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ON A NON-LINEAR DIFFERENTIAL EQUATION

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ABSTRACT

In an attempt to generalize and unify several results established earlier by Garde [4], Saxena and Kushwaha [8], Srivastava and Singh [9], Munot and Mathur [6], and many others, we have applied the linear orthonormal polynomial approximation to solve quite a general non-linear differential equation

$$\ddot{x} + f(x) = NF(t),$$

where $\ddot{x} \equiv \frac{d^2x}{dt^2}$. In particular, Jacobi polynomials have been used to obtain the solutions of the above equation and the non-linear differential equations associated with the multivariable H-function defined by Srivastava and Pandy [11] and a generalized hypergeometric function.

1. INTRODUCION

Many non-linear differential equations frequently occur in Physics and Engineering. Various authors have attempted to solve the non-linear differential equations of the type

$$\ddot{x} + f(x) = 0,$$

by considering different $f(x)$.

In 1959, considering such an equation, Denmann [1] obtained an amplitude dependent approximation to frequency of the simple pendulum whose amplitude of motion is A,by the application of Tchebicheff polynomial approximation to $\sin\theta$ in the interval $(-A, A)$. Later, in 1964, Denmann and Howard [2] and Denmann and Liu [3] applied ultraspherical polynomials to the same problem. Garde [4], Saxena and Kushwaha([7],[8]) and Srivastava and Singh [9] have applied Jacobi polynomials to obtain amplitude dependent approximate solutions of certain non-linear differential equations.

Recently, we have applied the general orthogonal polynomials to obtain the amplitude dependent linear approximate solution of the differential equation (cf. [6])

$$\ddot{x} + w_0^2 x^r \, {}_pF_q \, [\, (a_p); \, (b_q); \, cx^s] = 0 \qquad \ldots(1.1)$$

where r and s are positive integers, subject to the initial conditions $x = A$, $\dot{x} = 0$ when $t = 0$, A being the amplitude of the motion.

In the present paper we have made an attempt to make use of a system of general orthonormal polyomials $\{\phi_n(x)\}$ to give an ampli tude dependent linear approximate solution of a general differential equation

$$\ddot{x} + f(x) = NF(t) \qquad \ldots(1.2)$$

The initial conditions of motion are $x = A$, $\dot{x} = 0$ when $t = 0$, A being the amplitude of the motion under which the solution of the proposed problem will be obtained.

The importance of this result lies in the fact that the general orthonormal polynomials can be specialized to any of the known orthogonal polynomials to provide a special solution of the differential eqnation(1.2).In particular Jacobi polynomials have been used to obtain the solution of(1.2)and the solutions of non-linear differential equations associated with the multivariable function defined by Srivastava and Panda [11] and the generalized hypergeometric function. The results obtained generalize the known results established earlier by Garde [4], Saxena and Kushwaha [8], Srivastava and Singh [9] and the authors [6].

2. Orthonormal polynomials and linear approximation

Let $\{\phi_n(x)\}$ be a set of orthonormal polynomials in the interval (a,b) with weight function $w(x) > 0$. Then it can be easily seen that the set of polynomials $\left\{\frac{1}{\sqrt{A}}\phi_n\left(\frac{x}{A}\right)\right\}$ is also orthonormal in the interval (aA, bB) with weight function $w\left(\frac{x}{A}\right)$.

Let L_w^2 be the class of functions $f(x)$ for which

$$\int_a^b f^2(x)\, w(x)\, dx < \infty \qquad \ldots(2.1)$$

and the orthonormal system $\left\{\frac{1}{\sqrt{A}}\phi_n\left(\frac{x}{A}\right)\right\}$ belongs to L_w^2.

Then the sypstem is closed and for every $f \varepsilon L_w^2$, $\sum_n a_n^2$ is convergent and the sequence $f_n(x)$ expressed by

$$f_n(x) = \sum_1^n a_m\, \phi_m\left(\frac{x}{A}\right) \qquad \ldots(2.2)$$

converges in mean to f, where a_n is given by

$$a_n = \int_a^b f(Ax)\, \phi_n(x)\, w(x)\, dx \qquad \ldots(2.3)$$

Now $f_2(x) = [f(x)]_*$ can be taken as the desired linear approximation of $f(x)$ so that

$$[\,f(x)\,]_* = a_0\,\phi_0\left(\frac{x}{A}\right) + a_1\,\phi_1\left(\frac{x}{A}\right) \qquad \ldots(2.4)$$

3. Solution of the main problem

Here we have made an attempt to solve the non-linear differential equation

$$\ddot{x} + f(x) = NF(t) \qquad \ldots(3.1)$$

by making use of the linear orthonormal polynomial approximation of $f(x)$.

From (2.4), we have

$$[\,f(x)\,]_* = -K^2 + K^{*2}x \qquad \ldots(3.2)$$

where for a_j, $j=0, 1$ defined by (2.3), $\phi_0(x) = 1$ and $\phi_1(x) = c + dx$, we have

$$K^{*2} = a_1\,\frac{d}{A},$$

and $\quad -K^2 = a_0 + a_1 c.$

On replacing $f(x)$ by its approximation $[\,f(x)\,]_*$ in (3.1), the equation transforms into

$$\ddot{x} + K^{*2}x = K^2 + NF(t). \qquad \ldots(3.3)$$

Thus the approximate solution of (3.1) subject to the initial conditions $x = A$, $\dot{x} = 0$ when $t = 0$ can easily be seen to be

$$x_* = \left(A - \frac{K^2}{K^{*2}}\right)\cos K^*t + \frac{K^2}{K^*} + \frac{N}{K^*}$$

$$\int_0^t F(u) \sin \{K^*(t-u)\}\, du \qquad \dots(3.4)$$

4. Deductions

(i) Let $\phi_n(x) = g_n^{-1/2} P_n^{(\alpha,\beta)}(x)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial and

$$g_n = \frac{2^{1+\alpha+\beta}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{n!\,\Gamma(1+\alpha+\beta+n)\,(1+\alpha+\beta+2n)}, \qquad \dots(4.1)$$

in (3.3). Then, on using the formulae (*Szegö* [*13*], *p. 71*)

$$P_0^{(\alpha,\beta)}(x) = 1,\; P_1^{(\alpha,\beta)}(x) = \tfrac{1}{2}(\alpha+\beta+2)\,x + \frac{(\alpha-\beta)}{2},$$

the result (3.4) reduces to

$$x_* = \left[A + \frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left(1-\frac{\lambda^{*2}}{\lambda^2}\right)\right]\cos\lambda t - \frac{A(\alpha-\beta)}{(\alpha+\beta+2)}\left(1-\frac{\lambda^{*2}}{\lambda^2}\right)$$

$$+ \frac{N}{\lambda}\int_0^t F(u)\sin\{\lambda(t-u)\}\,du, \qquad \dots(4.2)$$

where

$$\lambda^2 = \frac{2+\alpha+\beta}{2A}\,a_1, \qquad \dots(4.3)$$

and

$$\lambda^{*2} = \frac{2+\alpha+\beta}{A(\beta-\alpha)}\,a_0, \qquad \dots(4.4)$$

with

$$a_n = \frac{1}{g_n}\int_{-1}^{1}(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\, f(Ax)\,dx,\; n=0,\,1. \qquad \dots(4.5)$$

(ii) Now if in the differential equation (3.1), we take

$$f(x) = w_0 \left(1+\frac{x}{A}\right)^{\rho} H \begin{matrix} 0, \lambda: (\mu', \nu') ;\dots; (\mu^{(r)}, \nu^{(r)}) \\ A, C: [B', D'] ;\dots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [(a): \theta', \dots, \theta^{(r)}] : [(b'): \phi'] ;\dots; [(b^{(r)}) : \phi^{(r)}]; \\ [(c): \Psi', \dots, \Psi^{(r)}] : [(d'): \delta'] ;\dots; [(d^{(r)}): \delta^{(r)}]; \end{matrix} \right.$$

$$\left. z_1 \left(1+\frac{x}{A}\right)^{\sigma_1} ;\dots; z_r \left(1+\frac{x}{A}\right)^{\sigma_r} \right), \qquad \dots(4.6)$$

where the H-function occurring in the right-hand side of (4.6) is the multivariable H-function defined by Srivastava and Panda [11] (for details of this function, one is referred to Srivastava and Panda [11], [12]; see also [10], pp. 251–254) and use the following modified form of the result given by Srivastava and Panda [12, p. 131, Fq. (2.2)] viz.

$$\int_{-1}^{1} (1+x)^{\rho} (1-x)^{\alpha} (\alpha,\beta)\, H \begin{matrix} 0, \lambda: (\mu', \nu') ;\dots; \mu^{(r)}, \nu^{(r)}) \\ A, C: [B', D'] ;\dots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [(a): \theta', \dots, \theta^{(r)}]: [(b'): \phi'] ;\dots; [(b^{(r)}): \phi^{(r)}]; \\ [(c); \Psi', \dots, \Psi^{(r)}]: [(d'): \delta'] ;\dots; [(d^{(r)}); \delta^{(r)}]; \end{matrix} \right.$$

$$\left. z_1 (1+x)^{\sigma_1}, \dots, z_r (1+x)^{\sigma_r} \right)$$

$$= \frac{2^{\rho+\alpha+1}\, \Gamma(\alpha+n+1)}{n!} H \begin{matrix} 0, 2+\lambda: (\mu', \nu') ;\dots; (\mu^{(r)}, \nu^{(r)}) \\ 2+A, 2+C: [B', D'] ;\dots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [-\rho: \sigma_1, \dots, \sigma_r], [\beta-\rho; \sigma_1, \dots, \sigma_r], [(a): \theta', \dots, \theta^{(r)}]: \\ [-\alpha-\rho-n-1: \sigma_1, \dots, \sigma_r], [\beta-\rho+n; \sigma_1, \dots, \sigma_r]\ [(c): \Psi', \dots, \Psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'): \phi'] ;\dots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta'] ;\dots; [(d^{(r)}); \delta^{(r)}]; \end{matrix}\ z_1 2^{\sigma_1}, \dots, z_r 2^{\sigma_r} \right). \qquad \dots(4.7)$$

its approximate solution is given by

$$x_* = \left[A + \frac{(\alpha-\beta)A}{(\alpha+\beta+2)} \left(1 - \frac{\lambda_1^{*2}}{\lambda_1^2}\right) \right] \cos \lambda_1 t$$

$$- \frac{A(\alpha-\beta)}{(\alpha+\beta+2)} \left(1 - \frac{\lambda_1^{*2}}{\lambda_1^2}\right) + \frac{N}{\lambda_1} \int_0^t F(u) \sin \{\lambda_1 (t-u)\} \, du, \qquad \ldots(4.8)$$

where

$$\lambda_1^2 = \frac{2^{\rho-1} w_0 \Gamma(\alpha+\beta+4)}{A\,\Gamma(2+\beta)} H^{\,0,\,2+\lambda:\,(\mu',\,\nu');\ldots;(\mu^{(r)},\,\nu^{(r)})}_{\,2+A,\,2+C:\,[B',\,D'];\ldots;[(B^{(r)}):D^{(r)}]}$$

$$\left\{ \begin{matrix} [-\rho-\beta:\sigma_1,\ldots,\sigma_r],\ [-\rho:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [-\alpha-\beta-\rho-2:\sigma_1,\ldots,\sigma_r],\ [-\rho+1:\sigma_1,\ldots,\sigma_r],\ [(c):\Psi',\ldots,\Psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad z_1 2^{\sigma_1},\ldots,z_r 2^{\sigma_r} \right\} \qquad \ldots(4.9)$$

and

$$\lambda_1^{*2} = \frac{2^{\rho} w_0 \Gamma(3+\alpha+\beta)}{A(\beta-\alpha)\,\Gamma(1+\beta)} H^{\,0,\,\lambda+1:\,(\mu',\,\nu');\ldots;(\mu^{(r)},\,\nu^{(r)})}_{\,A+1,\,C+1:\,[B',\,D'];\ldots;[B^{(r)},\,D^{(r)}]}$$

$$\left\{ \begin{matrix} [-\rho-\beta:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots; \\ [-\alpha-\beta-\rho-1:\sigma_1,\ldots,\sigma_r],\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad z_1 2^{\sigma_1};\ldots;z_r 2^{\sigma_r} \right]. \qquad \ldots(4.10)$$

λ_1^2 and λ_1^{*2} exist when $\mathrm{Re}(\beta) > -1$, $\mathrm{Re}\,(\rho + \sum_{i=1}^{r} \sigma_i \alpha_i) > -1$,

$\sigma_i > 0$ and

$$\Delta_i = \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

$|\arg (z_i 2^{\sigma_i})| < \frac{1}{2} \pi \Delta_i$, where $a_i = \dfrac{d_j^{(i)}}{\delta_j^{(i)}}$, $j=1, \ldots, \mu^{(i)}$,

$i=1, \ldots, r$.

Further, on taking

$$F(t) = t^{\rho_1 - 1} H_{P,\,Q}^{M,\,N} \left[z\, t^u \,\middle|\, \begin{matrix} (a_P, A_P) \\ (b_Q, B_Q) \end{matrix} \right], \qquad \ldots(4.11)$$

where $H_{P,\,Q}^{M,\,N} \left[x \right]$ is the well-known Fox's H– function and using

a special case of the known result (Goyal and Aggarwal [5]) viz.

$$\int_0^t x^{\rho_1 - 1} \sin \{\lambda_1 (t-x)\} H_{P,\,Q}^{M,\,N} \left[z x^u \,\middle|\, \begin{matrix} (a_P, A_P) \\ (b_Q, B_Q) \end{matrix} \right] dx$$

$$= t^{\rho_1 + 1} \sum_{k=0}^{\infty} (-1)^k \lambda_1^{2k+1} t^{2k} H_{P+1,\,Q+1}^{M,\,N+1} \left[z t^u \,\middle|\, \begin{matrix} (1-\rho_1, u), (a_P, A_P) \\ (b_Q, B_Q), (-1-2k-\rho_1, u) \end{matrix} \right], \qquad \ldots(4.12)$$

under its conditions of convergence, the solution (4.8) becomes

$$x_* = \left[A + \frac{(\alpha-\beta) A}{(\alpha+\beta+2)} \left(1 - \frac{\lambda_1^{*2}}{\lambda_1^2}\right) \right] \cos \lambda_1 t$$

$$- \frac{(\alpha-\beta) A}{(\alpha+\beta+2)} \left(1 - \frac{\lambda_1^{*2}}{\lambda_1^2}\right)$$

$$+ N t^{\rho_1+1} \sum_{k=0}^{\infty} (-1)^k \lambda_1^{2k} t^{2k}$$

$$H_{P+1,\, Q+1}^{M,\, N+1} \left[z t^u \,\middle|\, \begin{matrix} (1-\rho_1, u), (a_P, A_P) \\ (b_Q, B_Q), (-1-2k-\rho_1-, u) \end{matrix} \right]. \quad \ldots(4.13)$$

If we take $x = A(A-1)$, $\rho = 0$, $r = 1$, $\lambda = A = C = 0$ and make necessary changes in λ_1^2 and λ_1^{*2}, the result (4.8) reduces to a known result given earlier by Srivastava and Singh [9, p. 29, Eq. (3. 8)].

(iii) If we take

$$f(x) = w_0^2 x^r \, {}_pF_q \, [\, (\alpha_p); \, (\beta_q); \, kx^s], \quad \ldots(4.14)$$

where r and s are positive integers,

$$F(t) = \cos pt,$$

and use the result (see [14], p. 466)

$$\int_{-1}^{1} x^s (1-x)^\alpha \, (1+x)^\beta \, P_n^{(\alpha,\beta)} \, (x) \, dx$$

$$= \frac{2^{n+\alpha+\beta+1} \, s! \, \Gamma(\alpha+n+1) \, \Gamma(\beta+s+1)}{n! \, s-n! \, \Gamma(n+s+\alpha+\beta+2)}$$

$${}_2F_1 \left(\begin{matrix} n-s, \alpha+n+1; \\ -\beta-s; \end{matrix} -1 \right), \; s > n, \quad \ldots(4.15)$$

the approximate solution of the non-linear differential equation

$$\ddot{x} + w_0^2 x^r \, {}_pF_q \, [\, (\alpha_p); \, (\beta_q); \, kx^s] = N \cos pt \quad \ldots(4.16)$$

is given by

$$x_* = \left[A + \frac{(\alpha-\beta)A}{(\alpha+\beta+2)} \left(1- \frac{\lambda_2{}^{*2}}{\lambda_2{}^2}\right) \right] \cos \lambda_2 t$$

$$- \frac{(\alpha-\beta)A}{(\alpha+\beta+2)} \left(1 - \frac{\lambda_2{}^{*2}}{\lambda_2{}^2}\right)$$

$$+ \frac{N}{p^2-\lambda_2{}^2} (\cos \lambda_2 t - \cos pt), \qquad \ldots(4.16)$$

where

$$\lambda_2{}^{*2} = \frac{A^{r-1} w_0{}^2}{(\beta-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n k^n A^{sn} \Gamma(\alpha+\beta+3) \Gamma(\beta+r+sn+1)}{\prod_{j=1}^{q} (\beta_j)_n n! \Gamma(\beta+1) \Gamma(\alpha+\beta+r+sn+2)}$$

$$\cdot {}_2F_1 \left({-r-sn, \alpha+1; \atop -\beta-r-sn;} -1 \right), \qquad \ldots(4.18)$$

and

$$\lambda_2{}^2 = w_0{}^2 A^{r-1}$$

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n k^n A^{sn} (r+sn) \Gamma(\beta+r+sn+1) \Gamma(\alpha+\beta+4)}{\prod_{j=1}^{q} (\beta_j)_n n! \Gamma(\beta+2) \Gamma(\alpha+\beta+r+sn+3)}$$

$$\cdot {}_2F_1 \left({1-r-sn, \alpha+2; \atop -\beta-r-sn \;\;;} -1 \right). \qquad \ldots(4.19)$$

The series for $\lambda_2{}^{*2}$ and $\lambda_2{}^2$ are convergent for $p \leqslant q$ and for $p=q+1$ if $| kA^s | < 1$.

Again, for $N=0$, the result (4.17) reduces to the known result given earlier by the authors [6] which further includes the results of Saxena and Kushwaha [8] and Garde [4].

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