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FIXED POINT THEOREM FOR MAPPINGS SATISFYING RATIONAL INEQUALITIES

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Several fixed point theorems are obtained by Fisher [1] and Khan [2] in the metric spaces for mappings satisfying different types of rational inequalities. In the present paper we use a new type of rational inequalities and obtain two fixed point theorems in complete metric space, Our first result is stated as

Theorem 1. *If T is self mapping of a complete metric space (X,d) and satisfies the inequality*

$$d(Tx, Ty) \leqslant \frac{\beta(x, Tx)\, d(x, Ty)}{d(x, Tx) + d(x, Ty)}$$

where $d(x, Tx) + d(x, Ty) \neq 0$ for all $x, y \in X$ and $0 < \beta < \frac{1}{2}$, then T has a unique fixed point.

Proof: Let x_0 be any arbitrary point of X. We define a sequence $\{x_n\}$ of elements of X such that $x_n = Tx_{n-1}$, for $n = 1, 2, 3, \ldots$. Then we have

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

$$\leqslant \beta \frac{d(x_0, x_1)\, d(x_0, x_2)}{d(x_0, x_1) + d(x_0, x_2)}$$

$$\leqslant \beta \frac{d(x_0, x_1) [d(x_0, x_1) + d(x_1, x_2)]}{d(x_1, x_2)}$$

or

$$[d(x_1, x_2)]^2 - \beta d(x_1, x_2) d(x_0, x_1) + \tfrac{1}{4}\beta^2 [d(x_0, x_1)]^2$$

$$\leqslant \tfrac{1}{4} (\beta^2 + 4\beta) [d(x_0, x_1)]^2$$

or

$$d(x_1, x_2) \leqslant \tfrac{1}{2} (\beta + \sqrt{\beta^2 + 4\beta}) d(x_0, x_1)$$

or

$$d(x_1, x_2) \leqslant \lambda d(x_0, x_1)$$

where $\lambda = \frac{1}{2} (\beta + \sqrt{\beta^2 + 4\beta})$; for $0 < \beta < \frac{1}{2}$, $0 < \lambda < 1$.

Similarly,

$$d(x_2, x_3) \leqslant \lambda d(x_0, x_1) \leqslant \lambda^2 d(x_0, x_1),$$

and in general,

$$d(x_n, x_{n+1}) \leqslant \lambda^n d(x_0, x_1).$$

Since $\lambda < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X.

Then, assuming $z \neq Tz$, we have

$$d(z, Tz) \leqslant d(z, x_n) + d(Tx_{n-1}, Tz)$$

$$\leqslant d(z, x_n) + \beta \frac{d(x_{n-1}, x_n) \cdot d(x_{n-1}, Tz)}{d(x_{n-1}, x_n) + d(x_{n-1}, Tz)}.$$

Letting n tend to infinity, we get

$$d(z, Tz) \leqslant 0.$$

Thus, we arrive at a contradiction which implies $z = Tz$ i.e. z is a fixed point of T. Now to show the uniqueness of z, let us consider that $z' \neq z$ is another fixed point of T. Then

$$d(z, z') = d(Tz, Tz')$$

$$\leqslant \beta \frac{d(z, Tz)\, d(z, Tz')}{d(z, Tz) + (z, Tz')}$$

or

$$d(z, z') \leqslant 0.$$

Thus, we arrive at a contradiction which implies that $z = z'$ proving the uniqueness of z. This completes the proof of the theorem 1.

Next we state

Theorem 2. *If S and T are self mappings of a complete metric space (X, d) and satisfy*

$$d(Sx, Ty) \leqslant \beta \frac{d(x, Sx)\, d(x, Ty) + d(y, Sx)\, d(y, Ty)}{d(x, Sx) + d(x, Ty)}$$

where $d(x, Sx) + d(x, Ty) \neq 0$ for all $x, y \in X$ and $0 < \beta < \frac{1}{2}$, then S and T have a unique common fixed point.

Proof: Let x_0 is any arbitrary point of X. We define a sequence $\{x_n\}$ of elements of X such that $X_{2n+1} = Sx_{2n}$, and $X_{2(n+1)} = Tx_{2n+1}$ for $n = 0, 1, 2, 3, \ldots$. Then, we have

$$d(x_{2n+1}, x_{2(n+1)}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leqslant \beta \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})\, d(x_{2n+1}, x_{2(n+1)})}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1})}$$

$$\leqslant \beta \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2(n+1)})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2(n+1)})}$$

$$\leqslant \beta \frac{d(x_{2n}, x_{2n+1}) \, [\, d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2(n+1)}) \,]}{d(x_{2n+1}, x_{2(n+1)})}$$

or

$$[\, d(x_{2n+1}, x_{2(n+1)}) \,]^2 - \beta \, d(x_{2n+1}, x_{2(n+1)}) \, d(x_{2n}, x_{2n+1})$$

$$+ \tfrac{1}{4} \beta^2 \, [\, d(x_{2n}, x_{2n+1}) \,]^2$$

$$\leqslant \tfrac{1}{4} (\beta^2 + 4\beta) \, [\, d(x_{2n}, x_{2n+1}) \,]^2$$

or

$$d(x_{2n+1}, x_{2(n+1)}) \leqslant \tfrac{1}{2} \left(\beta + \sqrt{\beta^2 + 4\beta}\right) d(x_{2n}, x_{2n+1})$$

Putting $\lambda = \frac{1}{2} (\beta + \sqrt{\beta^2 + 4\beta})$.

Since $0 < \beta < \frac{1}{2}$, we can find that $0 < \lambda < 1$.

Therefore,

$$d(x_{2n+1}, x_{2(n+1)}) \leqslant \lambda \, d(x_{2n}, d_{2n+1})$$

$$\leqslant \lambda^{2n+1} d(x_0, x_1)$$

for $n = 0, 1, 2, \ldots$

Similarly, for $n = 1, 2, 3, \ldots$, we have

$$d(x_{2n}, x_{2n+1}) \leqslant \lambda^{2n} d(x_0, x_1).$$

Now by routine calculation for any $k > n$, we observe that

$$d(x_n, x_{n+k}) \leqslant \frac{\lambda^n}{1-\lambda} \cdot d(x_0, x_1)$$

since $\lambda < 1$, the right hand side of the above inequality tends to zero as $n \to \infty$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence with the limit z in X. Let us consider $z \neq Tz$, so that

$$d(z, Tz) = d(z, x_{2n+1}) + d(Sx_{2n}, Tz)$$

$$\leqslant d(z, x_{2n+1}) + \beta \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, Tz) + d(z, Sx_{2n})\, d(z, Tz)}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tz)}$$

Letting n tend to infinity we get

$$d(z, Tz) \leqslant 0.$$

Thus, we arrive at a contradiction. Hence $z = Tz$ i.e. z is the fixed point of T. Similarly, we can show that z is the fixed point of S. Therefore z is the common fixed point of S and T.

Now to show the uniqueness of z, let us consider that $z' \neq z$ is another common fixed point of S and T. then,

$$d(z, z') = d(Sz, Tz')$$

$$\leqslant \beta \frac{d(z, Sz)\, d(z, Tz') + d(z', Sz)\, d(z', Tz')}{d(z, Sz) + d(z, Tz')} \leqslant 0.$$

Thus, we arrive at a contradiction. Hence $z = z'$. Thus, it follows that z is the unique common fixed point of S and T. This completes the proof of the theorem 2.

Now consider the following example

Let $S: [0, 1] \rightarrow [0, 1]$ be defined by

$Sx = x/2$ for all x.

$T: [0, 1] \rightarrow [0, 1]$ be defined by

$Tx = x/4, x \varepsilon [0, \frac{1}{2}]$

$= x/5, x \varepsilon [\frac{1}{2}, 1]$.

Clearly, T is discontinuous at $x = \frac{1}{2}$ and also T is not a contraction map but T has a unique fixed point $x = 0$. Also we find that 0 is the unique fixed point of S. Therefore *0* is the unique common fixed point of S and T.

The above example satisfies the conditions and inequality of our theorem for $\frac{1}{7} < \beta < \frac{1}{2}$ and we find that 0 is the common fixed point of S and T.

REFERENCES

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