Jñānābha, Vol. 13, 1983

AN INTEGRAL ANALOGUE OF TAYLOR'S SERIES FOR THE MULTIVARIABLE *H*-FUNCTION

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(*Received : July 2, 1982; Revised : December 10, 1982*)

ABSTRACT

In this paper we obtain an integral analogue of Taylor's series for the multivariable *H*-function introduced and studied in a series of papers by Srivastava and Panda ([10] and [11]) by the application of the fractional derivative of arbitrary order. The corresponding results for Lauricella's; hypergeometric function of r variables [3] and the (Srivastava and Daoust) generalized Lauricella function [8] follow as particular cases of our main result. The integral analogue of Taylor's series for the well-known *H*-function of Fox [1] also follows as a special case of our main result; this last result, in turn, yields as its particular cases various results obtained recently by Osler [5].

1. INTRODUCTION AND DEFINITIONS

An integral analogue of the familiar Taylor series was given by Osler [5, p. 449, eq. (1)] in the following general form:

$$(1.1)\quad f(z) = \int_{-\infty}^{\infty} \frac{D_z^{\omega+\gamma} f(z)\ \big|_{z=z_0}}{\Gamma(\omega+\gamma+1)} (z-z_0)^{\omega+\gamma}\, d\omega$$

where γ is an arbitrary complex number and z is restricted to the circle $| z - z_0 | = | z_0. |$ Further, the symbol $D_z^q\ f(z)$ stands for *qth* order fractional derivative of $f(z)$. We use here the following definitions for the fractional derivative given by Ross [6].

$$(1.2)\quad D_z^q f(z) = \frac{1}{\Gamma(-q)} \int_0^z (z-t)^{-q-1} f(t)\, dt, \text{ for } q < 0$$

and

$$(1.3)\quad D_z^q f(z) = \frac{d^n}{dz^n} [D_z^{q-n} f(z)] \qquad \text{for } q \geqslant 0$$

where q is an arbitrary real or complex number and n is a positive integer such that $n \geqslant q$.

The following multivariable H-function introduced by Srivastava and Panda [10, p. 271, eqn. (4.1)] will also be used in the sequel

$$(1.4)\quad H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H\begin{matrix} 0,\, N: (m', n') ;\dots; (m^{(r)}, n^{(r)}) \\ P,\, Q: [p', q'] ;\dots; [p^{(r)}, q^{(r)}] \end{matrix}$$

$$\left[\begin{matrix} [(a): A' ;\dots; A^{(r)}]: [(b'): B'] ;\dots; [(b^{(r)}): B^{(r)}]; \\ [(c): C' ;\dots; C^{(r)}]: [(d'): D'] ;\dots; [(d^{(r)}): D^{(r)}]; \end{matrix}\; z_1, \dots, z_r\right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \prod_{k=1}^{r} \left\{ \phi_k(\zeta_k) z_k^{\zeta_k} \right\} \Psi(\zeta_1, \dots, \zeta_r)\, d\zeta_1 \dots d\zeta_r$$

$$(1.5)\quad \phi_k(\zeta_k) = \frac{\prod_{j=1}^{m^{(k)}} \Gamma[d_j^{(k)} - D_j^{(k)} \zeta_k] \prod_{j=1}^{n^{(k)}} \Gamma(1 - b_j^{(k)} + B_j^{(k)} \zeta_k]}{\prod_{j=m^{(k)}+1}^{q^{(k)}} \Gamma[1 - d_j^{(k)} + D_j^{(k)} \zeta_k] \prod_{j=n^{(k)}+1}^{p^{(k)}} \Gamma(b_j^{(k)} - B_j^{(k)} \zeta_k]}$$

$$k = 1, \dots, r$$

$$(1.6)\quad \Psi(\zeta_1, \dots, \zeta_r) = \frac{\prod_{j=1}^{N} \Gamma[1 - a_j + \sum_{k=1}^{r} A_j^{(k)} \zeta_k]}{\prod_{j=N+1}^{P} \Gamma[a_j - \sum_{k=1}^{r} A_j^{(k)} \zeta_k] \prod_{j=1}^{Q} \Gamma[1 - c_j + \sum_{k=1}^{r} C_j^{(k)} \zeta_k]}$$

For various conditions on the parameters, and for conditions of convergence, etc., the paper of Srivastava and Panda [11, p. 130, eqns.

(1.3) to (1.10)] is referred to. See also the recent book by Srivastava, Gupta and Goyal ([9], pp. 251–254).

2. MAIN RESULT

The main result obtained in this paper is the following integral analogue of the multivariable H-function:

$$(2.1)\quad z^{\lambda}\, H\begin{pmatrix} \alpha_1 z^{\lambda_1} \\ \vdots \\ \alpha_r z^{\lambda_r} \end{pmatrix} = \int_{-\infty}^{\infty} \frac{z_0^{\lambda-\omega-\gamma}}{\Gamma(\omega+\gamma+1)} (z-z_0)^{\omega+\gamma}$$

$$H^{0,\,N+1:\,(m',\,n')\,;\ldots;\,(m^{(r)},\,n^{(r)})}_{P+1,\,Q+1:\,[p',\,q']\,;\ldots;\,[p^{(r)},\,q^{(r)}]} \left[\begin{matrix} (-\lambda;\lambda_1,\ldots,\lambda_r),\,[\,(a):A',\ldots,A^{(r)}]:\,[\,(b'):B'];\ \ldots,\,[\,(b^{(r)}:B^{(r)}\,]; \\ [\,(c):C';\ldots;C^{(r)}],\,(-\lambda+\omega+\gamma;\lambda_1,\ldots,\lambda_r):\,[\,(d'):D'];\ \ldots;\,[\,(d^{(r)}):D^{(r)}]; \end{matrix}\ \alpha_1 z_0^{\lambda_1},\ldots,\alpha_r z_0^{\lambda_r} \right] d\omega$$

for $|z-z_0| = |z_0|$ and provided that

$$Re\left\{\lambda + \sum_{k=1}^{r} \lambda_k \left(\frac{d_j^{(k)}}{D_j^{(k)}}\right) + 1\right\} \geqslant 0,\ 1 \leqslant j \leqslant m^{(k)},$$

$\lambda_k > 0,\ k=1,\ldots,r$, Also the multivariable H-function occurring in (2.1) must satisfy the conditions corresponding appropriately to those given by Srivastava and Panda [11, p. 130, eqn. (1.6) and (1.7)].

Proof. To establish the result (2.1), we take

$$(2.2)\quad f(z) = z^{\lambda}\, H\begin{pmatrix} \alpha_1\, z^{\lambda_1} \\ \vdots \\ \alpha_r\, z^{\lambda_r} \end{pmatrix} \lambda_k > 0,\ k=1,\ldots,r$$

in (1.1). Now, on following the method of Raina and Koul [6], we obtain

$$(2.3)\quad D_z^{\omega+\gamma}\left\{z^\lambda H\left(\begin{matrix}\alpha_1 z^{\lambda_1}\\ \vdots\\ \alpha_r z^r\end{matrix}\right)\right\} = z^{\lambda-\omega-\gamma}$$

$$H\begin{matrix}0,\ N+1:\ (m',\ n')\ ;\dots;\ (m^{(r)},\ n^{(r)})\\ P+1,\ Q+1:\ [p',\ q']\ ;\dots;\ [p^{(r)},\ q^{(r)}]\end{matrix}$$

$$\left[\begin{matrix}(-\lambda;\ \lambda_1,\dots,\lambda_r,\ [\ (a):\ A',\dots,A^{(r)}]:\ [\ (b'):\ B\ ];\\ \dots;\ [\ (b^{(r)}:\ B^{(r)}];\\ \hspace{6em}\alpha_i z^{\lambda_1},\ \dots,\ \alpha_r z^{\lambda_r}\\ [\ (c):\ C',\dots,C^{(r)}],\ (-\lambda+\omega+\gamma;\ \lambda_1,\dots,\lambda_r):\ [\ (d):\ D'];\\ \dots[\ (d^{(r)}):\ D^{(r)}\ ];\end{matrix}\right]$$

provided that $\operatorname{Re}\left\{\lambda + \sum_{k=1}^{r} \lambda_k \left[\frac{d_j^{(k)}}{D_j^{(k)}}\right] + 1\right\} > 0,\ 1 \leqslant j \leqslant m^{(k)}$

On substituting the values of f(z) and $D_z^{\omega+\gamma}$ f(z) from (2.2) and (2.3) respectively, into (1.1), we obtain the required result (2.1).

3. PARTICULAR CASES

(i) If in (2.1), we let each of $\lambda_2, \lambda_3, \dots, \lambda_r \to 0$, we obtain the following integral analogue of Taylor's series for the well-known H-function of Fox [1].

$$(3.1)\quad z^\lambda H_{P,\,Q}^{M,\,N}\left[\alpha z^h \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\ (b_j,\beta_j)_{1,Q}\end{matrix}\right] = \int_{-\infty}^{\infty} \frac{z_0^{\lambda-\omega-\gamma}}{\Gamma(\omega+\gamma+1)}$$

$$(z-z_0)^{\omega+\gamma} \times$$

$$\times H_{P+1,\,Q+1}^{M,\,N+1}\left[\alpha z_0^h \,\middle|\, \begin{matrix}(-\lambda,\ h),\ (a_j,\ \alpha_j)_{1,P}\\ (b_j,\ \beta_j)_{1,Q},\ (-\lambda+\omega+\gamma,\ h)\end{matrix}\right] d\omega$$

for $|z-z_0| = |z_0|$ and provided that

$$Re \left\{ \lambda + h \left[\frac{b_j}{\beta_j} \right] + 1 \right\} > 0,\ 1 \leqslant j \leqslant M,\ h > 0$$

Also

$$|\arg \alpha| < A\pi/2, \text{ where } A = \left\{ \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P} \alpha_j \right.$$

$$\left. + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j \right\} > 0$$

(ii) On reducing the multivariable H--function occurring in (2.1) to the generalized Lauricella function introduced by Srivastava and Doust [8, p. 454], we obtain the following result:

$$(3.2)\quad z^{\lambda}\, F^{N:\, n'\, ;\ldots;\, n^{(r)}}_{Q:\, q-1\, ;\ldots;\, q^{(r)}-1} \left[\begin{matrix} [(a): A', \ldots, A^{(r)}] : [(b'): B']; \ldots, [(b^{(r)}): B^{(r)}]; \\ [(c): C', \ldots, C^{(r)}] : [(d'): D']; \ldots; [(d^{(r)}): D^{(r)}]; \end{matrix}\ z^{\lambda_1}, \ldots, z^{\lambda_r} \right]$$

$$= \int_{-\infty}^{\infty} \frac{z_0^{\lambda-\omega-\gamma}\, \Gamma(1+\lambda)}{\Gamma(1+\lambda-\omega-\gamma)\, \Gamma(\omega+\gamma+1)} (z-z_0)^{\omega+\gamma}$$

$$F^{N+1\, :\, n'\, ;\ldots;\, n^{(r)}}_{Q+1\, :\, q'-1\, ;\ldots;\, q^{(r)}-1} \left[\begin{matrix} (1+\lambda\, ;\, \lambda_1\, ;\ldots;\, \lambda_r),\ [(a) : A', \ldots, A^{(r)}];\ [(b') : B']; \ldots; [(b^{(r)}) : B^{(r)}]; \\ [(c): C'\, ;\ldots,\, C^{(r)}],\ (1+\lambda-\omega-\gamma\, ;\, \lambda_1 \ldots \lambda_r),\ [(d'): D']; \ldots; [(d^{(r)}): D^{(r)}]; \end{matrix}\ z_0^{\lambda_1}, \ldots, z_0^{\lambda_r} \right] d\omega$$

(iii) If in (3.2), we take $\lambda_1 = \ldots = \lambda_r = 1$, $N = Q = 0$ and reduce the generalized Lauricella function to one of Lauricella's hypergeometric functions of r variables [3, p. 113), we obtain the following results after a little simplification:

$$(3.3)\quad \int_{-\infty}^{\infty} \frac{z_0^{\lambda-\omega-\gamma}\,\Gamma(1+\lambda)}{\Gamma(1+\lambda-\omega-\gamma)\,\Gamma(\omega+\gamma-1)}\,(z-z_0)^{\omega+\gamma}$$

$$F_D^{(r)}\,(1+\lambda, b_1, \ldots, b_r; 1+\lambda-\omega-\gamma; z_0, \ldots, z_r)\,d\omega$$

$$= z^{\lambda}\,(1-z)^{-(b_1+\ldots b_r)},\quad |\,z_0\,| < 1$$

which on using the reduction formula ([3, p. 150]; see also [4, p. 163])

$$(3.4)\quad F_D^{(r)}\,(a, b_1, \ldots, b_r\,;\,c\,;\,z, \ldots, z) = {}_2F_1\,(a, b_1+\ldots+b_r; c; z)$$

yields the result given below:

$$(3.5)\quad \int_{-\infty}^{\infty} \frac{z_0^{\lambda-\omega-\gamma}\,\Gamma(1+\lambda)}{(1+\lambda-\omega-\gamma)\,\Gamma(\omega+\gamma+1)}\,(z-z_0)^{\omega+\gamma}$$

$${}_2F_1\,(1+\lambda, b_1+\ldots+b_r\,;\,1+\lambda-\omega-\gamma\,;\,z_0)\,d\omega$$

$$= z^{\lambda}\,(1-z)^{-(b_1+\ldots+b_r)}$$

(iv) If we reduce the generalised Lauricella function occurring in (3.2) to generalised Kampé de Fériet function and Lauricella's hypergeometric function $F_D^{(r)}$, occurring in (3.3) to Appell's function F_1, we get two known results obtained by the author recently [2, eqn. (3.2) and (3.3)].

(v) The result (3.1) is also sufficiently general in nature, and of interest in itself. On reducing Fox's H-function to simpler functions it gives certain results obtained by Osler[5, Table 5.1].

ACKNOWLEDGEMENTS

The author wishes to express her thanks to Dr. K. C. Gupta for his guidance and useful suggestions during the preparation of this paper and also to Prof. H. M. Srivastava for his valuable suggestions for the improvement of the paper.

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