda ([7] p. 175, eq. (3.12)), and we arrive at the desired result.

3. Solution of the Problem Posed

We obtain the solution of (1.1) in the following form:

$$\xi\,(y, \theta, t) = \sum_{a,\, s=0}^{\infty} \sum_{N=1}^{m} \frac{(-1)^s\, 2^{2k\eta_s+\lambda}\, M^{\eta_s}\, a\, J_\alpha\,(\beta\, y)}{s!\, \beta^{2k\eta_s+\lambda}\, f_N}$$

$$\cdot\ \frac{\cos\,(a\,\theta - a\,\phi)\,\cos\,(\alpha\,wt - \alpha\,\Psi)\,\phi\,(\eta_s)}{\cos\,(a\,\phi)\,\cos\,(a\,\Psi)}$$

$$H\begin{matrix}0,1:(u',v')\ ;\ \ ;(u^{(r)},v^{(r)})\\ A+2,C:[B',D']\ ;\ldots;\ [B^{(r)},D^{(r)}]\end{matrix}\left(\begin{matrix}[1-\lambda/2-\nu/2-k\eta_s : h_1,\ldots,h_r],\\ [(c):\Psi',\ldots,\Psi^{(r)}]\end{matrix}\right. :$$

$$[(a):\theta',\ldots,\theta^{(r)}],\ [1+\nu/2-\lambda/2+k\eta_s : h_1,\ldots,h_r]:$$

$$\left.\begin{matrix}[(b'):\phi']\ ;\ldots;\ [(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta']\ ;\ldots;\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\ (2/\beta)^{2h_1} z_1,\ldots,(2/\beta)^{2h_r} z_r\right) \qquad (3.1)$$

valid under the conditions obtainable from (2.1).

Derivation on (3.1.) For $f(y)$ defined by (1.4), let

$$f(y)=\sum_{a=0}^{\infty} R_\alpha J_\alpha(\beta y)\cos(\alpha\phi)\cos(\alpha\Psi) \qquad \ldots(3.2)$$

Multiply both sides of (3.2) by $y^{-1} J_\nu(\beta y)$, integrate with respect to y from 0 to ∞, and use (2.1) and the orthogonality property of the Bessel function.

we thus obtain

$$R_\nu=\sum_{N=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s\ 2^{2k\eta_s+\lambda}\ M^{\eta_s}\ \nu\ \phi(\eta_s)}{s!\ \beta^{2k\eta_s+\lambda}\ f_N\cos(\nu\phi)\cos(\nu\Psi)}$$

$$.\ H\begin{matrix}0,1:(u',v')\ ;\ldots;\ (u^{(r)},v^{(r)})\\ A+2,C:[B',D']\ ;\ldots;\ [B^{(r)},D^{(r)}]\end{matrix}\left(\begin{matrix}[1-\lambda/2-\nu/2-k\eta_s : h_1,\ldots,h_r],\\ [(c):\Psi',\ldots,\Psi^{(r)}]\end{matrix}\right. :$$

$$[(a):\theta',\ldots,\theta^{(r)}],\ [1+\nu/2-\lambda/2+k\eta_s : h_1,\ldots,h_r]:$$

$$\left.\begin{matrix}[(b'):\phi']\ ;\ldots;\ [(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta']\ ;\ldots;\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\ (2/\beta)^{2h_1} z_1,\ldots,z_r\,(2/\beta)^{2h_r} z_r\right) \qquad (3.3)$$

Substituting the value of R_α from (3.3) in (3.2) we get the desired solution of (3.1).

In view of the generality of the multivariable H-function involved, the result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature on applied mathematics and mathematical physics For various known special cases of our results, see recent book by Srivastava, Gupta and Goyal ([5], pp. 202–203).

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