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GENERATING FUNCTIONS FOR CERTAIN POLYNOMIAL SYSTEMS OF SEVERAL VARIABLES

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ABSTRACT

A number of generating functions for certain classes of polynomials of several variables, whose co-efficients involve H-functions of one or several variables, are obtained.

1. INTRODUCTION

Recently, in an attempt to unify and generalise results of Srivastava and Panda [6] and Rania [3], Srivastava and Raina [7] gave a number of generating functions for certain polynomial systems having H-functions in their co-efficients. In this paper we further generalise these results and obtain a number of generating functions for certain classes of polynomials of several variables whose co-efficients involve the H-functions of one or several variables.

Fox's H-function is defined and represented as usual in the following manner :

$$(1.1)\quad H_{u,\,v}^{f,\,g}\left[x \,\middle|\, \begin{matrix}(c_u, \gamma_u)\\(d_v, \delta_v)\end{matrix}\right] = \frac{1}{2\pi w}\int_L \theta(s)\, x^s\, ds$$

where $w = \sqrt{-1}$ and

$$(1.2)\quad \theta(s) = \frac{\prod_{j=1}^{f} \Gamma(d_j - \delta_j s) \prod_{j=1}^{g} (1 - c_j + \gamma_j s)}{\prod_{j=f+1}^{v} \Gamma(1 - d_j + \delta_j s) \prod_{j=g+1}^{u} \Gamma(c_j - \gamma_j s)}.$$

The conditions of convergence of (1.1) are given in detail at several places including, for example, in a recent book by Srivastava, Gupta and Goyal [5].

A further extension of (1.1) to n variables was given by Srivastava and Panda [6] and is defined and represented in the following manner:

$$(1.3)\quad H^{0,\lambda:(u',v');\ldots;(u^{(n)},v^{(n)})}_{A,C:(B',D');\ldots;(B^{(n)},D^{(n)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c):\Psi',\ldots\Psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\ldots,x_n\right)$$

$$= \frac{1}{(2\pi w)^n}\int_{L_1}\cdots\int_{L_n} \phi_1(s_1)\ldots\phi_n(s_n)\,\Psi(s_1,\ldots,s_n)\, x_1^{s_1}\ldots x_n^{s_n}\,ds_1\ldots ds_n$$

where $w = \sqrt{-1}$ and

$$(1.4)\quad \phi_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)}$$

$i = 1, \ldots, n$

and

$$(1.5)\quad \Psi(s_1,\ldots,s_n)=\frac{\prod\limits_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum\limits_{i=1}^{n}(\theta_j^{(i)}s_i)\right)}{\prod\limits_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum\limits_{i=1}^{n}(\theta_j^{(i)}s_i)\right)\prod\limits_{j=1}^{C}\Gamma\left(1-c_j+\sum\limits_{i=1}^{n}(\Psi_j^{(i)}s_i)\right)},$$

Here (a) stands for the A parameters $a_1,\ldots,a_A$; $(b^{(i)})$ stands for the set of $B^{(i)}$ parameters $b_1^{(i)},\ldots,b^{(i)}_{B^{(i)}}$, $i=1,\ldots,n$, etc.

The contours $L_1,\ldots,L_n$ are suitably defined. For the conditions on the parameters and various other conditions of existence of (1.3) together with several particular cases, we refer to the paper of Srivastava and Panda [6]; see also Srivastava Gupta and Goyal ([5], pp 251–254).

These conditions are assumed to be satisfied throughout this paper. Also, the appearance of an asterisk (*) at a particular place indicates that parameters at that place are made available appropriately.

In what follows, we have used the symbol (a_r) to represent the sequence of parameters $a_1,\ldots,a_r$ and $(a_r x_r)$ to represent the sequence $a_1x_1,\ldots,a_rx_r$ etc. Also, we assume that the various multiple series occurring in this paper are absolutely convergent.

2. THE GENERATING FUNCTIONS

Let $\theta(x_1,\ldots,x_r)$ possess a power series expansion given by

$$(2.1)\quad \theta(x_1,\ldots,x_r)=\sum_{m_1,\ldots,m_r=0}^{\infty} A(m_1,\ldots,m_r)\,x_1^{m_1}\ldots x_r^{m_r}$$

where $A(0,\ldots,0)\neq 0$ and $A(m_1,\ldots,m_r)$ are complex sequences.

The generating functions for the following polynomial systems of several variables, which have as co-efficients Fox's H-functions as given by (1.1), are established below :

For $i = 1, \ldots, r$, a_i, b_i, h_i, μ_i complex and q_i (integers) $\geqslant 1$, $e_i > 0$, we have :

$$(2.2)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} \left\{ T^{(a_r),\,(n_r)}_{(b_r),\,(q_r)} [\,(x_r);\,(y_r)\,] \prod_{i=1}^{r} \frac{(t_i^{n_i})}{(n_i)!} \right\}$$

$$= \prod_{i=1}^{r} \left\{ \frac{(1+\eta_i)^{a_i}}{(1-b_i\eta_i)} H^{f_i,\,g_i}_{u_i,\,v_i} \left[y_i(1+\eta_i)^{e_i} \left| \begin{matrix} (c^{(i)}_{u_i}, & \gamma^{(i)}_{u_i}) \\ (d^{(i)}_{v_i}, & \delta^{(i)}_{v_i}) \end{matrix} \right. \right] \right\}$$

$$.\,\theta\,[\,x_1\,(-\eta_1)^{q_1}, \ldots, x_r\,(-\eta_r)^{q_r}\,]$$

for the polynomial system T given by

$$(2.3)\quad T^{(a_r),\,(n_r)}_{(b_r),\,(q_r)} [\,(x_r);\,(y_r)\,] = \sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} A\,(k_1, \ldots, k_r)$$

$$.\prod_{i=1}^{r} \left\{ (-n_i)_{q_i k_i}\, x_i^{k_i}\, H^{f_i,\,g_i+1}_{u_i+1,\,v_i+1} \left[y_i \left| \begin{matrix} (-a_i-(b_i+1)\,n_i, e_i), & (c_{u_i}^{(i)}, & \gamma_{u_i}^{(i)}) \\ (d_{v_i}^{(i)}, \ \delta_{v_i}^{(i)}), & (-a_i - b_i n_i - q_i k_i, & e_i) \end{matrix} \right. \right] \right\}.$$

where

$$(2.4)\quad \eta_i = t_i\,(1+\eta_i)^{b_i+1}, \quad \eta_i\,(0) = 0.$$

$$(2.5)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} \left\{ U^{(a_r),\,(n_r)}_{(b_r),\,(q_r)} [\,(\mu_r);\,(x_r);\,(y_r)\,] \prod_{i=1}^{r} \left(\frac{t_i^{n_i}}{(n_i)!} \right) \right\}$$

$$= \prod_{i=1}^{r} \left\{ \frac{(1+\eta_i)^{a_i+1}}{(1-b_i\eta_i)} H^{f_i,\,g_i}_{u_i,\,v_i} \left[y_i\,(1+\eta_i)^{e_i} \left| \begin{matrix} (c_{u_i}^{(i)}, & \gamma_{u_i}^{(i)}) \\ (d_{v_i}^{(i)}, & \delta_{v_i}^{(i)}) \end{matrix} \right. \right] \right\}$$

$$.\,\theta\,[x_1\,(-\eta_1)^{q_1}\,(1+\eta_1)^{\mu_1}, \ldots, x_r\,(-\eta_r)^{q_r}\,(1+\eta_r)^{\mu_r}]$$

for the polynomial system U given by

(2.6) $$U^{(a_r),\,(n_r)}_{(b_r),\,(q_r)}\left[\,(\mu_r);\,(x_r);\,(y_r)\,\right] = \sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} A\,(k_1,\ldots,k_r) \prod_{i=1}^{r} \Big\{ (-n_i)_{q_i k_i}$$

$$\cdot\, x_i^{k_i}\, H^{f_i\,,\,g_i+1}_{u_i+1,\,v_i+1}\left[\, y_i \,\middle|\, \begin{matrix} (-a_i-(b_i+1)n_i-\mu_i k_i,\, e_i)\,,\ (c^{(i)}_{u_i},\ \gamma^{(i)}_{u_i}) \\ (d^{(i)}_{v_i},\ \delta^{(i)}_{v_i}),\ (-a_i-b_i n_i-(\mu_i+q_i)k_i, e_i) \end{matrix} \right] \Big\}.$$

(2.7) $$\sum_{n_1,\ldots,n_r=0}^{\infty} \prod_{i=1}^{r} \left\{ \frac{h_i}{h_i+(b_i+1)n_i} \frac{t_i^{n_i}}{(n_i)\,!} \right\}$$

$$U^{(a_r),\,(n_r)}_{(b_r),\,(q_r)}\left[\,(\mu_r);\,(x_r);\,(y_r)\,\right]$$

$$= \prod_{i=1}^{r} \left\{ (1+\eta_i)^{a_i} \right\} E\left[\,(\sigma_r\, x_r);\,(\zeta_r\, y_r);\,(\xi_r)\,\right]$$

where

(2.7a) $$\sigma_i = (-\eta_i)^{q_i}\,(1+\eta_i)^{\mu_i}$$

$$\left.\begin{aligned} \zeta_i &= (1+\eta_i)^{e_i} \\ \xi_i &= -\,\eta_i/(1+\eta_i) \end{aligned}\right\} i=1,\ldots,r$$

and

(2.8) $$E\left[\,(x_r);\,(y_r);\,(z_r)\,\right]$$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \left(\phi \left[\begin{matrix} (n_r),\,(a_r),\,(h_r);\,(x_r) \\ (q_r),\,(b_r),\,-\,;\,(y_r) \end{matrix} \right] \prod_{i=1}^{r} \left\{ \frac{(z_i)^{n_i}}{(n_i)\,!} \right\} \right)$$

and

$$\phi \left[\begin{matrix} (n_r),\,(a_r),\,(h_r)\,;\,(x_r) \\ (q_r),\,b_r)\quad -\quad ;\,(y_r) \end{matrix} \right]$$

$$= \sum_{k_1,\,\ldots\,,\,k_r\,=\,0}^{\infty} \Big[A\,(k_1\,,\,\ldots\,k_r) \prod_{i=1}^{r} \Big\{ \frac{h_i}{h_i+(b_i+1)\,q_i k_i}$$

$$\cdot \binom{n_i + q_i k_i + h_i \mid (b_i + 1)}{n_i}^{-1} [x_i (-\eta_i)^{q_i} (1 + \eta_i)^{\mu_i}]^{k_i}$$

$$(2.9) \quad \cdot H^{f_i, g_i + 1}_{u_i + 1, v_i + 1} \left[y_i (1 + \eta_i) e_i \middle| \begin{matrix} (h_i - a_i - \mu_i k_i, e_i), (c_{u_i}^{(i)}, \gamma_{u_i}^{(i)}) \\ (d_{v_i}^{(i)}, \delta_{v_i}^{(i)}) (h_i - a_i - \mu_i k_i + n_i, e_i) \end{matrix} \right] \Bigg\}.$$

$$(2.10) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} V^{(a_r), (n_r)}_{(b_r), (q_r)} [(\mu_r); (x_r); (y_r)] \prod_{i=1}^{r} \left\{ \frac{(t_i^{n_i})}{(n_i)!} \right\}$$

$$= \prod_{i=1}^{r} \{ (1 + \eta_i)^{a_i} \} B [(\sigma_r x_r); (\zeta_r y_r)]$$

(σ_i, ζ_i for $i = 1, \ldots, r$ being given by (2.7a))

for the polynomial system V given by

$$(2.11) \quad V^{(a_r), (n_r)}_{(b_r), (q_r)} [(\mu_r); (x_r); (y_r)]$$

$$= \sum_{k_1 = 0}^{[n_1/q_1]} \cdots \sum_{k_r = 0}^{[n_r/q_r]} A(k_1, \ldots, k_r) \prod_{i=1}^{r} \Bigg\{ (-n_i)_{q_i k_i} x_i^{k_i}$$

$$, H^{f_i, g_i + 1}_{u_i + 1, v_i + 1} \left[y_i \middle| \begin{matrix} (1 - a_i - (b_i + 1) n_i - \mu_i k_i, e_i), \\ (d_{v_i}^{(i)}, \delta_{v_i}^{(i)}), \end{matrix} \begin{matrix} (c_{u_i}^{(i)}, \gamma_{u_i}^{(i)}) \\ (-a_i - b_i n_i - (\mu_i + q_i) k_i, e_i) \end{matrix} \right] \Bigg\}$$

where

$$B [(x_r); (y_r);]$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} [A(k_1, \ldots k_r) \prod_{i=1}^{r} \{ [x_i (-\eta_i)^{q_i} (1 + \eta_i)^{\mu_i}]^{k_i}$$

$$H^{f_i, g_i + 1}_{u_i + 1, v_i + 1} \left[y_i (1 + \eta_i)^{e_i} \middle| \begin{matrix} (1 - a_i - (b_i + 1) q_i k_i - \mu_i k_i, e_i), \\ d_{v_i}^{(i)}, \delta_{v_i}^{(i)}), \end{matrix} \right.$$

$$\left.\left.\begin{array}{l}(c_{u_i}^{(i)}, \gamma_{u_i}^{(i)}) \\ (-a_i-(b_i+1)q_ik_i-\mu_ik_i, e_i)\end{array}\right]\right\}\Bigg]$$

3. METHOD OF DERIVATION

To derive the generating function (2.2), we first substitute for the polynomial system T from (2,3), use (1.1) to replace the H–function occurring on the left hand side, change the order of summations and integrations(which are valid because of the conditions assumed), applying [3, p. 232, eqn. (11)] and then use (2. 1) to arrive at the desired result (2. 2).

The generating function (2. 5) is established on proceeding on similar lines as indicated above and using [4, p. 361, Th. 2].

Gould's identity [1, p. 196], viz.

$$\sum_{n=0}^{\infty} \frac{h}{h+(b+1)n}\binom{a+(b+1)n}{n} t^n$$

$$= (1+\eta)^a \sum_{n=0}^{\infty}\left[(-1)^n\binom{a-h}{n}\binom{n+h/(b+1)}{n}^{-1}\left(\frac{\eta}{1+\eta}\right)^n\right]$$

(where a, b and h are arbitrary and η is defined by (2.4)), is used to arrive at the generating function (2. 7), while (2. 10) is obtained by using (3. 1) with h = a.

4. PARTICULAR CASES

The generating relations (2. 2), (2. 5), (2. 7) and (2. 10) are capable of yielding a number of particular cases involving polynomials of several variables. In particular ,the generating relations of Srivastava and Raina [7, eqns. (1. 14), (3. 3), (4.6) and (5.3)], which include as special cases a number of interesting generating functions for various polynomials, can be easily obtained from these results.

5. Extensions Involving The Multivariable H-Function

For the polynomial system given by

$$P_{(b_m),\,(q_m)}^{(a_m),\,(n_m)}\left[(\mu_m);\,(x_m);\,(y_r'),\,\ldots,\,(y_r^{(m)})\right]$$

$$=\sum_{k_1=0}^{[n_1/q_1]}\cdots\sum_{k_m=0}^{[n_m/q_m]}\left[\prod_{j=1}^{m}\left\{(-n_j)_{q_jk_j}\,x_j^{k_j}\,S^{(j)}\right\}A\,(k_1,\ldots,k_m)\right]$$

where, for $j=1,\ldots,m$, we have assumed that

$$S^{(j)}=H_{P^{(j)}+1,\;Q^{(j)}+1\;:\;*}^{O,\;\lambda^{(j)}+1\;\;\;\;\;\;\;\;:\;*}\left[\begin{matrix}L^{(j)}\;;\;*\\ M^{(j)}\;;\;*\end{matrix}\;;\,y_1^{(j)},\,\ldots,\,y_r^{(j)}\right]$$

and

$$L^{(j)}:\;[-a_j-(b_j+1)\,n_j-\mu_jk_j\;;\;e_1^{(j)},\ldots,e_r^{(j)}],\;[\,(p^{(j)})\;;$$
$$\theta_1^{(j)},\ldots,\theta_r^{(j)}\,]$$

$$M^{(j)}:[\,-a_j-b_j\,n_j-(\mu_j+q_j)\,k_j;\,e_1^{(j)},\ldots,e_r^{(j)}],$$
$$[\,(q^{(j)}),\,\Psi_1^{(j)},\ldots,\Psi_r^{(j)}\,]$$

having for co-efficients product of m H-functions of r variables, the following generating function is established:

$$(5.3)\qquad\sum_{n_1,\ldots,n_m=0}^{\infty}\left[\prod_{i=1}^{m}\left\{\frac{h_i}{h_i+(b_i+1)n_i}\,\frac{(t_i)^{n_i}}{(n_i)\,!}\right\}\right.$$

$$\left.P_{(b_m),\,(q_m)}^{(a_m),\,(n_m)}\left[(\mu_m);\,(x_m);\,(y_r')\,,\ldots,\,(y_r^{(m)})\right]\right.$$

$$=\prod_{i=1}^{r}\left\{(1+\eta_i)^{a_i}\,\phi_1\left[(\zeta_m\,x_m);\,(\sigma_r'\,y_r')\,,\ldots,\,(\sigma_r^{(m)}\,y_r^{(m)});\,(\xi_m)\right]\right.$$

where

$$(5.4)\qquad\left.\begin{aligned}\zeta_i&=(-\eta_i)^{q_i}\,(1+\eta_i)^{\mu_i}\\ \sigma_j^{(i)}&=(1+\eta_i)^{e_j^{(i)}}\\ \xi_i&=-\,\eta_i/(1+\eta_i)\end{aligned}\right\}\text{ for } j=1,\ldots,r;\; i=1,\ldots,m,$$

(5.5) $\Phi_1 [(x_m); (y_r') ,\ldots, (y_r^{(m)}); (w_m)]$

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \Psi_1 \left[\begin{matrix} (n_m), (a_m), (h_m); \\ (q_m), (b_m), - ; \end{matrix} (x_m); (y_r') ,\ldots, (y_r^{(m)}) \right]$$

$$\prod_{i=1}^{r} \left\{ \frac{w_i}{(n_i)!} \right\}$$

and further

$$(5.6.) \quad \Psi_1 \left[\begin{matrix} (n_m), (a_m), (h_m) ; \\ (q_m), (b_m) , - ; \end{matrix} (x_m); (y_r') ,\ldots, (y_r^{(m)}) \right]$$

$$= \sum_{k_1,\ldots,k_m=0}^{\infty} A (k_1 \ldots, k_m) \prod_{i=1}^{m} \left\{ S_i \frac{h_i}{h_i + (b_i + 1)q_i k_i} \right.$$

$$\left. \cdot \begin{pmatrix} n_i + q_i k_i + h_i/(b_i + 1) \\ n_i \end{pmatrix}^{-1} [x_1 (-\gamma_1)^{q_i} (1+\gamma_i)^{\mu_i}]^{k_i} \right\}$$

with

$$(5.7) \quad S_j = H \begin{matrix} o, \lambda^{(j)} + 1 ; : * \\ P^{(j)} + 1, Q^{(j)} + 1 : * \end{matrix} \left[\begin{matrix} L_j ; * \\ M_j ; * \end{matrix} ; y_1^{(j)} ,\ldots, y_r^{(j)} \right],$$

$$L_j = [h_j - a_j - \mu_j k_j ; e_1^{(j)} ,\ldots, e_r^{(j)}], [(p^{(j)}); \theta_1^{(j)} ,\ldots, \theta_r^{(j)}],$$

$$M_j = [h_j - a_j - \mu_j k_j + n_j ; e_1^{(j)} ,\ldots, e_r^{(j)}], [(q^{(j)}); \Psi_1^{(j)} ,\ldots, \Psi_r^{(j)}]$$

for $j = 1 ,\ldots, m$.

Results similar to (2.2), (2.5) and (2.10) for polynomials having as co-efficients product of m H-functions of r variables can also be given but are not mentioned here for lack of space.

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