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EULER SUMMABILITY OF FOURIER SERIES AND THE SEQUENCE OF FOURIER COEFFICIENTS

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1. Definitions and Notations

An infinite series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{S_n\}$ of its partial sums of first $(n+1)$ terms is said to be summable by Euler means or more presisely, summable (E,q), $(q>0)$ to s, if

$$P_n = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k$$

tends to a finite limit s, as $n \to \infty$.

It is obvious that $q=0$ makes this summability equivalent to convergence and $q = 1$ reduces it to well known summability $(E, 1)$.

Let $f(x)$ be a function integrable in the sense of Lebesgue over $(-\pi, \pi)$ and periodic with period 2π. Let the Fourier series associated with $f(x)$ be

$$(1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + \sum_{1}^{\infty} A_n(x).$$

The Derived Series of the Fourier series (1.1) is

$$(1.2) \quad \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} n\, B_n(x).$$

The sequence $\{n B_n(x)\}$ is known as the sequence of Fourier coefficients We shall use, $0 < q < 1$, the following notations :

$$\phi(t) = f(x+t) + f(x-t) - 2s \quad , \quad S \equiv constant.$$

$$\Psi(t) = f(x+t) - f(x-t) - l \quad , \quad l \equiv constant.$$

$$P(q, t) = 1 + q^2 + 2q \cos t.$$

$$Q(q, t) = \tan^{-1}\left(\frac{\sin t}{q + \cos t}\right).$$

$$E(n, t) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin kt.$$

$$H(n, t) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (k \sin kt).$$

2. INTRODUCTION

Initiating the Borel summability of Fourier series Hardy and Littlewood [3] proved the following :

Theorem A. *If*

(2.1) $$\phi(t) = o\left\{\frac{1}{\log(1/t)}\right\}, \text{ as } t \to 0,$$

then the Fourier series (1.1) *is summable* (B) *to s.*

Recently, Chandra [1] studied the Euler summability of Fourier series and established :

Theorem B. *If* (2,1) *is satisfied, then the Fourier series* (1.1) *is summable* (E, q), *for* $0 < q < 1$, *to s.*

In view of Lemma 2, section 3, the summability (E, q) implies summability (B). Therefore, Chandra's theorem implies Hardy and Littlewood's theorem.

The object of this note is to generalise the above theorem of

Chandra and thereby also the theorem of Hardy and Littlewood by proving the following Theorem 1. We also establish, in Theorem 2, the summability (E, q), $(q > 0)$ of the sequence $\{n B_n(x)\}$ under an analogous condition :

Theorem 1. *If*

(2.2) $\phi(t) = o(1)$, *as* $t \to 0$,

then the Fourier series (1.1) *is summable* (E, q), *for* $0 < q < 1$, *to* s.

Theorem 2. *If*

(2.3) $\Psi(t) = o(1)$, *as* $t \to 0$,

then the sequence of Fourier coefficients $\{n B_n(x)\}$ *is summable* (E, q), *for* $0 < q < 1$, *to* $1/\pi$.

3. We shall need the following lemmas in the sequel :

Lemma 1. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* (E, q), *for* $q > 0$, *then* $\sum_{n=0}^{\infty} u_n$ *is also summable* (E, q'), *for* $q' > q$. (See Hardy [2], Theorem 119).

Lemma 2. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* (E, q), *for* $q > 0$, *then* $\sum_{n=0}^{\infty} u_n$ *is also* summable (B). (See Hardy [2], Theorem 128).

Lemma 3. $E(n,t) = \{P(q, t)\}^{n/2} \sin\{n\, Q(q, t)\}$.

For the proof compare with Chandra ([1], (3.2)).

Lemma 4. *For* $0 < q < 1$ *and* $0 < t \leqslant \pi/2$,

$$(q+1)^{-n}\{P(q, t)\}^{n/2} = O\left\{\exp\left(-\frac{n\,q\,t^2}{2\pi^2}\right)\right\}.$$

The proof has been given by Ray ([4], Lemma 2).

Lemma 5 $\int_0^{\pi} H(n, t)\, dt = 1 - o(1)$, *as* $n \to \infty$.

Proof. We have

$$\int_0^\pi H(n, t)\, dt = \int_0^\pi (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (k \sin kt)\, dt$$

$$= (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^\pi (k \sin kt) dt)$$

since the series is uniformly convergent.

$$= (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} [1-(-1)^k]$$

$$= 1 - (q+1)^{-n} (q-1)^n$$

$$= 1 - o(1), \text{ as } n \to \infty,$$

since $0 < q < 1$.

Lemma 6.

$$H(n, t) = (q+1)^{-n}\, n \{ P(q, t) \}^{\frac{n-1}{2}} \sin \{ (n-1)\, Q(q, t) + t \}.$$

Proof. We have

$$H(n, t) = (q+1)^{-n}\, n \sum_{k=1}^{n} \binom{n-1}{k-1} q^{n-k} \sin kt$$

$$= (q+1)^{-n}\, n\, I_m \left[e^{it} \sum_{k=1}^{n} \binom{n-1}{k-1} q^{n-k} e^{i(k-1)t} \right]$$

$$= (q+1)^{-n}\, n\, I_m [e^{it} (q+e^{it})^{n-1}],$$

where $I_m [F(t)]$ denotes the imaginary part of $F(t)$.

$$= (q+1)^{-n}\, n \{P(q, t)\}^{\frac{n-1}{2}} \sin \{ (n-1)\, Q(q, t) + t\}$$

where $q+e^{it} = \{P(q, t)\}^{\frac{1}{2}} e^{iQ(q, t)}$.

Lemma 7. *If* $1/3 \leqslant a < \frac{1}{2}$ *and*

$$\int_0^t |\phi(t)|\, dt = o(t), \text{ as } t \to 0,$$

then

$$(q+1)^{-n} \int_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\alpha}} \frac{\phi(t)}{t} \{P(q,t)\}^{n/2} \left[\begin{matrix} \sin \\ \cos \end{matrix} \left\{ (n+\gamma)\, Q(q,t) \right\} - \begin{matrix} \sin \\ \cos \end{matrix} \left(\frac{n+\gamma}{1+q} \right) t \right] dt$$

$= o(1)$, *as* $n \to \infty$, *where r is a finite integer or zero.*

Proof. We prove the 'sine' case only. The 'cosine' case follows on similar lines.

It is easy to obtain

$$\left| \sin \{ (n+\gamma)\, Q(q,t) \} - \sin \left(\frac{n+\gamma}{1+q} \right) t \right| = O(n\, t^3).$$

Then applying Lemma 4 and the second mean value theorem, we have

$$(q+1)^{-n} \int_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\alpha}} \frac{\phi(t)}{t} \{P(q,t)\}^{n/2} \left[\sin \{(n+\gamma)\, Q(q,t)\} - \sin \left(\frac{n+\gamma}{1+q} \right) t\right] dt$$

$$= \int_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\alpha}} \frac{|\phi(t)|}{t}\, O \{ \exp(-\frac{nqt^2}{2\pi^2}) \}\, O(nt^3)$$

$$= O \left[n \exp \{ -\frac{nq}{2\pi^2} \frac{1}{(n+\gamma)^2} \} \right] \int_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\beta}} |\phi(t)|\, t^2\, dt$$

where $\alpha < \beta < 1$.

$$= O(n) \left[\{ o(t)\, t^2 \}_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\beta}} + 2 \int_{(n+\gamma)^{-1}}^{(n+\gamma)^{-\beta}} o(t)\, t\, dt \right]$$

$$= o(n)\,[t^3]\begin{matrix}(n+\gamma)^{-\beta}\\ (n+\gamma)^{-1}\end{matrix}$$

$$= o(1), \text{ as } n \to \infty, \text{ since } 1/3 \leqslant \alpha < \beta.$$

4. Proof of Theorem 1

The partial sum S_n of first $(n+1)$ terms of Fourier series (1.1) is given by

$$S_n - S = 1/\pi \int_0^\pi \frac{\phi(t)}{t} \sin nt\, dt + o(1).$$

Therefore, for $0 < q < 1$, the (E, q) transform ρ_n of S_n will be given, in view of Lemma 3, by

$$\rho_n - S = \frac{(q+1)^{-n}}{\pi} \int_0^\pi \frac{\phi(t)}{t} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin kt \right\} dt + o(1)$$

$$= \frac{(q+1)^{-n}}{\pi} \int_0^\pi \frac{\phi(t)}{t} E(n, t)\, dt + o(1)$$

$$= \frac{(q+1)^{-n}}{\pi} \int_0^\pi \frac{\phi(t)}{t} \{ P(q,t) \}^{n/2} \sin \{ n\, Q(q,t) \}\, dt + o(1)$$

$$= 1/\pi \left[\int_0^{1/n} + \int_{1/n}^{1/n^\alpha} + \int_{1/n^\alpha}^{\pi} \right] \frac{\phi(t)}{t} \frac{\{P(q, t)\}^{n/2}}{(q+1)^n}$$

$$\sin \{n\, Q(q, t)\}\, dt + o(1)$$

$$= 1/\pi\, [I_1 + I_2 + I_3] + o(1), \text{ say, where } 1/3 \leqslant \alpha < 1/2. \tag{4.1}$$

Using (2.2), Lemma 4 and the estimate, $\sin \{n\, Q(q, t)\} = O(nt)$, which is easily obtained, we get

$$| I_1 | = \int_0^{1/n} \frac{o(1)}{t}\, O \left\{ \exp \left(- \frac{n\, q\, t^2}{2\pi^2} \right) \right\} O(nt) dt$$

$$= \int_0^{1/n} o(n)\, dt$$

(4.2) $= o(1)$, *as* $n \to \infty$.

Applying Lemma 4 and the second mean value theorem, we have

$$| I_3 | \leqslant \frac{n^{\alpha} \{P(q, \frac{1}{n^{\alpha}})\}^{n/2}}{(q+1)^n} \int_{1/n^{\alpha}}^{\pi} | \phi(t) | \, dt$$

$$= O \left\{ \frac{n^{\alpha}}{exp\left(\frac{nq}{2\pi^2} \cdot \frac{1}{n^{2\alpha}}\right)} \right\} \int_{1/n^{\alpha}}^{\pi} | \phi(t) | \, dt$$

$$= O \left\{ \frac{n^{\alpha}}{exp\,(An^{1-2\alpha})} \right\}$$

(4.3) $= o(1)$, *as* $n \to \infty$, since $\alpha < \frac{1}{2}$, where A is a positive constant not necessarily the same at each occurrence.

Lastly, writing in view of (2.2), $\phi(t) = \eta$, where $| \eta | < \varepsilon$, for a given small $\varepsilon > 0$, and using the second mean value theorem, Lemmas 7 and 4, we have

$$I_2 = \frac{n \{ P(q, \frac{1}{n}) \}^{n/2}}{(q+1)^n} \int_{1/n}^{1/n^{\alpha'}} \phi(t)\, sin\left(\frac{nt}{1+q}\right) dt$$

where $\alpha < \alpha' < 1$.

$$= O \left\{ \frac{n}{exp\left(\frac{nq}{2\pi^2}, \frac{1}{n^2}\right)} \right\} \left| \int_{1/n}^{1/n^{\alpha'}} \eta\, sin\left(\frac{nt}{1+q}\right) dt \right|$$

$$= O(n) \left| \left[\eta \left(\frac{1+q}{n}\right) cos\left(\frac{nt}{1+q}\right) \right]_{1/n}^{1/n^{\alpha'}} \right|$$

$$= O\left[| \eta | \frac{2(1+q)}{n} \right]$$

$$= O\,[\,2\,\varepsilon\,(1+q)\,]$$

(4.4) $= o\,(1)$, as $n \to \infty$, since ε may be taken as small as we please.

Thus (4.1) (4.2), (4.3) and (4.4) complete the proof of Theorem 1.

5. PROOF OF THEOREM 2

The (E, q) transform ρ_n, for $0 < q < 1$, of sequence $\{n\,B_n\,(x)\}$ is given by

$$\rho_n = \frac{(q+1)^{-n}}{\pi} \int_0^{\pi} \{ f\,(x+t) - f\,(x-t) \} \, \{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k}\,(k \sin kt) \} dt$$

(5.1) $$= \frac{1}{\pi} \int_0^{\pi} \{ f\,(x+t) - f\,(x-t) \} \, H\,(n,\,t)\,dt.$$

Also from Lemma 5, we have

(5.2) $$\frac{1}{\pi} \{ 1 - o\,(1) \} = \frac{1}{\pi} \int_0^{\pi} H\,(n,\,t)\,dt$$

Subtracting (5.2) from (5.1) and using Lemma 6, we obtain

$$\rho_n - \frac{1}{\pi} + o\,(1) = \frac{1}{\pi} \int_0^{\pi} \{ f\,(x+t) - f\,(x-t) - l \} \, H(n,t)\,dt$$

$$= \frac{n\,(q+1)^{-n}}{\pi} \int_0^{\pi} \Psi\,(t)\,\{P\,(q,t)\}^{\frac{n-1}{2}} \sin\{\,(n-1)\,Q\,(q,t) + t\}\,dt$$

$$= \frac{n}{\pi} \left[\int_0^{(n-1)^{-1}} + \int_{(n-1)^{-1}}^{(n-1)^{-\alpha}} + \int_{(n-1)^{-\alpha}}^{\pi} \right]$$

$$\Psi\,(t)\,\frac{\{P\,(q,t)\}^{\frac{n-1}{2}}}{(q+1)^n} \sin\{\,(n-1)\,Q\,(q,t) + t\,\}\,dt$$

(5.3) $= 1/\pi\,[R_1 + R_2 + R_3]$, say where $1/3 \leqslant \alpha < 1/2$.

Using hypothesis (2.3) and Lemma 4, we get

$$| R_1 | = n \int_0^{(n-1)^{-1}} \frac{| \Psi (t) |}{\{ P (q, t)\}^{1/2}} \; O \{\exp (- \frac{nqt^2}{2\pi^2}) \} dt$$

$$= O \left[\frac{n}{\{P (q, \frac{2}{(n-1)})\}^{1/2}} \right] \int_0^{(n-1)^{-1}} | \Psi (t) | \, dt$$

$$= O (n) \int_0^{(n-1)^{-1}} o (1) \, dt$$

(5.4) $= o (1)$, *as* $n \to \infty$.

Further, by Lemma 4 and the second mean value theorem

$$| R_3 | \leqslant \frac{n \, [P \{q, \frac{1}{(n-1)^{\alpha}} \}]^{n/2}}{(q+1)^n} \int_{(n-1)^{-\alpha}}^{\pi} \frac{| \Psi (t) | \, dt}{\{ P (q, t)\}^{1/2}}$$

$$= O \left[\frac{n}{\{ P (q, \pi)\}^{1/2} \exp \{ \frac{nq}{2\pi^2 (n-1)^{1/2}} \}} \right]$$

$$\int_{(n-1)^{-\alpha}}^{\pi} | \Psi (t) | \, dt$$

$$= O \left[\frac{n}{(1-q) \exp (A \, n^{1-2\alpha})} \right]$$

(5.5) $= o (1)$, *as* $n \to \infty$, *since* $\alpha < \frac{1}{2}$.

Next, we write

$$R_2 = n \int_{(n-1)^{-1}}^{(n-1)^{-\alpha}} \Psi (t) \frac{\{ P (q, t) \}^{\frac{n-1}{2}}}{(q+1)^n} \cos t \sin \{(n-1) \, Q \, (q,t)\} dt$$

$$+ n \int_{(n-1)^{-1}}^{(n-1)^{-\alpha}} \Psi(t) \frac{\{P(q,t)\}^{\frac{n-1}{2}}}{(q+1)^n} \sin t \cos \{(n-1) Q(q,t)\} dt$$

$= R_{2\cdot1} + R_{2\cdot2}$, *say.*

Assuming in view of (2.3), $\Psi(t) = \eta$, where $|\eta| < \varepsilon$, for a given small $\varepsilon > 0$, and applying the second mean value theorem and Lemma 7, we have

$$|R_{2\cdot1}| = \left| \frac{n \cos \frac{1}{(n+1)} \{P(q, \frac{1}{n-1})\}^{(n-1)/2}}{(q+1)^n} \int_{(n-1)^{-1}}^{(n-1)^{-\alpha}} \Psi(t) \sin \{\frac{(n-1)t}{q+1}\} dt \right|$$

$$= A n \left| \int_{(n-1)^{-1}}^{(n-1)^{-\alpha}} \eta \sin \{ (\frac{n-1}{q+1}) t \} dt \right|$$

$$= A n \left| \left[\eta \left(\frac{1+q}{n-1} \right) \cos \left(\frac{n-1}{q+1} \right) t \right]_{(n-1)^{-1}}^{(n-1)^{-\alpha}} \right|$$

$$\leqslant A n |\eta| 2 \left(\frac{1+q}{n-1} \right)$$

$$\leqslant 2 A \varepsilon \left(\frac{1+q}{1-1/n} \right)$$

$= o(1)$, as $n \to \infty$, since ε can be taken as small as we please. Similarly, $|R_{2\cdot2}| = o(1)$, as $n \to \infty$.

Thus

(5.6) $R_2 = o(1)$, as $n \to \infty$.

In view of (5.3), (5.4) and (5.6) the proof of Theorem 2 is complete.

[6] **REMARKS.** In the light of Lemmas 1 and 2, it is intersting to note that under conditions (2,2) and (2.3) of Theorems 1 and 2, the Fourier series (1.1) and the sequence $\{ n\, B_n(x) \}$ are summable (E,1) and (B) both to s and $1/\pi$ respectively. Independent proofs of such theorems can be attempted .

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