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FIXED POINTS OF A PAIR OF MAPPINGS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

Fixed point theorems for a pair of commuting mappings in a probabilistic metric space (PM-space) are proved.

INTRODUCTION

Jungck [6] has generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting mappings on metric spaces. Recently, Jungck's result has been generalized on several settings (e.g.,see [2], [3], [8] and [9]).

We prove a common fixed point theorem (cf. Theorem 1 below) for a pair of commuting mappings in a PM-space, and then we consider the convergence of a pair of sequences of mappings and the convergence of their common fixed points. Several known results in metric spaces and PM- spaces may be obtained as corollaries to our results.

Let X be a nonempty set and f a mapping from $X \times X$ to $\mathcal{L}$, the collection of all distribution functions. The ordered pair (X, f) is called a PM- space if it satisfies the following conditions in which $F_{p,q}$ denotes the distribution function $f(p,q)$:

(a) $F_{p,q}(x) = 1,$ *for all* $x > 0$ *iff* $p = q$;

(b) $F_{p,q}(0) = 0,$ *for all* $p,q \in X$;

(c) $F_{p,q} = F_{q,p}$ *for all* $p,q \in X$;

(d) *If* $F_{p,q}(x) = 1$ *and* $F_{q,r}(y) = 1$ *then* $F_{p,r}(x + y) = 1.$

A Menger space is a triplet (X, f, t) consisting of a probabilistic metric space (X, f) and a T-norm t [7] satisfying the inequality

(1) $F_{p,r}(x + y) \geqslant t\{F_{p,q}(x), F_{q,r}(y)\}$

for all $p,q,r \in X$ *and for all* $x \geqslant 0, y \geqslant 0.$

For details of the topological preliminaries refer to Schweizer and Sklar [7].

2. RESULTS

Lemma. *Let* $\{u_n\}$ *be a sequence in a complete Menger space* (X, f, t) *where* t *is continuous and satisfies* $t(x, x) \geqslant x$ *for* $x \in [0, 1]$. *If there exists a* $q \in (0, 1)$ *such that*

(2) $F_{u_n,u_{n+1}}(qx) \geqslant F_{u_{n-1},u_n}(x), \quad n = 1, 2, \ldots,$

for all $x > 0$, *then* $\{u_n\}$ *converges to a point* u *in* X.

Proof. Let ε, λ be positive reals. Then for $m > n$, we have by (1),

$$F_{u_n,u_m}(\varepsilon) \geqslant t\{F_{u_n,u_{n+1}}(\varepsilon - q\varepsilon), F_{u_{n+1},u_m}(q\varepsilon)\}$$

$$\geqslant t\{F_{u_0,u_1}(\varepsilon - q\varepsilon)q^{-n}, F_{u_{n+1},u_m}(q\varepsilon)\}, \text{ by (2)}.$$

Taking $(\varepsilon - q\varepsilon)q^{-n} = a$, it follows that

$$F_{u_n,u_m}(\varepsilon) \geqslant t\{F_{u_0,u_1}(a), t(F_{u_{n+1},u_{n+2}}(q\varepsilon - q^2\varepsilon),$$

$$F_{u_{n+2},u_m}(q^2\varepsilon))\}$$

$$\geqslant t\,\{F_{u_0, u_1}(a),\ t\,(F_{u_0, u_1}(a), F_{u_{n+2}, u_m}(q^2\varepsilon))\}.$$

By the associativity of t and $t\,(x, x) \geqslant x$,

$$F_{u_n, u_m}(\varepsilon) \geqslant t\,\{F_{u_0, u_1}(a),\ F_{u_{n+2}, u_m}(q^2\,\varepsilon)\}.$$

Using the same argument repeatedly, we have

$$F_{u_n, u_m}(\varepsilon) \geqslant t\,\{F_{u_0, u_1}(a),\ F_{u_{m-1}, u_m}(q^{m-n-1}\,\varepsilon)\}$$

$$\geqslant t\,\{F_{u_0, u_1}(a), F_{u_0, u_1}(q^{-n}\,\varepsilon)\}$$

$$\geqslant t\,\{F_{u_0, u_1}(a),\ F_{u_0, u_1}(a)\}$$

$$\geqslant F_{u_0, u_1}((\varepsilon-q\varepsilon)g^{-n}).$$

Therefore, if N be so chosen that $F_{u_0, u_1}(\varepsilon-q\varepsilon)q^{-N} > 1-\lambda$, it follows that $F_{u_n, u_m}(\varepsilon) > 1-\lambda$ for all $n \geqslant N$. Hence $\{u_n\}$ is a fundamental sequence. Then, (X, f, t) being a complete Menger space, there exists a $u \in X$ such that $u_n \to u$.

Theorem 1. *Let (x, f, t) be a complete Menger space, where t is continuous and satisfies $t\,(x, x) \geqslant x$ for $x \in [0, 1]$. Let P and T be commuting mappings from X to itself such that*

(3) $P\,(X) \subseteq T\,(X)$

and

(4) $$F_{Pu,\,Pv}(qx) \geqslant \min\{F_{Pu,\,Tu}(x), F_{Pv,\,Tv}(x), F_{Pu,\,Tv}(x), F_{Tu,\,Tv}(x), F_{Pv,\,Tu}(2x)\}$$

for all $u, v, \in X$ and for all $x > 0$, where $q \in (0,1)$. If T be continuous, then P and T have a unique common fixed point.

Proof. Pick $u_0 \in X$. In view of (3), we can construct a sequence $\{u_n\}$ in X such that

(5) $Tu_n = Pu_{n-1}, \quad n = 1, 2, \ldots$

By (4),

$$F_{Tu_n,\, Tu_{n+1}}(qx) = F_{Pu_{n-1},\, Pu_n}(qx)$$

$$\geqslant \min\{F_{Tu_n,\, Tu_{n-1}}(x),\ F_{Tu_{n+1}\, Tu_n}(x),\ F_{Tu_n,\, Tu_n}(x),$$

$$F_{Tu_{n+1},\, Tu_{n-1}}(2x), F_{Tu_{n-1},\, Tu_n}(x)\}.$$

So $\quad F_{Tu_n, Tu_{n+1}}(qx) \geqslant F_{Tu_{n-1},\, Tu_n}(x)$

since $\quad F_{Tu_{n-1},\, Tu_{n+1}}(2x) \geqslant \min\{F_{Tu_{n-1},\, Tu_n}(x),\ F_{Tu_{n+1}}(x)\}.$

In view of the above lemma, $\{Tu_n\}$ converges to some point $u \in X$. By (5), $\{Pu_n\}$ also converges to u. Further,

$$F_{PTu_n,\, PTu_{n+1}}(qx) \geqslant \min\{F_{PTu_n,\, PTu_{n-1}}(x), F_{PTu_{n+1},\, PTu_n}(x),$$

$$F_{PTu_n,\, PTu_n}(x),\ F_{PTu_{n-1},\, PTu_n}(x),$$

$$F_{PTu_{n+1},\, PTu_{n-1}}(2x)\}.$$

So $\quad F_{PTu_n,\, PTu_{n+1}}(qx) \geqslant F_{PTu_{n-1},\, PTu_n}(x),$

since $\quad F_{PTu_{n-1}, PTu_{n+1}}(2x) \geqslant \min\{F_{PTu_{n-1},\, PTu_n}(x),$

$$F_{PTu_n,\, PTu_{n+1}}(x)\}.$$

Thus $\{PTu_n\}$ is also fundamental in X. Consequently

$$\lim_n F_{PTu_n,\, TTu_n}(x) = \lim_n F_{PTu_n,\, PTu_{n-1}}(x) = 1.$$

PROOF. Pick $u_0 \in X$. In view of (3), we can construct a sequence $\{u_n\}$ in X such that

(5) $\quad Tu_n = Pu_{n-1}, \quad n = 1, 2, \ldots$

By (4),

$$F_{Tu_n, Tu_{n+1}}(qx) = F_{Pu_{n-1}, Pu_n}(qx)$$

$$\geqslant min\{F_{Tu_n, Tu_{n-1}}(x), \; F_{Tu_{n+1}, Tu_n}(x), \; F_{Tu_n, Tu_n}(x),$$

$$F_{Tu_{n+1}, Tu_{n-1}}(2x), F_{Tu_{n-1}, Tu_n}(x)\}.$$

So $\quad F_{Tu_n, Tu_{n+1}}(qx) \geqslant F_{Tu_{n-1}, Tu_n}(x),$

since $\quad F_{Tu_{n-1}, Tu_{n+1}}(2x) \geqslant min\{F_{Tu_{n-1}, Tu_n}(x), \; F_{Tu_{n+1}}(x)\}.$

In view of the above lemma, $\{Tu_n\}$ converges to some point $u \in X$. By (5), $\{Pu_n\}$ also converges to u. Further,

$$F_{PTu_n, PTu_{n+1}}(qx) \geqslant min\{F_{PTu_n, PTu_{n-1}}(x),$$

$$F_{PTu_{n+1}, PTu_n}(x),$$

$$F_{PTu_n, PTu_n}(x), \; F_{PTu_{n-1}, PTu_n}(x),$$

$$F_{PTu_{n+1}, PTu_{n-1}}(2x)\}.$$

So $\quad F_{PTu_n, PTu_{n+1}}(qx) \geqslant F_{PTu_{n-1}, PTu_n}(x),$

since $\quad F_{PTu_{n-1}, PTu_{n+1}}(2x) \geqslant min\{F_{PTu_{n-1}, PTu}(x),$

$$F_{PTu_n, PTu_{n+1}}(x)\}.$$

Thus $\{PTu_n\}$ is also fundamental in X. Consequently

$$lim_n F_{PTu_n, TTu_n}(x) = lim_n F_{PTu_n, PTu_{n-1}}(x) = 1.$$

By the continuity of T, $PTu_n = TPu_n = TTu_{n+1} \to Tu$. Further,

$$F_{PTu_n,\ Pu}(qx) \geqslant min\,\{F_{PTu_n,\ TTu_n}(x),\ F_{Pu,\ Tu}(x),$$
$$F_{PTu_n,\ Tu}(x),\ F_{TTu_n,\ Tu}(x),$$
$$F_{Pu,\ TTu_n}(2x)\}$$

in the limit leads to $F_{Tu,\ Pu}(qx) \geqslant min\ \{ 1,\ F_{Pu, Tu}(x),$
$$1,\ 1,\ 1\},$$

whence

(6) $\quad Tu = Pu$

Consequently,

(7) $\quad T\ (Tu) = T\ (Pu) = P\ (Tu) = P\ (Pu)$.

By (4),

$$F_{Pu,\ P(Pu)}(qx) \geqslant min\ \{F_{Pu,\ Tu}(x),\ F_{P\ (Pu),\ T\ (Pu)}(x),$$
$$F_{Pu,\ T\ (Pu)}(x),\ F_{Tu,\ T\ (Pu)}(x),$$
$$F_{P\ (Pu),\ Tu}(2x)\}.$$

Using (6) and (7) we get

$$F_{Pu,\ P\ (Pu)}(qx) \geqslant min\,\{1,\ 1,\ F_{Pu,\ P\ (Pu)}(x),\ F_{Pu,\ P\ (Pu)}(x),$$
$$F_{Pu,\ P\ (Pu)}(2x)\}$$
$$= F_{Pu,\ P\ (Pu)}(x),$$

yielding $P\ (Pu) = Pu$. Thus Pu is a common fixed point of P and T.

To prove the unicity of the common fixed point of P and T, let P and T have two common fixed points y and z. Then

$$F_{y,\ z}(qx) = F_{Py,\ Pz}(qx)$$

$$\geqslant min\ \{ F_{Py,\ Ty}(x),\ F_{Pz,\ Tz}(x),\ F_{Py,\ Tz}(x),$$

$$F_{Ty,\ Tz}(x),\ F_{Pz,\ Ty}(2x)\ \}$$

$$= min\ \{ 1, 1, F_{y,\ z}(x), F_{y,\ z}(x), F_{z,\ y}(2x)\ \}$$

$$= F_{y,\ z}(x),$$

proving $y = z$.

Remark 1. The following result is the metric analogue of Theorem 1:

Let two commuting mappings P and T from a complete metric space (X, d) to itself satisfy

$$P(x) \subseteq T(x)$$

and

$$d(Pu, Pv) \leqslant q\ max\ \{ d(Pu, Tu),\ d(Pv, Tv),\ d(Pu, Tv),$$

$$d(Tu, Tv),\ \tfrac{1}{2}d(Pv, Tu)\ \}$$

for all $u, v \in X$ and $q < 1$. If T be continuous, then P and T have a unique common fixed point.

This result generalizes several fixed point theorems in metric spaces; *e. g.*, see [4], [8] and [10]. If T be an identity mapping in (4) then we obtain a slightly superior form of 'generalized contraction' studied by Ćirić [1].

Definiton 1 [5]. A sequence of mappings $T_n : X \to X$ on a PM-space X converges uniformly to a mapping $T : X \to X$ iff for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $K = K(\varepsilon, \lambda)$ such that

$$F_{Tp,\ T_n p}(\varepsilon) > 1-\lambda$$

for every $p \in X$ and all $n \geqslant K$.

Theorem 2. *Let P, T, P_n and T_n ($n = 1, 2, \ldots$) be mappings from a Menger space (X, f, t) to itself, where t is continuous and satisfies $t(x,x) \geqslant x$, for $x \in [0, 1]$. Let P_n and T_n have at least one common fixed point z_n ($n = 1, 2, 3, \ldots$).*

Further, let $\{P_n\}$ and $\{T_n\}$ converge uniformly to P and T, respectively. If z be a common fixed point of P and T and there exists a positive number $q < 1$ such that for every $u, v \in x$

$$F_{Pu,\, Pv}(qx) \geqslant \min\{F_{Pu,\, Tu}(x), F_{Pv,\, Tv}(x), F_{Pu,\, Tv}(x),$$

$$F_{Tu,\, Tv}(x), F_{Pv,\, Tu}(2x)\},$$

for all $x > 0$, then $z_n \to z$,

Proof. Since $\{P_n\}$ and $\{T_n\}$ converge uniformly to P and T, there exist $\varepsilon > 0$, $\lambda > 0$ such that

$$F_{P_n z_n,\, Pz_n}\left(\frac{1-q}{4}\varepsilon\right) > 1-\lambda \text{ and } F_{T_n z_n,\, Tz_n}\left(\frac{1-q}{4}\varepsilon\right) > 1-\lambda$$

for all $n \geqslant K(\varepsilon, \lambda) = K$.

For any n,

$$F_{z_n,\, z}(\varepsilon) = F_{P_n z_n,\, Pz}(\varepsilon)$$

$$= F_{P_n z_n,\, Pz}\left(\frac{1-q}{2}\varepsilon + \frac{1+q}{2}\varepsilon\right)$$

$$\geqslant \min\left\{F_{P_n z_n,\, Pz_n}\left(\frac{1-q}{2}\varepsilon\right), F_{Pz_n,\, Pz}\left(\frac{1+q}{2}\varepsilon\right)\right\}$$

$$(8) \geqslant \min\left\{F_{P_n z_n,\, Pz_n}\left(\frac{1-q}{4}\varepsilon\right), F_{Pz_n,\, Pz}\left(\frac{1+q}{2}\varepsilon\right)\right\}.$$

Noting that $Pz = z = Tz$ and $P_n z_n = z_n = T_n z_n$, we have

$$F_{Pz_n, Pz}\left(\frac{1+q}{2}\varepsilon\right) = F_{Pz_n, Pz}\left(\frac{1+q}{2q}\varepsilon\right)$$

$$\geqslant \min\left\{F_{Pz_n, Tz_n}\left(\frac{1+q}{2q}\varepsilon\right), F_{Pz, Tz}\left(\frac{1+q}{2q}\varepsilon\right),\right.$$

$$F_{Pz_n, Tz}\left(\frac{1+q}{2q}\varepsilon\right),$$

$$\left.F_{Tz_n, Tz}\left(\frac{1+q}{2q}\varepsilon\right), F_{Pz, Tz_n}\left(\frac{1+q}{q}\varepsilon\right)\right\}$$

$$= \min\left\{F_{Pz_n, Tz_n}\left(\frac{1+q}{4q}\varepsilon + \frac{1+q}{4q}\varepsilon\right),\right.$$

$$\left.F_{Tz_n, z}\left(\frac{1-q}{2q}\varepsilon + \frac{1+2q}{2q}\varepsilon\right)\right\}$$

$$\geqslant \min\left\{F_{Pz_n, P_nz_n}\left(\frac{1+q}{4q}\varepsilon\right), F_{T_nz_n, Tz_n}\left(\frac{1+q}{4q}\varepsilon\right),\right.$$

$$\left.F_{Tz_n, T_nz_n}\left(\frac{1-q}{2q}\varepsilon\right), F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)\right\}$$

$$\geqslant \min\left\{F_{Pz_n, P_nz_n}\left(\frac{1-q}{4}\varepsilon\right) F_{T_nz_n, Tz_n}\left(\frac{1-q}{4}\varepsilon\right),\right.$$

$$\left.F_{Tz_n, T_nz_n}\left(\frac{1-q}{4}\varepsilon\right), F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)\right\}.$$

So from (8),

$$F_{z_n, z}(\varepsilon) \geqslant \min\left\{F_{Pz_n, P_nz_n}\left(\frac{1-q}{4}\varepsilon\right),\right.$$

$$\left.F_{T_nz_n, Tz_n}\left(\frac{1-q}{4}\varepsilon\right), F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)\right\}$$

$$= \min\left\{F_{Pz_n, P_nz_n}\left(\frac{1-q}{4}\varepsilon\right), F_{T_nz_n, Tz_n}\left(\frac{1-q}{4}\varepsilon\right)\right\},$$

since $F_{z_n, z}(\varepsilon) \leqslant F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)$.

Hence $F_{z_n, z}(\varepsilon) > 1-\lambda$ *for all* $n \geqslant K$.

Thus $z_n \to z$.

Defiention 2, [5]. A sequence of mappings $T_n : X \to X$ on a PM–space X converges pointwise to a mapping $T : X \to X$ iff for every $u \in X$, $\{T_n(u)\}$ converges to $T(u)$.

Theorem 3. *Let* (X, f, t) *be a Menger space, where* t *is continuous and* $t(x, x) \geqslant x$ *for* $x \in [0, 1]$. *Let* P_n *and* T_n *be mappings from* X *to itself with at least one common fixed point* z_n *for each* $n = 1, 2, 3, \ldots$. *Suppose* q (< 1) *be a positive number such that*

$$(9) \quad F_{P_n u, P_n v}(qx) \geqslant \min\{F_{P_n u, T_n u}(x),\ F_{P_n v, T_n v}(x),\ F_{P_n u, T_n v}(x),\ F_{T_n u, T_n v}(x),\ F_{P_n v, T_n u}(2x)\}$$

for all $u, v, \in X$ *and* $n = 1, 2, \ldots$. *If the sequences* $\{P_n\}$ *and* $\{T_n\}$ *converge, respectively, pointwise to* $P, T : X \to X$ *with common fixed point* z, *then* $z_n \to z$.

PROOF. Since $\{P_n\}$ and $\{T_n\}$ converge pointwise to P and T, there exist positive numbers ε, λ and a positive integer $K = K(\varepsilon, \lambda)$ corresponding to z such that

$$F_{P_n z, Pz}\left(\frac{1-q}{4}\varepsilon\right) > 1-\lambda \text{ and } F_{T_n z, Tz}\left(\frac{1-q}{4}\varepsilon\right) > 1-\lambda$$

for all $n \geqslant K$.

For any n,

$$F_{z_n, z}(\varepsilon) = F_{P_n z_n, Pz}(\varepsilon)$$

$$= F_{P_n z_n,\, Pz}\left(\frac{1+q}{2}\varepsilon + \frac{1-q}{2}\varepsilon\right)\Big\}$$

$$\geqslant \min\left\{ F_{P_n z_n,\, P_n z}\left(\frac{1+q}{2}\varepsilon\right),\ F_{P_n z,\, Pz}\left(\frac{1-q}{2}\varepsilon\right)\right\}$$

(10) $$\geqslant \min\left\{ F_{P_n z_n,\, P_n z}\left(\frac{1+q}{2}\varepsilon\right), F_{P_n z,\, Pz}\left(\frac{1-q}{4}\varepsilon\right)\right).$$

By (9),

$$F_{P_n z_n,\, P_n z}\left(\frac{1+q}{2}\varepsilon\right) = F_{P_n z_n,\, P_n z}\left(q\,\frac{1+q}{2q}\varepsilon\right)$$

$$\geqslant \min\{F_{z_n,\, z_n}\left(\frac{1+q}{2q}\varepsilon\right), F_{P_n z,\, T_n z}\left(\frac{1+q}{2q}\varepsilon\right),$$

$$F_{z_n,\, T_n z}\left(\frac{1+q}{2q}\varepsilon\right), F_{z_n,\, T_n z}\left(\frac{1+q}{2q}\varepsilon\right).$$

$$F_{P_n z,\, z_n}\left(\frac{1+q}{q}\varepsilon\right)\}$$

$$= \min\{F_{P_n z,\, T_n z}\left(\frac{1+q}{4q}\varepsilon + \frac{1+q}{4q}\varepsilon\right),$$

$$F_{z_n,\, T_n z}\left(\frac{1+2q}{2q}\varepsilon + \frac{1-q}{2q}\varepsilon\right)\}$$

$$\geqslant \min\{F_{P_n z,\, Pz}\left(\frac{1+q}{4q}\varepsilon\right), F_{Tz,\, T_n z}\left(\frac{1+q}{4q}\varepsilon\right),$$

$$F_{P_n z_n,\, Pz}\left(\frac{1+2q}{2q}\varepsilon\right), F_{Tz,\, T_n z}\left(\frac{1-q}{2q}\varepsilon\right)\}$$

$$\geqslant \min\{ F_{P_n z,\, Pz}\left(\frac{1-q}{4}\varepsilon\right), F_{Tz,\, T_n z}\left(\frac{1-q}{4}\varepsilon\right),$$

$$F_{z_n,\, z}\left(\frac{1+2q}{2q}\varepsilon\right)\},$$

So from (10),

$$F_{z_n, z}(\varepsilon) \geqslant min\left\{ F_{P_n z, Pz}\left(\frac{1-q}{4}\varepsilon\right), F_{Pz, T_n z}\left(\frac{1-q}{4}\varepsilon\right),\right.$$

$$\left. F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)\right\}$$

$$\geqslant min\left\{ F_{P_n z, Pz}\left(\frac{1-q}{4}\varepsilon\right), F_{Tz, T_n z}\left(\frac{1-q}{4}\varepsilon\right)\right\},$$

since $F_{z_n, z}(\varepsilon) \leqslant F_{z_n, z}\left(\frac{1+2q}{2q}\varepsilon\right)$.

Hence $F_{z_n, z}(\varepsilon) > 1-\lambda$ for $n \geqslant K$.

Thus $z_n \to z$.

REMARK 2. A slightly improved version of a result of Ćirić [1,Th.2] is obtained by taking $Tx = x \ \forall x \in X$ in Theorem 2 of this paper. The results of [9] are particular cases of the metric analogues of Theorem 2 and Theorem 3 of this paper.

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