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GENERALIZED ALMOST PERIODIC FUNCTION DEPENDING ON PARAMETERS

By

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The concept of almost periodic functions depending on parameters has been introduced by C. Corduneanu [1]. Later on, it has been generalized to weakly almost periodic functions by Sharma & Reddy [2]. The aim of this paper is to define certain generalized almost periodic functions uniformly depending on parameters on $\Omega \times R$ with values in Banach space.

Let Ω be a set of n-dimensional complex space and $Z=(z_1,\ldots,z_n)$ be the point with coordinates $z_1,\ldots,z_n$. We shall assume that any function appearing in theorems or in definitions is continuous on the set $\Omega \times R$, where R is the real line.

Definition 1. A function $f(Z, x)$ defined on $\Omega \times R$, with values in Banach space X, is called a generalized almost periodic function uniformly with respect to $Z \in \Omega$, if, to any $\varepsilon > 0$, there corresponds a number $l(\varepsilon)$ such that any interval of length $l(\varepsilon)$ of the real line contains at least one point with abscissa T for which

$$\| f(Z, x+T) - f(Z, x) \| < \varepsilon.\ Z \in \Omega, x \in R.$$

The number T is called an ε - translation number of $f(Z, x)$. The uniform dependence on parameters follows from the fact that $l(\varepsilon)$

and T are independent of Z.

Theorem 1. Let $f(Z, x)$ be a generalized almost periodic function uniformly depending on parameters. If c is a complex number and a is real, then $c.\ f(Z, x)$; $f(Z, x+a)$; $\bar{f}(Z, x)$ and $|\ f(Z, x)\ |$ are generalized almost periodic functions uniformly depending on parameters. If $\|\ f(Z, x)\ \| \geqq m > 0$, then the function $1/f(Z, x)$ is almost periodic uniformly depending on parameters.

Proof. The proof of first four assertion made in the theorem follows immediately from the definition and the proof of the last assertion follows from the following inequality:

$$\left\| \frac{1}{f(Z, x+T)} - \frac{1}{f(Z, x)} \right\| \leqq \frac{1}{m^2} \left\| f(Z, x+T) - f(Z, x) \right\|.$$

This inequality shows that any $m^2\ \varepsilon$ – translation number of $f(Z, x)$ is an ε – translation number for $1/f(Z, x)$.

Theorem 2. If Ω is a closed bounded set, then the function $f(Z, \bar{x})$, generalized almost periodic uniformly depending on parameters, is bounded on $\Omega \times R$.

Proof. Let $\varepsilon = 1$ and $l = l\ (1)$ be the corresponding length described in the definition of generalized almost periodic functions uniformly with respect to Z. The function $f(Z, x)$ is bounded with respect to the norm (in X) on the set $\Omega \times [0, l)$ since it is continuous. Set $m = \text{Sup}\ \|\ f(Z, x)\ \|$, $Z \in \Omega$, $x \in [0, l]$. Consider new any real number x and a 1-translation number T of $f(Z, x)$ which belongs to the interval $[-x, -x + l]$. It follows that $0 \leqq x + T \leqq l$. But

$$\|\ f(Z, x)\ \| \leqq \|\ f(Z, x) - f(Z, x + T)\ \| + \|\ f(Z, x + T)\ \|$$

$$< 1 + m = M \text{ (say)}.$$

This proves the theorem.

Corollary. If Ω is a closed and bounded set and $f(Z, x)$ is a generalized almost periodic function uniformly with respect to $Z \in \Omega$, then $f^2(Z, x)$ is generalized almost periodic uniformly with respect to $Z \in \Omega$.

Theorem 3. If Ω is a closed and bounded set and $f(Z, x)$ is generalized almost periodic function uniformly with respect to $Z \in \Omega$, then $f(Z, x)$ is uniformly continuous on the set $\Omega X R$.

Proof. Let ε be a positive number, and $l=l\ (\varepsilon/3)$ the length associated with $\varepsilon/3$ according to the definition of the generalized almost periodic functions. The function $f(Z, x)$ is continuous on the set $\Omega \times [-1, 1 + l]$ and is therefore uniformly continuous. Thus we can determine a positive number $\delta = \delta\ (\varepsilon/3) < 1$, such that

$$\| f(Z_2, y_2) - f(Z_1, y_1) \| < \varepsilon/3,$$

provided that

$$| Z_2, - Z_1 | < \delta, | y_2 - y_1 | < \delta,$$

$$(Z_2, y_2), (Z_1, y_1) \in \Omega \times [-1, 1 + l].$$

Consider two points (Z_2, X_2) and (Z_1, X_1) of $\Omega \times R$ such that

(1) $\quad | Z_2 - Z_1 | < \delta, | x_2 - x_1 | < \delta.$

If T is an $\varepsilon/3$- translation number of $f(Z, x)$ belonging to the interval $[-x_1, -x_1 + l]$, then we have $0 \leqq x_1 + T \leqq 1, -1 \leqq x_2 + T \leqq 1+l.$ The last double inequality follows from the first one, according to (1) and the fact that $\delta < 1$. Therefore,

$$(Z_2, x_2 + T), (Z_1, x_1 + T) \in \Omega \times [-1, 1 + l]$$

and consequently

$$\| f(Z_2, x_2) - f(Z_1, x_1) \|$$

$$\leqq \| f(Z_2, x_2) - f(Z_2, x_2 + T) \|$$

$$+ \| f(Z_2, x_2 + T) - f(Z_1, x_1 + T) \|$$

$$+ \| f(Z_1, x_1 + T) - f(Z_1, x_1) \|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

provided that

$$| Z_2 - Z_1 | < \delta, \ | x_2 - x_1 | < \delta.$$

Theorem 4. If a sequence $f_n(Z, x)$ of generalized almost periodic functions uniformly with respect to $Z \in \Omega$, is uniformly convergent on $\Omega \times R$ to $f(Z, x)$, then $f(Z, x)$ is a generalized almost periodic function uniformly with respect to Z.

Proof. Consider a positive number ε, and let n be a sufficiently large number so that

$$\| f(Z, x) - f_n(Z, x) \| < \varepsilon/3, \ (Z, x) \in \Omega \times R.$$

Let T be an $\varepsilon/3$— translation number of $f_n(Z, x)$, we have

$$\| f(Z, x + T) - f(Z, x) \|$$

$$\leqq \| f(Z, x + T) - f_n(Z, x + T) \|$$

$$+ \| f_n(Z, x + T) - f_n(Z, x) \|$$

$$+ \| f_n(Z, x) - f(Z, x) \|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This proves the theorem.

Definition 2. A continuous function $f(Z, x)$: $\Omega \times R \rightarrow X$ is called normal if and only if any set of translates of $f(Z, x)$ has a subsequence uniformly convergent on $\Omega \times R$ in the sense of norm.

We shall now show that the set of generalized almost periodic functions uniformly with respect to Z concides with set of normal functions.

Theorem 5. The function $f(Z, x)$: $\Omega \times R \rightarrow X$ is a generalized almost periodic function uniformly with respect to $Z \in \Omega$, where Ω is a bounded and closed set, if and only if the family $\{f(Z, x + h)\}$ is normal on $\Omega \times R$.

Proof. Necessity Let $f(Z, x)$ be generalized almost periodic function uniformly depending on parameters. Consider a sequence of real numbers $\{h_k\}$. What we are proving is that from the sequence $\{ f(Z, x + h_k) \}$ one can extract a subsequence which converges uniformly on the set $\Omega \times R$. Let (Z_k, x_k) be a countable set of points every where dense in $\Omega \times R$. Since $f(Z, x)$ is bounded, then from the numerical sequence $\{ f(Z_1, x_1 + h_k) \}$ we can choose a convergent subsequence $\{ (Z_1, x_1 + h_{1k}) \}$. Then from the sequence $\{f(Z_2, x_2 + h_{1k})\}$ we can choose a convergent subsequence $\{ f(Z_2, x_2 + h_{2k}) \}$ and so on. Consider the diagonal sequence $\{ f(Z, x + h_{kk}) \}$, since the numerical sequence $\{h_{kk}\}$ is, except for a finite number of terms, a subsequence of any sequence $\{h_{mk}\}$, this means that $\{ f(Z, x + h_{kk}) \}$ converges at all the points (Z_k, x_k). Let us show that $\{ f(Z, x + h_{kk}) \}$ is uniformly convergent on $\Omega \times R$. Let (Z_0, x_0) be any point from $\Omega \times R$, an ε be a positive number. Consider the numbers $l = l\ (\varepsilon/5)$ and $\delta = \delta\ (\varepsilon/5)$ corresponding to $\varepsilon/5$, by the definition of almost periodic functions uniformly with respect to $Z \in \Omega$ and according to Theorem 3, the set

$\Omega \times [0, l]$ is bounded and closed. Therefore, there exists certain points $(Z_1, y_1), (Z_2, y_2), \ldots, (Z_p, y_p)$ which belong to this set, such that for any point $(Z, y) \in \Omega \times [0, l]$ there exists a sub script i, $1 \leqq i \leqq p$ for which $| Z - Z_i | < \delta$, $| y - y_i | < \delta$. On the other hand, there exists an integer $N(\varepsilon)$ depending only on ε, such that

(1) $\| f(Z_i, y_i + h_{rr}) - f(Z_i, y_i + h_{ss}) \| < \varepsilon/5, \quad i=1, 2, \ldots, p,$

whenever $r, s \geqq N(\varepsilon)$. Consider an $(\varepsilon/5)$ – translation number T of $f(Z, x)$ belonging to the interval $[-x_0, -x_0 + l]$. Then the number $y_0 = x_0 + T$ belongs to the interval $[0, l]$ and consequently, $(Z_0, y_0) \in \Omega \times [0, l]$. There exists an i, $1 \leqq i \leqq p$, for wich $| Z_i - Z_0 | < \delta$, $| y_i - y_0 | < \delta$. This means that

(2) $\| f(Z_i, y_i + h_{kk}) - f(Z_0, y_0 + h_{kk}) \| < \varepsilon/5, \; k = 1, 2 \ldots$

From (1) and (2) one finds

$$\| f(Z_0, x_0 + h_{rr}) - f(Z_0, x_0 + h_{ss}) \|$$
$$\leqslant \| f(Z_0, x_0 + h_{rr}) - f(Z_0, y_0 + h_{rr}) \|$$
$$+ \| f(Z_0, y_0 + h_{rr}) - f(Z_i, y_i + h_{rr}) \|$$
$$+ \| f(Z_i, y_i + h_{rr}) - f(Z_i, y_i + h_{ss}) \|$$
$$+ \| f(Z_i, y_i + h_{ss}) - f(Z_0, y_0 + h_{ss}) \|$$
$$+ \| f(Z_0, y_0 + h_{ss}) - f(Z_0, x_0 + h_{ss}) \|$$
$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon,$$

if $r, s \geqq N$. The five distances are smaller that $\varepsilon/5$ for the following reasons: the first and the last are less then $\varepsilon/5$ due to the almost periodicity of $f(Z, x)$, the second and the fourth due to the uniform continuity of $f(Z, x)$, and the third due to the condition (1). Since ε is arbitrary and N depends only on ε, the condition is necessary.

Sufficiency Suppose that the condition is not sufficient. Then there exists atleast an $\varepsilon > 0$, such that for any $l > 0$, we can determine an interval of length l which contains no ε - translation number of $f(Z, x)$. Consider an arbitrary number h_1 and an interval (a_1, b_1) of the real line of the length greater then $2 \mid h_1 \mid$ which does not contain any ε - translation number of $f(Z, x)$. If we write $h_2 = 1/2 (a_1 + b_1)$, then $h_2 - h_1 \in (a_1, b_1)$ and consequently, $h_2 - h_1$ can not be an ε - translation number of $f(Z, x)$. Then there exists an interval (a_2, b_2) of the real line of the length greater then $2 \mid h_1 \mid + \mid h_2 \mid)$ which does not contain any ε - translation numder of $f(Z, x)$. Letting $h_3 = \frac{1}{2} (a_2 + b_2)$, we get that $h_3 - h_2, h_3 - h_1 \in (a_2, b_2)$ and thus $h_3 - h_2, h_3 - h_1$ are not ε - translation number of the function $f(Z, x)$. Therefore, for any i and j

$$\text{Sup} \| f(Z, x + h_i) - f(Z, x + h_j) \|$$

$$= \text{Sup} \| f(Z, x + h_i - h_j) - f(Z, x) \| \geqq \varepsilon,$$

which proves that the sequence $\{f(x + h_n)\}$ can not contain any uniformly convergent subsequence. This contradicts the fact that $f(Z, x)$ is normal. Hence the condition is sufficient.

Theorem 6. If Ω is a bounded closed set, and $f(Z, x)$ and $g(Z, x)$ are generalized almost periodic functions uniformly depending on parameters, then $f(Z, x) + g(Z, x)$ and $f(Z, x). (gZ, x)$ are generalized almost, periodic functions. If $\| g(Z, x) \| \geqq m > 0$, then $f(Z, x)/g(Z, x)$, is a generalized almost periodic function uniformly depending on parameters.

Proof. To prove the almost periodicity of the sum it is sufficient to show that from each sequence $\{ f(Z, x + h_k) + g(Z, x + h_k) \}$ one can extract a subsequence converging unifoamly on $\Omega \times R$. We shall extract from $\{h_k\}$ a subsequence $\{h_{1k}\}$ such that the sequence $\{ g(Z, x + h_{1k}) \}$ will be uniformly convergent on $\Omega \times R$. Then, from $\{h_{1k}\}$ we extract a subsequence $\{h_{2k}\}$ such that the sequence $\{f(Z, x + h_{2k})\}$ will be uniformly convergent on $\Omega \times R$. The sequence $\{ f(Z, x + h_{2k}) + g(Z, x + h_{2k}) \}$ is uniformly convergent on $\Omega \times R$. This proves the assertion made in the theorem regarding the sum.

We have, $f(Z, x). g(Z, x) = \frac{1}{4} [(f(Z, x) + g(Z, x))^2$

$- (f(Z, x) - g(Z, x))^2].$

Using this equality and the above fact and the Corollary one can easily prove the almost periodicity of the product $f(Z, x). g(Z, x)$. Agiain the function $f(Z, x)/g(Z, x)$ is generalized almost periodic according to Theorem 1, and the fact that the product of generalized almost periodic functions is also generalized almost periodic depending on parameters.

Definition 3. The fuuction $Z = Z(x)$ is an almost periodic, if all the components $Z_i = Z_i(x)$, $i = 1, 2, \ldots, n$ are almost periodic.

Theorem 7. If $f(Z, x)$ is generalized almost periodic uniformly depending on parameters, if $Z = Z(x)$ is generalized almost periodic and $Z(x) \in \Omega$ when $x \in R$, then $f(Z(x), x)$ is a generalized almost periodic function uniformly depending on parameters.

Proof. Without loss of generality, we can assume that Ω is bounded and closed due to the fact that the set of point $Z(x), x \in R$ is bounded. Let $\{h_k\}$ be an arbitrary sequence of real numbers. We must show that the sequence $\{ f(Z(x + h_k), x + h_k) \}$ contains a subsequence converging uniformly on the real line. Since $\{Z(x + h) \}$ is normal, it follows from Theorem 6 that there exists a subscquence $\{h'_k\}$ of the sequence $\{h_k\}$ such that $\{Z(x + h'_k)\}$ is uniformly convergent on the real line, and $\{f(Z, x + h'_k)\}$ is uniformly convergent on $\Omega \times R$. Let $\varepsilon > 0$ be an arbitrary number. Then there exists a number $\delta(\varepsilon) > 0$ such that

$$\| f(Z_1, x) - f(Z_2, x) \| < \varepsilon/2,$$

provided that

$$| Z_1 - Z_2 | < \delta, Z_1, Z_2 \in \Omega, x \in R.$$

But now we have,

$$\| f(Z, x + h'_r) - f(Z, x + h'_s) \| < \varepsilon/2, \text{ if } r, s \geqq N_1(\varepsilon),$$

and also

$$| Z(x + h'_r) - Z(x + h'_s) | < \delta, \text{ for } r\ s \geqq N_2(\varepsilon).$$

Set $N = \max\{N_1, N_2\}$. It follows from the above equations that

$$\| f(Z(x + h'_r), x + h'_r) - f(Z(x + h'_s), x + h'_s) \|$$

$$\leqq \| f(Z(x + h'_r), x + h'_r) - f(Z(x + h'_r), x + h'_s) \|$$

$$+ \| f(Z(x + h'_r), x + h'_s) - f(Z(x + h'_s), x + h'_s) \|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ for } r, s \geqq N(\varepsilon).$$

This complets the proof.

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