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HEAT CONDUCTION AND THE *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

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The object of this paper is to make use of the *H*-function of several complex variables in obtaining a solution of the partial differential equation related to a problem of heat conduction,

1. INTRODUCTION

As an example of the application of the *H*-function of several complex variables introduced by Srivastava and Panda [9] in applied mathematics, we shall consider the problem of obtaining a solution of a problem of heat conduction. Consider the partial differential equation

$$(1.1) \quad \frac{\partial \phi}{\partial t} = \xi \frac{\partial^2 \phi}{\partial x^2} - \xi \phi x^2$$

where $\phi(x, t) \to 0$ for large value of t and when $|x| \to \infty$; this equation is related to the problem of heat conduction given by Churchill [3]

$$(1.2) \quad \frac{\partial \phi}{\partial t} = \xi \frac{\partial^2 \phi}{\partial x^2} - \eta (\phi - \phi_0)$$

provided that $\phi_0 = 0$ and $\eta = \xi x^2$.

In this paper we shall assume that

$$(1.3)\ f(x) = x^{2\sigma} e^{-x^2} H^{m,n}_{p,q}\left[yx^{2h} \middle| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right]$$

$$H^{0,0:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]: \\ [(c):\Psi',\dots,\Psi^{(r)}]: \end{matrix} \begin{matrix} [(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 x^{2k_1}, \dots, z_r x^{2k_r} \right)$$

and require a series representation of the H–function given by

Skibiński [7]

$$(1.4)\quad H^{m,n}_{p,q}\left[y \middle| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right] = \sum_{g=1}^{m} \sum_{s=1}^{\infty} \frac{(-1)^s \phi(\eta_s)\, y^{\eta_s}}{s!\, F_g}$$

where

$$\phi(\eta_s) = \prod_{\substack{i=1 \\ i \neq g}}^{m} \Gamma(f_i - F_i \eta_s) \prod_{i=1}^{n} \Gamma(1 - e_i + E_i \eta_s)$$

$$\left\{ \prod_{i=m+1}^{q} \Gamma(1 - f_i + F_i \eta_s) \prod_{i=n+1}^{p} \Gamma(e_i - E_i \eta_s) \right\}^{-1}$$

and

$$\eta_s = \frac{f_g + s}{F_g}.$$

The multivariate H–function defined recently by Srivastava and Panda [9] by means of the multiple contour integral as follows (see also Srivastava, Gupta and Goyal [8, p. 251, Eq. (C. 1)]) :

$$(1.5)\quad H^{0,0:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]; \\ [(c):\Psi',\dots,\Psi^{(r)}]; \end{matrix} \right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\dots,z_r\right)$$

$$= (2\pi\,\omega)^{-r}\int_{L_1}\dots\int_{L_r}\dots U_1(s_1)\dots U_r(s_r)\,V(s_1,\dots,s_r)$$

$$z_1^{s_1}\dots z_r^{s_r}\,ds_1\dots ds_r\,,\ \omega=\sqrt{-1},$$

where

$$(1.6)\quad U_i(s_i) = \prod_{j=1}^{u^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}s_i]\prod_{j=1}^{v^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}s_i]$$

$$\cdot\left\{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}s_i]\right.$$

$$\left.\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}s_i]\right\}^{-1},$$

$$(1.7)\quad V(s_1,\dots,s_r) = \prod_{j=1}^{\varepsilon}\Gamma[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i]$$

$$\cdot\left\{\prod_{j=\varepsilon+1}^{A}\Gamma[a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{r}\Psi_j^{(i)}s_i]\right\}^{-1},$$

an empty product is interpreted as 1, the coefficients $\theta_j^{(i)}$, $j=1,\dots,A$; $\phi_j^{(i)}$, $j=1,\dots,B^{(i)}$; $\Psi_j^{(i)}$, $j=1,\dots,C$; $\delta_j^{(i)}$, $j=1,\dots,D^{(i)}$; and $i=1,\dots,r$, are positive numbers, and ε, $u^{(i)}$, $v^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are all integers such that $0\leqslant\varepsilon\leqslant A$, $0\leqslant u^{(i)}\leqslant D^{(i)}$, $C\geqslant 0$, and $0\leqslant v^{(i)}\leqslant B^{(i)}$, $i=1,\dots,r$. The contour L_i in the complex s_i-plane is of the Mellin–Barnes type which from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma[d_j^{(i)}-\delta_j^{(i)}s_i]$ $j=1,\dots,u^{(i)}$, are to the right, and those of $\Gamma[1-b_j^{(i)}+\phi_j^{(i)}s_i]$,

$j = 1, \ldots, \nu^{(i)}$, and $\Gamma\left[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right]$, $j = 1, \ldots, \varepsilon$,

to the left, of L_i, the various parameters so restricted that these poles are all simple and none of them coincide ; and with the points $z_i = 0$, $i = 1, \ldots, r$ being tacitly excluded, the multiintegral in (1.5) converges absolutely if

$$(1.8) \quad | \arg z_i | < \tfrac{1}{2} \Delta_i \pi, \; i = 1, \ldots, r$$

where

$$(1.9) \quad \Delta_i = -\sum_{j=\varepsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

$$i = 1, \ldots, r.$$

The conditions corresponding to the aforementioned ones will be assumed to hold throughout this paper.

Also, following Srivastava and Panda [9], we use the abbreviation (a) to denote the sequence of A parameters $a_1, \ldots, a_A$; for each $i = 1, \ldots, r$, $b^{(i)}$ abbreviates the sequence of $B^{(i)}$ parameters $b_j^{(i)}$, $j = 1, \ldots, B^{(i)}$, with similar interpretations for (c), $(d^{(i)})$, etc., $i = 1, \ldots, r$, it being understood, for example, that $b^{(1)} = b'$, $b^{(2)} = b''$, and so on.

In section 2 of this paper we have evaluated an integral involving Fox's H–function and the H–function of several complex variables which is required in the proof detailed in the subsequent section.

2. THE MAIN INTEGRAL

$$(2.1) \quad \int_{-\infty}^{\infty} x^{2\sigma} e^{-x^2} H_{2v}(x) \, H_{p,q}^{m,n}\left[y x^{2h} \middle| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right]$$

$$H^{0,\,0:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1 x^{2k_1},\ldots,\ z_r x^{2k_r}\right)dx$$

$$= \sum_{g=1}^{m}\ \sum_{s=1}^{\infty}\ \frac{(-1)^s \sqrt{\pi}\,\phi\ (\eta_s)\ y^{\eta_s}\ 2^{2\,(\nu-\sigma-h\,\eta_s)}}{s\,!\,F_g}$$

$$H^{0,\,1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix}[-2\sigma-2h\,\eta_s : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}],\ [\nu-\sigma-h\,\eta_s : k_1,\ldots,k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}:\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1\,2^{-2k_1},\ldots,\ z_r\ 2^{-2k_r}\right),$$

where $h > 0,\ k > 0,\ \sigma > 0,\ Re\,(1+h\,\frac{f_l}{F_l} + \sum_{i=1}^{r} k_i\,\frac{d_j^{(i)}}{\delta_j^{(i)}}) > 0,$

$l = 1,\ldots,m;\ i = 1,\ldots,r\ ;\ j = 1,\ldots,u^{(i)},\ \triangle_i > 0,\ \triangle > 0$

$|\arg z_i| < \frac{1}{2}\,\triangle_i\,\pi,\ |\arg y| < \frac{1}{2}\,\nabla\pi$

$$(\ \nabla = \sum_{1}^{n} E_i - \sum_{n+1}^{p} E_i + \sum_{1}^{m} F_i - \sum_{m+1}^{q} F_i\ ).$$

The integral formula (2.1) can be established by making use of (1.4), (1.5) and the following integral :

$$(2.2)\qquad \int_{-\infty}^{\infty} x^{2\sigma}\ e^{-x^2}\ H_{2\nu}\,(x)\ dx = \frac{\sqrt{\pi}\ 2^{2(\nu-\sigma)}\ \Gamma\,(2\,\sigma+1)}{\Gamma\,(1+\sigma-\nu)}\ ,$$

$$\nu = 0,\ 1,\ 2,\ \ldots\ .$$

3. SOLUTION

The solution of (1.1) to be established is

$$(3.1)\quad \phi(x,t) = \sum_{\alpha=0,\ s=1}^{\infty} \sum_{g=1}^{m} \frac{(-1)^s\, \phi(\eta_s)\, y^{\eta_s}\, 2^{\alpha-2\sigma-2h\,\eta_s-\frac{1}{2}}}{s\,!\,\alpha\,!\,F_g}$$

$$e^{-(1+2\alpha)\,\xi\,t - x^2/2}$$

$$H^{0,\,1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+1\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left[\begin{array}{l} [-2\sigma - 2h\,\eta_s : 2k_1,\ldots,2k_r],\ [(a):\theta',\ldots,\theta^{(r)}] : \\ [(c):\Psi',\ldots,\Psi^{(r)}],\ [\alpha/2-\sigma-h\,\eta_s : k_1,\ldots,k_r] : \end{array}\right.$$

$$\left.\begin{array}{l} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\,; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\,; \end{array}\ z_1\, 2^{-2k_1},\ldots,z_r^{-2k_r}\right] H_\alpha(x),$$

where $h > 0,\ k > 0,\ \sigma > 0,\ \mathrm{Re}\,(1 + h\,\frac{f_l}{F_l} + \sum_{i=1}^{r} k_i\,\frac{d_j^{(i)}}{\delta_j^{(i)}}) > 0,$

$l = 1,\ldots,m\ ;\ i = 1,\ldots,r\ ;\ j = 1,\ldots,u^{(i)},\ \triangle_i > 0,\ \nabla > 0,$

$|\arg z_i| < \frac{1}{2}\triangle_i\,\pi;\ |\arg y| < \frac{1}{2}\nabla\,\pi$

$(\nabla = \sum_{1}^{n} E_i - \sum_{n+1}^{p} E_i + \sum_{1}^{m} F_i - \sum_{m+1}^{q} F_i).$

PROOF :

The solution of (1.1) can be written as [*1, p. 360, Eq.* (2.3)]

$$(3.2)\quad \phi(x,t) = \sum_{\alpha=0}^{\infty} T_\alpha\, e^{-(1+2\alpha)\,\xi\,t - x^2/2}\, H_\alpha(x)$$

where $H_\alpha(x)$ is well-known Hermite polynomial.

If $t = 0$, then by virtue of (1.3), we have

$$(3.3)\quad x^{2\sigma} e^{-x^2} H^{m,n}_{p,q}\left[yx^{2h} \middle| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right]$$

$$H^{0,0:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left\{ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 x^{2k_1},\ldots,z_r x^{2k_r} \right\}$$

$$= \sum_{a=o}^{\infty} T_\alpha\, e^{-x^2/2} H_\alpha(x).$$

Multiplying both sides of (3.3) by $H_\beta(x)$ and integrating from $-\infty$ to ∞ with respect to x, and using (2.1) and the orthogonality property of Hermite polynomials [4], we find

$$(3.4)\quad T_\beta = \sum_{g=1}^{m} \sum_{s=1}^{\infty} \frac{(-1)^s \phi(\eta_s)\; y^{\eta_s}\; 2^{\beta-2\sigma-2h\eta_s-\frac{1}{2}}}{F_g}$$

$$H^{0,1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left\{ \begin{matrix} [-2\sigma-2h\,\eta_s: 2k_1,\ldots,2k_r], \\ [(c):\Psi',\ldots,\Psi^{(r)}], \end{matrix} \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [\beta/2-\sigma-h\,\eta_s: k_1,\ldots,k_r]: \end{matrix} \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 2^{-2k_1},\ldots,z_r 2^{-2k_r} \right\}$$

With the help of (3.2) and (3.4), the soultion (3.1) is established.

4- CONCLUSION

On specializing the parameters, Fox's H-function and the multivariable H-Function may be transformed into G-functions, E-functions, Lauricella's functions, Appell's functions, Kampé de Fériet's functions, hypergeometric functions, Legendre functions, Bessel functions, and several other higher transcendental functions, in one or more arguments. Therefore, $f(x)$ given in (1.3) is of general character and hence encompasses several cases of interest.

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