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STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE AND MISSING COEFFICIENTS

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ABSTRACT

Let $S(A, B, K)$ be the class of functions $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$ $(a_1 > 0, k \geqslant 2)$ regular and univalent in the unit disc $U = \{z : |z| < 1\}$ and satisfying

$$|\{zf'(z)/f(z) - 1\}/\{A - Bzf'(z)/f(z)\}| < 1, \; z \in U,$$

where $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant B \leqslant 0$. Let $0 < z_0 < 1$. We denote by $S_1(A, B, k, z_0)$ and $S_2(A, B, k, z_0)$, two subclasses of $S(A, B, k)$, consisting of functions which satisfy $f(z_0) = z_0$ and $f'(z_0) = 1$ respectively. In this paper we obtain coefficient estimates, distortion and closure theorems and radius of convexity of order $\rho (0 \leqslant \rho < 1)$ for the classes $S_1(A, B, k, z_0)$ and $S_2(A, B, k, z_0)$.

1. INTRODUCTION

Let S denote the class of functions $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$ $(a_1 > 0, k \geqslant 2)$

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regular and univalent in the unit disc $U = \{z: |z| < 1\}$. Let $S(A, B, k)$ be the class of those functions f of S for which $zf'(z)/f(z)$ is subordinate to $(1+Az)/(1+Bz)$, $z \varepsilon U$, here $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant B \leqslant 0$. That is $f \varepsilon S(A, B, k)$ if there exists a function ω which is regular in U and satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$(1.1) \quad z\frac{f'(z)}{f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \; z \varepsilon U.$$

The requirement (1.1) is equivalent to

$$(1.2) \quad \left|\frac{zf'(z)/f(z) - 1}{A - Bzf'(z)/f(z)}\right| < 1, \; z \varepsilon U.$$

Let us denote by $K(A, B, k)$, the class of functions f for which zf' $\varepsilon S(A,B,k)$. Clearly, the functions in $S(A,B,k)$ and $K(A,B,k)$ are starlike and convex respectively and hence are univalent in U. The author and Shukla [3] have recently studied these two classess when $a_1 = 1$. For $0 < z_0 < 1$, let T_1 and T_2 be the subclasses of S consisting of the functions which satisfy $f(z_0) = z_0$ and $f'(z_0) = 1$ respectively. We define

$$S_m(A, B, k, z_0) = S(A, B, k) \cap T_m, \; m = 1, 2$$

and

$$K_m(A, B, k, z_0) = K(A, B, k) \cap T_m, \; m = 1, 2.$$

In the present paper we obtain coefficient estimates, distortion and covering theorems for the classes $S_m(A, B, k, z_0)$ and $K_m(A,B,k,z_0)$, $m = 1, 2$, and the radius of convexity of order $\rho (0 \leqslant \rho < 1)$ for the classes $S_m(A, B, k, z_0)$, $m = 1, 2$. We also show that each of these four classes is closed under 'arithmetic mean' and 'convex linear combinations'. Section 2 deals with the results for the classes $S_1(A, B, k, z_0)$ and $K_1(A, B, k, z_0)$ whereas in section 3 we state the results for the classes

$S_2(A, B, k, z_0)$ and $K_2(A, B, k, z_0)$.

Two subclasses of S, obtained by replacing $zf'(z)/f(z)$ by $f'(z)/a_1$ in the definitions of $S_m(A, B, k, z_0)$, $m = 1, 2$, have been studied by author and Shukla in [2].

2. The classes $S_1(A, B, k, z_0)$ and $K_1(A, B, k, z_0)$

Coefficient estimates

Theorem 1. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. In f is regular in U and satisfies $f(z_0) = z_0$; then $f \in S_1(A, B, k, z_0)$ if and only if

$$(2.1) \qquad \sum_{n=k}^{\infty} \{(n-1) + (A-nB) - (A-B)\, z_0^{n-1}\}\, |a_n| \leqslant (A-B).$$

The result is sharp.

Proof. Author and Shukla [3] have shown that a function

$$g(z) = z - \sum_{n=k}^{\infty} |b_n| z^n, \text{ regular in } U, \text{ satisfies}$$

$$\left| \frac{zg'(z)/g(z) - 1}{A - Bzg'(z)/g(z)} \right| < 1, \; z \in U$$

if and only if

$$\sum_{n=k}^{\infty} \{(n-1) + (A-nB)\}\, |b \; [illegible] \;).$$

Applying that result to the [illegible]ion $g(z) = f(z)/a_1$, we find that f satisfies (1.2) if and only if

$$\sum_{n=k}^{\infty} \{(n-1) + (A-nB)\}\, |a_n| \leqslant (A-B)\, a_1.$$

Since $f(z_0) = z_0$, we also have from the representation of $f(z)$ that

$$a_1 = 1 + \sum_{n=k}^{\infty} |a_n| z_0^{n-1}.$$

Putting this value of a_1 in the above inequality we arrive at (2.1).

Since, in (2.1), equality is obtained for the function

$$f(z) = \frac{\{(n-1) + (A-nB)\} z - (A-B) z^n}{(n-1) + (A-nB) - (A-B)z_0^{n-1}},$$

it follows that the result is sharp.

Theorem 2. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. If f is regular in U and satisfies $f(z_0) = z_0$, then $f \in K_1(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} [n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}] \, |a_n| \leqslant (A-B).$$

The result is sharp.

Proof. Let $g(z) = z - \sum_{n=k}^{\infty} |b_n| z^n$ be regular in U. Then, as stated also in the proof of the preceeding theorem, g satisfies $|\{zg'(z)/g(z)-1\}/\{A-Bzg'(z)/g(z)\}| < 1$ for $z \in U$ if and only if

$$\sum_{n=k}^{\infty} \{(n-1) + (A-nB)\} |b_n| \leqslant (A-B).$$

Applying this coefficient inequality to the function $g(z) = zf'(z)/a_1$, we find that $zf'(z) \in S(A, B, k)$ if and only if

$$\sum_{n=k}^{\infty} n\{(n-1) + (A-nB)\} |a_n| \leqslant (A-B)a_1.$$

Hence $f \in K(A, B, k)$ if and only if this last inequality holds.

Now, in view of $f(z_0) = z_0$, we again have

$$a_1 = 1 + \sum_{n=k}^{\infty} | a_n | z_0^{n-1}.$$

Substituting the value of a_1 in the above inequality we conclude that $f \in K_1(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} [n \{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}] \, | a_n | \leqslant (A-B).$$

Sharpness of the result follows if we take

$$f(z) = \frac{n \{(n-1) + (A-nB)\}z - (A-B)z^n}{n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}} \quad (n=k, k+1,...).$$

Distortion properties

Theorem 3. If $f \in S_1(A, B, k, z_0)$ and $| z | = r$, then

(2.2) $Cr - Dr^k \leqslant | f(z) | \leqslant Cr + Dr^k$

and

(2.3) $C - Dkr^{k-1} \leqslant | f'(z) | \leqslant C + Dkr^{k-1}$,

where

$$C = \frac{(k-1) + (A-kB)}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}$$

and

$$D = \frac{A-B}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}.$$

The result is sharp.

Since $f(z_0) = z_0$, we also have from the representation of $f(z)$ that

$$a_1 = 1 + \sum_{n=k}^{\infty} |a_n| z_0^{n-1}.$$

Putting this value of a_1 in the above inequality we arrive at (2.1).

Since, in (2.1), equality is obtained for the function

$$f(z) = \frac{\{(n-1) + (A-nB)\} z - (A-B) z^n}{(n-1) + (A-nB) - (A-B) z_0^{n-1}},$$

it follows that the result is sharp.

Theorem 2. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. If f is regular in U and satisfies $f(z_0) = z_0$, then $f \in K_1(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} [n\{(n-1) + (A-nB)\} - (A-B) z_0^{n-1}] \, |a_n| \leqslant (A-B).$$

The result is sharp.

Proof. Let $g(z) = z - \sum_{n=k}^{\infty} |b_n| z^n$ be regular in U. Then, as stated also in the proof of the preceeding theorem, g satisfies $|\{zg'(z)/g(z)-1\}/\{A-Bzg'(z)/g(z)\}| < 1$ for $z \in U$ if and only if

$$\sum_{n=k}^{\infty} \{(n-1) + (A-nB)\} |b_n| \leqslant (A-B).$$

Applying this coefficient inequality to the function $g(z) = zf'(z)/a_1$, we find that $zf'(z) \in S(A, B, \mathbf{k})$ if and only if

$$\sum_{n=k}^{\infty} n\{(n-1) + (A-nB)\} |a_n| \leqslant (A-B)a_1.$$

Hence $f \in K(A, B, k)$ if and only if this last inequality holds.

Now, in view of $f(z_0) = z_0$, we again have

$$a_1 = 1 + \sum_{n=k}^{\infty} |a_n| z_0^{n-1}.$$

Substituting the value of a_1 in the above inequality we conclude that $f \in K_1(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} [n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}]\, |a_n| \leqslant (A-B).$$

Sharpness of the result follows if we take

$$f(z) = \frac{n\{(n-1) + (A-nB)\}z - (A-B)z^n}{n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}} \quad (n=k, k+1, \ldots).$$

Distortion properties

Theorem 3. If $f \in S_1(A, B, k, z_0)$ and $|z| = r$, then

(2.2) $Cr - Dr^k \leqslant |f(z)| \leqslant Cr + Dr^k$

and

(2.3) $C - Dkr^{k-1} \leqslant |f'(z)| \leqslant C + Dkr^{k-1}$,

where

$$C = \frac{(k-1) + (A-kB)}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}$$

and

$$D = \frac{A-B}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}.$$

The result is sharp.

Proof. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. Then, it follows from Theorem 1 that

$$(2.4) \quad \sum_{n=k}^{\infty} |a_n| \leqslant \frac{A-B}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}$$

and

$$(2.5) \quad \sum_{n=k}^{\infty} n\,|a_n| \leqslant \frac{k(A-B)}{(k-1) + (A-kB) - (A-B)z_0^{k-1}} .$$

Since $f \varepsilon S_1(A, B, k, z_0)$, we also have

$$(2.6) \quad a_1 = 1 + \sum_{n=k}^{\infty} |a_n|\, z_0^{n-1}.$$

But, it follows from the representation of f(z) that

$$(2.7) \quad |f(z)| \leqslant a_1 r + \sum_{n=k}^{\infty} |a_n|\, r^n$$

$$\leqslant a_1 r + r^k \sum_{n=k}^{\infty} |a_n| ,$$

$$(2.8) \quad |f(z)| \geqslant a_1 r - \sum_{n=k}^{\infty} |a_n|\, r^n$$

$$\geqslant a_1 r - r^k \sum_{n=k}^{\infty} |a_n|$$

and

$$(2.9) \quad |f'(z)| \leqslant a_1 + \sum_{n=k}^{\infty} n\,|a_n|\, r^{n-1}$$

$$\leqslant a_1 + r^{k-1} \sum_{n=k}^{\infty} n\,|a_n|\,,$$

$$(2.10)\quad |f'(z)| \geqslant a_1 - \sum_{n=k}^{\infty} n\,|a_n|\,r^{n-1}$$

$$\geqslant a_1 - r^{k-1} \sum_{n=k}^{\infty} n\,|a_n|\,.$$

The required inequalities follow now by using (2.4) — (2.6) in (2.7) — (2.10).

Equality in (2.2) and (2.3) is obtained for the function

$$(2.11)\quad f(z) = \frac{\{(k-1)-(A-kB)\}z-(A-B)z^k}{(k-1)+(A-kB)-(A-B)z_0^{k-1}}\,.$$

Note. For the above function equality on the left hand side of (2.2) is obtained at $z=r$ whereas the equality on the right hand side is obtained at $z=-r$ when $k=2, 4, 6 \ldots$; $z=ir$ when $k=3, 7, 11, \ldots$ and $z=re^{i\pi/(k-1)}$ when $k=5, 9, 13, \ldots$. Similarly, the points where equality holds in (2.3) can be obtained.

Letting $r \to 1$ in (2.2) we have the following:

Corollary 1. If $f \in S_1(A, B, \mathrm{k}, z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$|w| < \frac{(k-1)\,(1-B)}{(k-1)+(A-kB)-(A-B)z_0^{k-1}}\,.$$

The result is sharp with the extremal function given by (2.11).

The following theorem can be proved on the lines of the proof of Theorem 3.

Theorem 4. If $f \in K_1(A, B, k, z_0)$ and $|z| = r$, then

(2.12) $Mr - Nr^k \leqslant |f(z)| \leqslant Mr + Nr^k$

and

(2.13) $M - Nkr^{k-1} \leqslant |f'(z)| \leqslant M + Nkr^{k-1}$,

where

$$M = \frac{k\{(k-1) + (A-kB)\}}{k\{(k-1) + (A-kB)\} - (A-B)z_0^{k-1}} \text{ and}$$

$$N = \frac{A-B}{k\{(k-1) + (A-kB)\} - (A-B)z_0^{k-1}} .$$

The result is sharp with the extremal function

(2.14) $$f(z) = \frac{k\{(k-1) + (A-kB)\}z - (A-B)z^k}{k\{(k-1) + (A-kB)\} - (A-B)z_0^{k-1}} .$$

Letting $r \to 1$ in (2.12) we have the following:

Corollary 2. If $f \in K_1(A, B, k, z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$|w| < \frac{k\{(k-1) + (A-kB)\} - (A-B)}{k\{(k-1) + (A-kB)\} - (A-B)z_0^{k-1}} .$$

The result is sharp with the extremal function given by (2.14).

Radius of convexity of $S_1(A, B, k, z_0)$

Theorem 5. If $f \in S_1(A, B, k, z_0)$, then f is convex function of order $\rho (0 \leqslant \rho < 1)$ in the disc $|z| < R$, where

$$R = \inf_{n \geqslant k} \left[\left(\frac{1-\rho}{n(n-\rho)} \right) \left(\frac{(n-1) + (A-nB)}{A-B} \right) \right]^{1/(n-1)} .$$

The result is sharp.

Proof. In order to establish the required result, it suffices to show that $| zf''(z)/f'(z) | < (1-\rho)$ in $| z | < R$. Let

$f(z) = a_1 z - \sum_{n=k}^{\infty} | a_n | z^n$. Then, we have

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{- \sum_{n=k}^{\infty} n(n-1) | a_n | z^{n-1}}{a_1 - \sum_{n=k}^{\infty} n | a_n | z^{n-1}} \right|$$

$$\leqslant \frac{\sum_{n=k}^{\infty} n(n-1) | a_n | | z |^{n-1}}{a_1 - \sum_{n=k}^{\infty} n | a_n | | z |^{n-1}} .$$

Therefore, $| zf''(z)/f'(z) | < (1-\rho)$ holds if

$$\sum_{n=k}^{\infty} n(n-\rho) | a_n | | z |^{n-1} < (1-\rho)a_1.$$

Since $a_1 = 1 + \sum_{n=k}^{\infty} | a_n | z_0^{n-1}$, the above inequality can be written

as

$$(2.15) \quad \sum_{n=k}^{\infty} \{ n(n-\rho) | z |^{n-1} - (1-\rho)z_0^{n-1} \} | a_n | < (1-\rho).$$

Also, by Theorem 1, it follows that

$$(1-\rho) \sum_{n=k}^{\infty} \left\{ \frac{(n-1) + (A-nB)}{A-B} - z_0^{n-1} \right\} | a_n | \leqslant (1-\rho).$$

Hence (2.15) will be satisfied if

$$\{ n(n-\rho) \mid z \mid^{n-1} - (1-\rho)z_0^{n-1} \} \mid a_n \mid$$

$$< (1-\rho) \left\{ \frac{(n-1)\,(A-nB)}{A-B} - z_0^{n-1} \right\} |a_n|, \text{ for each } n=k,k+1, \ldots$$

or if

$$|z| < \left[\left(\frac{1-\rho}{n(n-\rho)} \right) \left(\frac{(n-1) + (A-nB)}{A-B} \right] \right]^{1/(n-1)},$$

for each $n=k,\ k+1,\ldots$

This completes the proof of theorem. Sharpness follows if we take the same extremal function for which Theorem *1* is sharp.

Colsure properties

Now, we show that the classes $S_1(A, B, k, z_0)$ and $K_1(A, B, k, z_0)$ are closed under 'arithmetic mean' and 'convex linear combinations'.

Theorem 6. Let $f_j(z) = a_{1j} - \sum_{n=k}^{\infty} | a_{nj} | z^n,\ j = 1, 2, \ldots, m.$

If $f_j \in S_1(A, B, k, z_0)$ for each $j = 1, 2, \ldots, m,$ then the function

$h(z) = b_1 z - \sum_{n=k}^{\infty} | b_n | z^n$ also belongs to $S_1(A, B, k, z_0)$, where

$$b_1 = \sum_{j=1}^{m} \lambda_j\, a_{1j},\ b_n = \sum_{j=1}^{m} \lambda_j\, a_{nj}\ (n = k, k+1, \ldots),$$

$$\lambda_j \geqslant 0 \text{ and } \sum_{j=1}^{m} \lambda_j = 1.$$

Proof. Since $f_j \in S_1(A, B, k, z_0)$, it follows from (2.1) that

$$\text{(2.16)} \quad \sum_{n=k}^{\infty} \{(n-1) + (A-nB)\} - (A-B)z_0^{n-1} \,] \mid a_{nj} \mid \leqslant (A-B),$$

$j = 1, 2, \ldots, m.$

Therefore

$$\sum_{n=k}^{\infty} \{ (n-1) + (A-nB) - (A-B)z_0^{n-1}\} \mid b_n \mid \leqslant$$

$$\leqslant \sum_{n=k}^{\infty} [\{ (n-1) + (A-nB) - (A-B)z_0^{n-1}\} \{ \sum_{j=1}^{m} \lambda_j \mid a_{nj} \mid \}]$$

$$\leqslant \sum_{j=1}^{m} [\lambda_j \sum_{n=k}^{\infty} \{ (n-1) + (A-nB) - (A-B)z_0^{n-1}\} \mid a_{nj} \mid]$$

$$\leqslant \sum_{j=1}^{m} [\lambda_j(A-B)], \quad \text{by (2.16)}$$

$$= (A-B).$$

Hence, by Theorem 1, $h \in S_1(A, B, k, z_0)$.

Theorem 7. Let $f_1(z) = z$ and

$$f_n(z) = \frac{n\{(n-1) + (A-nB)\}z - (A-B)z^n}{n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}} \quad (n = k, k+1, \ldots).$$

Then, $f \in S_1(A, B, k, z_0)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqslant 0$ and $\lambda_1 + \sum_{n=k}^{\infty} \lambda_n = 1.$

Proof. Let us suppose that

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z)$$

$$= a_1 z - \sum_{n=k}^{\infty} | a_n | z^n,$$

where

$$a_1 = \lambda_1 + \sum_{n=k}^{\infty} \frac{\{ (n-1) + (A-nB) \}\lambda_n}{(n-1) + (A-nB) \} - (A-B)z_0^{n-1}}$$

and

$$| a_n | = \frac{(A-B)\lambda_n}{(n-1) + (A-nB) - (A-B)z_0^{n-1}} \cdot (n=k, k+1, \ldots).$$

Then, it is easy to see that $f(z_0) = z_0$ and the condition (2.1) is satisfied. Hence $f \in S_1(A, B, k, z_0)$.

Conversely Let $f \in S_1(A, B, k, z_0)$, and $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$.

Then, from (2.1), it follows that

$$|a_n| \leqslant \frac{A-B}{(n-1) + (A-nB) - (A-B)z_0^{n-1}} \quad (n=k, k+1, \ldots).$$

Setting

$$\lambda_n = \left\{ \frac{(n-1) + (A-nB) - (A-B)z_0^{n-1}}{A-B} \right\} |a_n| \; (n=k, k+1, \ldots)$$

and

$$\lambda_1 = 1 - \sum_{n=k}^{\infty} \lambda_n,$$

we have

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z).$$

This completes the proof of theorem.

The following two theorem can be proved on the lines of proofs of Theorems 6 and 7, respectively. We omit the details of proof.

Theorem 8. Let $f_j(z) = a_{1j} z - \sum_{n=k}^{\infty} |a_{nj}| z^n$, $j = 1, 2, \dots, m.$

If $f_j \in K_1(A, B, k, z_0)$ for each $j = 1, 2, \dots, m$, then the function

$h(z) = b_1 z - \sum_{n=k}^{\infty} |b_n| z^n$ also belongs to $K_1(A, B, k, z_0)$, where

$$b_1 = \sum_{j=1}^{m} \lambda_j a_{1j},\ b_n = \sum_{j=1}^{m} \lambda_j a_{nj}\ (n=k, k+1, \dots),$$

$$\lambda_j \geqslant 0 \text{ and } \sum_{j=1}^{m} \lambda_j = 1.$$

Thorem 9. Let $f_1(z) = z$ and

$$f_n(z) \quad \frac{n\{(n-1) + (A-nB)\} z - (A-B)z^n}{n\{(n-1) + (A-nB)\} - (A-B)z_0^{n-1}} \ (n=k, k+1, \dots).$$

Then $f \in K_1(A, B, k, z_0)$, if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqslant 0$ and $\lambda_1 + \sum_{n=k}^{\infty} \lambda_n = 1.$

3. The classes $S_2(A, B, k, z_0)$ and $K_2(A, B, k, z_0)$

The results of this section are analogous to those of the preceeding section. Since the technique of their proof is similar to that of the results in seetion 2, we are stating the results only. However, as an illustration, we present the proof in brief of the first theorem of this section.

Theorem 10. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. If f is regular in U and satisfies $f'(z_0) = 1$, then $f \varepsilon S_2(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} \{ (n-1) + (A-nB) - n(A-B)z_0^{n-1}\} |a_n| \leqslant (A-B).$$

The result is sharp.

Proof. While proving Theorem 1, we have shown that the function f satisfies the condition (1.2) if and only if

$$\sum_{n=k}^{\infty} \{ (n-1) + (A-nB) \} |a_n| \leqslant (A-B)a_1.$$

Since $f'(z_0) = 1$, we have

$$a_1 = 1 + \sum_{n=k}^{\infty} n |a_n| z_0^{n-1}.$$

The required coefficient inequality follows now by substituting this value of a_1 in the above inequality.

Sharpness follows if we take

$$f(z) = \frac{\{ (n-1) + (A-nB) \}z - (A-B)z^n}{(n-1) + (A-nB) - (A-B)z_0^{n-1}} \; (n=k, k+1, \ldots).$$

Theorem 11. Let $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n| z^n$. If f is regular in U

and satisfies $f'(z_0) = 1$, then $f \in K_2(A, B, k, z_0)$ if and only if

$$\sum_{n=k}^{\infty} n\{ (n-1) + (A-nB) - (A-B)z_0^{n-1}\} \, | a_n | \leqslant (A-B).$$

The result is sharp with the extremal function

$$(3.1) \quad f(z) = \frac{n\{ (n-1) + (A-nB)\} z - (A-B)z^n}{n\{ (n-1) + (A-nB) - (A-B)z_0^{n-1}} \quad (n=k, k+1, \ldots)$$

Theorem 12. If $f \in S_2(A, B, k, z_0)$ and $|z| = r$, then

$$cr - dr^k \leqslant | f(z) | \leqslant cr + dr^k$$

and

$$c - dkr^{k-1} \leqslant | f'(z) | \leqslant c + dkr^{k-1},$$

where

$$c = \frac{(k-1) + (A-kB)}{(k-1) + (A-kB) - k(A-B)z_0^{k-1}}$$

and

$$d = \frac{A-B}{(k-1) + (A-kB) - k(A-B)z_0^{k-1}} \, .$$

The result is sharp with equality holding for the function

$$f(z) = \frac{\{ (k-1) + (A-kB) \}z - (A-B)z^k}{(k-1) + (A-kB) - k(A-B)z_0^{k-1}}.$$

Corollary 3. $f \in S_2(A, B, k, z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$| w | < \frac{(k-1)\,(1-B)}{(k-1) + (A-kB) - k(A-B)z_0^{k-1}}.$$ The result is sharp

Theorem 13. If $f \in K_2(A, B, k, z_0)$ and $| z | = r$, then

$$ar - br^k \leqslant | f(z) | \leqslant ar + br^k$$

and

$$a - bkr^{k-1} \leqslant | f'(z) | \leqslant a + bkr^{k-1},$$

where

$$a = \frac{(k-1) + (A-kB)}{(k-1) + (A-kB) - (A-B)z_0^{k-1}}$$

and

$$b = \frac{A-B}{k\{(k-1) + (A-kB) - (A-B)z_0^{k-1}\}}.$$

The result is sharp with equality holding for the function

$$f(z) = \frac{k\{(k-1) + (A-kB)\}z - (A-B)z^k}{k\{(k-1) + (A-kB) - (A-B)z_0^{k-1}\}}.$$

Corollary 4. If $f \in K_2(A, B, k, z_0)$, then the disc U is mapped by f onto a domain that contains the disc

$$| w | < \frac{k\{(k-1) + (A-kB)\} - (A-B)}{k\{(k-1) + (A-kB) - (A-B)z_0^{k-1}\}}.$$

The result is sharp.

Theorem 14. If $f \in S_2(A, B, k, z_0)$, then f is convex function of order $\rho (0 \leqslant \rho < 1)$ in the disc $| z | < R$, where

$$R = \inf_{n \geqslant k} \left[\left(\frac{1-\rho}{n(n-\rho)} \right) \left(\frac{(n-1) + (A-nB)}{A-B} \right) \right]^{1/(n-1)},$$

The result is sharp with the extremal function f given by (3.1).

Theorem 15 Let f_j and h have the same power series expansion as in theorem 6. Then $h \in S_2(A, B, k, z_0)$ if $f_j \in S_2(A, B, k, z_0)$ for each $j = 1, 2, \ldots, m$.

Theorem 16. Let $f_1(z) = z$ and

$$f_n(z) = \frac{\{(n-1)+(A-nB)\}z - (A-B)z^n}{(n-1)+(A-nB) - n(A-B)z_0^{n-1}} \quad (n = k, k+1, \ldots).$$

Then, $f \in S_2(A, B, k, z_0)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqslant 0$ and $\lambda_1 + \sum_{n=k}^{\infty} \lambda_n = 1$.

Theorem 17. Let f_j and h have the same power series expansion as in theorem 8. Then $h \in K_2(A, B, k, z_0)$ if $f_j \in K_2(A, B, k, z_0)$ for each $j = 1, 2, \ldots, m$.

Throrem 18. Let $f_1(z) = z$ and

$$f_n(z) = \frac{n\{(n-1)+(A-nB)\}z - (A-B)z^n}{n\{(n-1)+(A-nB) - (A-B)z_0^{n-1}\}}$$

$(n = k, k+1, \ldots)$.

Then $f \in K_2(A, B, k, z_0)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

$$\lambda_n \geqslant 0 \text{ and } \lambda_1 + \sum_{n=k}^{\infty} \lambda_n = 1.$$

Remark. Our results generalize the results of Gupta and Ahmad [1].

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