Jñānābha, Vol. 13, 1983

A NEW GENERATING FUNCTION FOR THE ASSOCIATED LAGUERRE POLYNOMIALS AND RESULTING EXPANSIONS

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(*Received: July 21, 1983*)

If the function

$$V(x, s, t) = exp\,(s+t-xt/s) \tag{1}$$

is expanded as a double series of powers of s and t, we have

$$V = \sum_{r=0}^{\infty} (-x)^r/r! \sum_{j=0}^{\infty} s^{j-r}/j! \sum_{k=0}^{\infty} t^{k+r}/k!. \tag{2}$$

Replace $j-r$ and $k+r$, respectively, by m and n, when after re-arrangement, justified by the absolute convergence of the above series, it follows that

$$exp\,(s+t-xt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} s^m t^n F_n^m(x), \tag{3}$$

where

$$F_n^m(x) = {}_1F_1\,(-n;\, m+1;\, x)/(m!n!) = L_n^m(x)/(m+n)!, \tag{4}$$

$L_n^m(x)$ being the associated Laguerre, a function of frequent occurrence in quantum mechanics and other branches of applied mathematics. See Schiff [1], page 84, for example.

The double generating function (3) is rather curious in that its right-hand member is partly bilateral and partly unilateral.

If this expression is multiplied by $s^p t^q$, where p and q are integers, it follows by double integration about small circuits about the origin in the s– and t– planes, that

$$L_n^m(x) = \frac{(m+n)!}{(2\pi i)^2} \int^{(0+)} \int^{(0+)} s^{-m-1}\ t^{-n-1}$$

$$exp\ (s+t-xt/s) ds\ dt. \tag{5}$$

See Whittaker and Watson [2], page 353.

This note is concluded by indicating formulae whereby integral powers of x may be expanded as double series of associated Laguerre polynomials.

Firstly, we note that

$$V(x, x/2, x/2) = 1, \tag{6}$$

and that

$$\frac{\partial^r V}{\partial t^r} = (1-x/s)^r\ V = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} (-1)^r\ (-n)_r\ s^m\ t^{n-r}\ F_n^m(x). \tag{7}$$

Put $s = t = x/2$ into this result, and obtain the expansion

$$x^r = 2^r \sum_{\substack{m=-\infty \\ n=0}}^{\infty} (-n)_r\ (x/2)^{m+n}\ L_n^m(x)/(m+n)!, \tag{8}$$

for $r = 0, 1, 2, \ldots$.

A second set of expansions also exists which may be obtained in a similar manner by taking successive partial derivatives with respect to s of the generating relation (3), again letting $s=t=x/2$ in the various results. The general fomula of these expansions has not, so far been obtained, and the expansions of the powers of x up to x^4 now follow:

$$3x/2 = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} m(x/2)^{m+n} L_n^m(x)/(m+n)!,$$

$$9x^2/4 - 2x = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} m(m-1)\ (x/2)^{m+n} L_n^m(x)/(m+n)!,$$

$$27x^3/8 - 9x^2 + 6x = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} m(m-1)\ (m-2)\ (x/2)^{m+n} L_r^m(x)/(m+n)!$$

and

$$81x^4/16 - 9x^3 + 48x^2 - 24x = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} m(m-1)\ (m-2)\ (m-3)\ (x/2)^{m+n} L_n^m(x)/(m+n)!. \qquad (9)$$

REFERENCES

[1] L. Schiff, Quantum Mechanics, McGraw–Hill, New York, 1955.

[2] E. T. Whittaker and G. N. Watson, Modern Analysis, Fourth ed., Cambridge Univ. Press, Cambridge, 1952.