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MATRIX SUMMABILITY OF THE DERIVED FOURIER SERIES

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1. DEFINITIONS AND NOTATIONS

Let $T \equiv \{\lambda_{n,k}\}$, $n, k = 0, 1, 2, \ldots$ be a triangular Toeplitz matrix so that it satisfies the Silverman–Toeplitz conditions of regularity (see Hardy [1], p. 43, Theorem 2) and $\lambda_{n,k} = 0$, for $k > n$.

An infinite series Σu_n with partial sums S_n is said to be summable by matrix-Cesáro product means or summable (T) $(C, 1)$ to s, if

$$(1.1) \qquad \sum_{k=0}^{n} (\lambda_{n,k})\, \sigma_k \to S, \text{ as } n \to \infty,$$

where σ_n stands for the $(C, 1)$ transform of S_n.

Let the Fourier series associated with the function $f(x)$, which is integrable is Lebesgue sense over $(-\pi, \pi)$ and periodic with period 2π, be

$$(1.2) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The derived series of the Fourier series (1.2) is

$$(1.3) \qquad \sum_{n=1}^{\infty} n\,(b_n \cos nx - a_n \sin nx) \equiv \sum_{1}^{\infty} n\, B_n\,(x).$$

The sequence $\{nB_n(x)\}$ is known as the sequence of Fourier coefficients.

We shall use, for a fixed x, the following notations :

$$\Psi(t) = f(x+t) - f(x-t).$$

$$g(t) = \Psi(t)/(4 \sin t/2)$$

$$h_k(t) = \left\{\frac{2(1-\cos kt)}{kt^2} + \frac{\sin kt}{kt} - \frac{\sin kt}{t}\right\}.$$

$$D_k(t) = \frac{\sin kt}{t}.$$

$$k_k(t) = D_1(t) + D_2(t) + \dots + D_k(t).$$

2. INTRODUCTION

The following result concerning the matrix summability of the sequence of Fourier coefficients is due to Kathal [2] :

Theorem A. If, for $\Psi_x(t) = f(x+t) - f(x-t) - l$,

(2.1) $\Psi_x(t) \equiv \int_0^t |\Psi_x(t)|\, dt = o(t)$, as $t \to 0$

and

(2.2) $\sum_{k=1}^{n} k\, |(\lambda_{n,k} - \lambda_{n,k+1})| = O(1)$, as $n \to \infty$,

then the sequence $\{n B_n(x)\}$ is summable $(T)(C, 1)$ to l/π.

The product summability $(E, q)(C, 1)$ of the derived Fourier series has recently been studied by Sachan and Kathal [3]. Our object here is to extend Theorem A to the derived Fourier series by proving

Theorem. *If*

(2.3) $G(t) \equiv \int_0^t |g(u)|\, du = o(t)$, *as* $t \to o$

and

(2.4) $\sum_{k=0}^{n-1} k\, |(\lambda_{n,k} - \lambda_{n,k+1})| = O(1)$, *as* $n \to \infty$,

then the derived Fourier series (1.3) is summable $(T)(C, 1)$, at the point x, to zero.

3. PRELIMINARY ESTIMATES

For the proof of above theorem we shall require the following estimates, which may easily be verified :

(3.1) $|h_k(t)| = O(k)$, for $0 < t < 1/k$.

(3.2) $K_k(t) = K_{k-1}(t) + D_k(t) = \dfrac{\sin (k+1)\, t/2 \sin kt/2}{t \sin t/2}$,

(3.3) $|K_k(t)| = O(1/t^2)$, for $t > 1/k$.

4. PRELIMINARY LEMMA

We shall also require the following lemma in the sequel :

If $g(t)$ be integrable (L) and δ, any positive real number less than π, then the $(C,1)$ transform σ_n of the nth partial sum S_n of the derived Fourier series (1.3) *is given by*

$$(4.1)\quad \sigma_n = \frac{2}{\pi} \int_0^\delta g(t) \left\{ \frac{2(1-\cos nt)}{nt^2} + \frac{\sin nt}{nt} - \frac{\sin nt}{t} \right\} dt$$
$$+ o(1).$$

PROOF. The proof of (4.1) is due to Sachan and Kathal ([3], Lemma 2) and is given here only for the sake of completeness.

The n th partial sum S_n of the derived Fourier series (1.3) may be written as

$$S_n = -\frac{1}{\pi} \int_0^\pi \Psi(t) \frac{d}{dt} \left\{ \frac{\sin (n+\frac{1}{2})t}{2 \sin t/2} \right\} dt.$$

Now, since $g(t)$ is integrable (L) and δ, fixed with $0 < \delta < \pi$, the $(C,1)$ transform σ_n of S_n is obtained as

$$\sigma_n = -\frac{1}{n\pi} \int_0^\pi \Psi(t) \left[\sum_{k=1}^{n} \frac{d}{dt} \left\{ \frac{\sin (k+\frac{1}{2})t}{2 \sin t/2} \right\} \right] dt$$

$$= -\frac{1}{n\pi} \int_0^\pi \Psi(t) \frac{d}{dt} \left[\frac{1}{4 \sin^2 t/2} \sum_{k=1}^{n} \right.$$

$$\{ \cos kt - \cos (k+1)t\} \Big] dt$$

$$= -\frac{1}{4n\pi}\int_0^{\pi} \Psi(t) \frac{d}{dt}\left[\frac{\cos t - \cos (n+1)t}{\sin^2 (t/2)}\right] dt$$

$$= -\frac{1}{n\pi}\int_0^{\pi} g(t) \left[-\frac{\cos t/2}{\sin^2 t/2} + \frac{\cos t/2 \cos (n+1)t}{\sin^2 t/2}\right.$$

$$\left. + \frac{(n+1) \sin (n+1)t}{\sin t/2}\right] dt$$

$$= \frac{1}{n\pi}\int_0^{\pi} g(t) \left[\frac{\cos t/2}{\sin^2 t/2} - \frac{\cos (n+\frac{1}{2})t}{\sin^2 t/2} - \frac{n \sin nt\,(1-2\sin^2 t/2)}{\sin t/2}\right.$$

$$\left. - 2n \cos nt \cos t/2 \right] dt$$

$$(4.2) \quad = \frac{1}{\pi}\int_0^{\pi} g(t) \left[\frac{\cos t/2\,(1-\cos nt)}{n \sin^2 t/2} + \frac{\sin nt}{n \sin t/2} - \frac{\sin nt}{\sin t/2}\right] dt$$

$$- \frac{2}{\pi}\int_0^{\pi} g(t) \cos (n+\tfrac{1}{2})t\, dt$$

$$= \frac{1}{\pi}\int_0^{\delta} g(t) \left[\frac{\cos t/2\,(1-\cos nt)}{n \sin^2 t/2} + \frac{\sin nt}{n \sin t/2} - \frac{\sin nt}{\sin t/2}\right] dt$$

$$+ o(1),$$

as $n \to \infty$, since in (4.2) the part of the first integral over (δ, π) and the second integral – each is $o(1)$ by Riemann-Lebesgue theorem.

$$= \frac{2}{\pi}\int_0^{\delta} g(t) \left[\frac{2(1-\cos nt)}{nt^2} + \frac{\sin nt}{nt} - \frac{\sin nt}{n}\right] dt + o(1),$$

as $n \to \infty$, where the last two terms in the above integral are due to the Riemann–Lebesgue theorem and the first, since

$$\frac{1}{n}\int_0^{\delta} g(t)\,(1-\cos nt)\left[\frac{\cos t/2}{\sin^2 t/2}-\frac{4}{t^2}\right]dt$$

$$\leqslant \frac{2}{n}\int_0^{\delta} |\, g(t)\,|\,\left|\left[\frac{1-\frac{t^2}{8}}{t^2/4}-\frac{4}{t^2}\right]\right| dt$$

$$=\frac{1}{n}\int_0^{\delta} |\, g(t)\,|\, dt$$

$= o\ (1)$, as $n \to \infty$, by continuity of $\int |\,g(t)\,|\,dt$.

5. PROOF OF THE THEOREM

Using (1.1), (4.1) and denoting the matrix transform of σ_n by t_n, we get

$$t_n = \frac{2}{\pi}\sum_{k=1}^{n}(\lambda_{n,k})\int_0^{\delta} g(t)\left\{\frac{2\,(1-\cos kt)}{kt^2}+\frac{\sin kt}{kt}\right.$$

$$\left.-\frac{\sin kt}{t}\right\}dt + o\ (1)$$

$$=\frac{2}{\pi}\sum_{k=1}^{n}(\lambda_{n,k})\left\{\int_0^{1/k}+\int_{1/k}^{\delta}\right\} g(t)\, h_k(t)\, dt + o\ (1)$$

(5.1) $\quad = \frac{2}{\pi}(\tau_1+\tau_2)+o\ (1)$, say, where δ is chosen such that (2.3) holds for $t<\delta$ ([4], § 13.34, *p.* 415).

By (3.1) and (2.3), we obtain

$$\left|\int_0^{1/k} g(t)\, h_k(t)\, dt\right| = \int_0^{1/k} |\, g(t)\,|\, O(k)\, dt$$

$$= o\ (1).$$

Consequently, by regularity of the method of summation

(5.2) $\quad \tau_1 = o\ (1)$, as $n \to \infty$.

Further, since δ is fixed, we can assume, for a fixed positive integer N

(5.3) $1/k < \delta$, for $k \geqslant N$ and

(5.4) $\lambda_{n,k} = 0$, for $k = 1, 2, \ldots, (N-1)$, without loss of generality.

Now, by virtue of regularity of the method of summation, (5.3) and (5.4), we have

$$| \tau_2 | = \left| \sum_{k=1}^{n} (\lambda_{n,k}) \int_{1/k}^{\delta} g(t) \left\{ \frac{2\,(1-\cos kt)}{kt^2} + \frac{\sin kt}{kt} - \frac{\sin kt}{t} \right\} dt \right|$$

$$= o\,(1) + \left| \sum_{k=N}^{n} (\lambda_{n,k}) \int_{1/k}^{\delta} g(t) \frac{\sin kt}{t}\, dt \right|$$

since the first term of the integral is easily $o\,(1)$, by partial integration and (2.3) and the second is also so, by the second mean value and Riemann–Lebesgue theorems.

Next, we write, in view of (3.2)

$$| \tau_2 | = \left| \sum_{k=N}^{n} (\lambda_{n,k}) \int_{1/k}^{\delta} g(t)\, [K_k(t) - K_{k-1}(t)]\, dt \right| + o\,(1)$$

$$\leqslant \left\{ \left| \sum_{k=N}^{n-1} (\lambda_{n,k} - \lambda_{n,\,k+1}) \int_{1/k}^{\delta} g(t)\, K_k(t)\, dt \right| + \left| \sum_{k=N+1}^{n} (\lambda_{n,k}) \int_{1/k}^{1/(k-1)} g(t)\, K_{k-1}(t)\, dt \right| + \left| (\lambda_{n,n}) \int_{1/n}^{\delta} g(t)\, K_n(t)\, dt \right| + \left| (\lambda_{n,N}) \int_{1/N}^{\delta} g(t) K_{N-1}(t)\, dt \right| \right\} + o\,(1)$$

$$= [J_1 + J_2 + J_3 + J_4] + o\,(1), \text{ say.}$$

Using (3.3), (2.3) and integrating by parts, we easily obtain

$$| \int_{1/k}^{\delta} g(t)\, h_k(t)\, dt | = \int_{1/k}^{\delta} | g(t) |\, O\left(\frac{1}{t^2}\right) dt$$

(5.5) $$= o\ (K).$$

Also, we observe that the regularity condition, $\Sigma \mid \lambda_{n,k} \mid = O\ (1)$ and the hypothesis, $\Sigma\, K \mid (\lambda_{n,k} - \lambda_{n,k+1}) \mid = O\ (1)$, imply

(5.6) $K\ (\lambda_{n,k}) = O\ (1)$, as $n \to \infty$.

By (5.5) and (2.4), we have

$$J_1 = \sum_{K=N}^{n-1} \mid (\lambda_{n,k} - \lambda_{n,k+1}) \mid o\ (K)$$

$$= o\ (1), \text{ as } n \to \infty.$$

By (5.5) and (5.6), we easily get

$J_\nu = o\ (1)$, as $n \to \infty$, for $\nu = 3, 4$.

Lastly, by (3.3), (2.3) and integrating by parts, we have

$$J_2 = \sum_{k=N+1}^{n} \mid \lambda_{n,k} \mid \int_{1/k}^{1/(k-1)} \mid g(t) \mid O\ (1/t^2)\ dt$$

$$= \sum_{k=N+1}^{n} \mid \lambda_{n,k} \mid o \left\{ \left| \left\{ \frac{1}{t} \right\}_{1/k}^{1/(k-1)} \right| + \left| \int_{1/k}^{1/(k-1)} 1/t^2\, dt \right| \right\}$$

$$= \sum_{k=N+1}^{n} \mid \lambda_{n,k} \mid o\ (1)$$

$$= o\ (1), \text{ as } n \to \infty, \text{ by regularity of the matrix } \{\lambda_{n,k}\}.$$

Thus

(5.7) $\mid \tau_2 \mid = o\ (1)$, as $n \to \infty$.

Finally, from (5.1), (5.2) and (5.7), we find

$t_n = o\ (1)$, as $n \to \infty$.

This completes the proof of the theorem.

6. REMARKS

It may be noted that on replacing $(\lambda_{n,k})$ by (i) $1/n$ (ii) $(1-x)x^k$, $x \uparrow 1$ (iii) $(1-x)^{\lambda+1} \dfrac{(\lambda+1)(\lambda+2)\ldots(\lambda+k)}{1.\,2\ldots k} x^k$, $\lambda+1 > 0$; $x \uparrow 1$ and (iv) $\{-\log(1-x)\}^{-1} \dfrac{x^k}{k}$, $0 < x < 1$; the matrix $\{\lambda_{n,k}\}$ is transformed into the (C,1), Abel, (A, λ) and (L) matrices respectively, which satisfy the condition, $\sum_{k=1}^{n-1} k \mid \lambda_{n,k} - \lambda_{n,\, k+1} \mid = O(1)$, as $n \to \infty$. Consequently, our theorem gives, in particular, the results on $(C,1)$ $(C,1)$, (A) $(C,1)$, (A,λ) $(C,1)$ and (L) $(C,1)$ summabilities of the derived Fourier series. However, it may be interesting to work out independent proofs for these results.

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REFERENCES

[1] G. H. Hardy, *Divergent Series*, Oxford University Press, 1949.

[2] P. D. Kathal, *Matrix Summability of the Sequence of Fourier Coefficients*, Ph. D. thesis, University of Saugar, India, 1968.

[3] M. P. Schan and P. D. Kathal, Euler–Cesàro product summability of the derived Fourier series, *Math. Chronicle* (Communicated)

[4] E. C. Titchmarsh, *The Theory of Functions*, E. L. B. S. and Oxford University Press, 1961.