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SOME REMARKS ON THE POLAR DERIVATIVE OF A POLYNOMIAL

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1. Let P(z) be a polynomial of degree n and $P'(z)$ denote its derivative. The polar derivative $P_\xi(z)$, for any complex number ξ, is defined by

$$(1) \quad P_\xi(z) = (\xi - z)\, P'(z) + nP(z).$$

Clearly $P_\xi(z)$ itself is a polynomial of degree $n-1$ for fixed ξ. Concerning the zeros of P(z) and $P_\xi(z)$, there is a celebrated result known as:

Laguerre's Theorem [2; p. 592]. *If $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $z \in C$ where C is a circular domain (i. e. the image of a unit disk under a linear transformation), then for any $\xi \in C$ and $z \in C$,*

$$(2) \qquad P_\xi(z) \neq 0.$$

Applying Laguerre's Theorem to the polynomial $P(z) - \lambda$, one immediately has

DeBruijn's Theorem [2; p. 592]. *If $P(z)$ is a polynomial of degree n and $S = \{P(z): z \in C\}$ where C is a circular domain, then for any $\xi \in C$ and $z \in C$*

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(3) $\frac{1}{n} P_{\xi}(z) = (\xi - z) \frac{P'(z)}{n} + P(z) \in S.$

In [2] DeBruijn proved several interesting results concerning the derivative of a polynomial as an application of these two above mentioned theorems. Por other, results in this direction see [4] and [5]. In this paper, we present some results along the same line as a consequence of Laguerre's and DeBruijn s Theorems.

We first recall a remarkable result on the estimate of the derivative of a polynomial known as:

Bernstein's Theorem [1; p. 338]. *If $P(z)$ is a polynomial of degree n and $|P(z)| \leqslant 1$ for $|z| \leqslant 1$, then*

(4) $|P'(z)| \leqslant n$

for $|z| \leqslant 1$. There is equality in (4) if and only if $P(z) = az^n, |a| = 1$.

The of proof this theorem, as given by Bernstein uses Gauss–Lucas Theorem: The zeros of P'(z) lies in the convex hull of the zeroes of P(z). But the discussion on equality in (4) involves an interpolation formula for trigonometric polynomial due to Riesz [6; p. 357]. We note in the following that this conclusion follows immediately from DeBruijn's Theorem:

Let the circular domain C be the unit disk $|z| \leqslant 1$. Since $\max_{|z|=1} |P(z)| = 1$, S is contained in $|w| \leqslant 1$. Suppose for some $|z_0| = 1$, $P'(z_0) = ne^{i\alpha}$, *i. e,* the equality is attained in (4), then from (3) one has

(5) $\left| \xi e^{i\alpha} - \frac{z_0 P'(z_0)}{n} + P(z_0) \right| \leqslant 1$

for any $|\xi| \leqslant 1$. This means that a disk of radius one with center $\frac{z_0 P'(z_0)}{n} - P(z_0)$, which also lies in $|w| \leqslant 1$ (take $\xi = 0$ in (3)), is entirely contained in the unit disk $|w| \leqslant 1$. This is possibly only if the image of the unit disk $|z| \leqslant 1$ under $P(z)$ is the entire disk $|w| \leqslant 1$. It further implies that the $\max_{|z|=1} |P(z)| = 1$ is attained for infinitely many points on the boundary $|z| = 1$. But $|P(z)|$ can not attain its maximum on $|z| = 1$ at more then n points unless $P(z) = \alpha z^n$; obviously $|\alpha| = 1$. This verifies the assertion.

We also note that DeBruijn's Theorem gives an interesting improvement of Bernstein's Theorem. In fact, we prove:

Theorem 1. *If* $P(z)$ *is a polynomial of degree n and* $|P(z)| \leqslant 1$ *for* $|z| \leqslant 1$, *then*

$$(6) \qquad |P'(z)| \leqslant n \frac{1 + |P(z)|}{2}$$

for $|z| = 1$. *There is equality in* (6) *for* $P(z) = \alpha z^n$, $|\alpha| = 1$.

Proof of Theorem 1. In DeBruijn's Theorem, let the circular domain C be the unit disk $|z| \leqslant 1$, by hypothesis S is contained in $|w| \leqslant 1$. Then for any $|\xi| \leqslant 1$ and $|z| \leqslant 1$, we have

$$(7) \qquad |(\xi - z) P'(z) + nP(z)| \leqslant n.$$

Choosing $\xi = -z$ in (7), one gets

$$(8) \qquad |-2zP'(z) + nP(z)| \leqslant n.$$

From where,

$$(9) \qquad |2zP'(z)| - n|P(z)| \leqslant n.$$

Hence for $|z| = 1$ in (9), we have (6). The proof is complete.

2. In this section, we note that Laguerre's Theorem can be used to give a proof of the following known result:

Theorem A [4; p. 59]. *If* P(z) *is a polynomial of degree n having all its zeros in the disk* $|z| \leqslant k, k \leqslant 1$ *and* $\max_{|z|=1} |P(z)| = 1$, *then*

(10) $$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k}.$$

There is equality in (9) for $P(z) = (z+k)^n/(1+k)^n$.

Alternate Proof of Theorem A. Since P(z) has no zeros in the disk $|z| > k, k \leqslant 1$, it follows from Laguerre's Theorem where the circular disk C is the region $|z| > k, k \leqslant 1$, that for any $|\xi| > k$ and $|z| = 1$ one has

(11) $$(\xi - z) P'(z) \neq -nP(z).$$

We note that for any fixed z, $|z| = 1$ a number ξ lying on the circle $C_\rho : |\xi - z| = 1 + \rho > 1 + k$ does not belong to the disk $|\xi| \leqslant k$. Moreover, we can find ξ on C_ρ such that $\arg(\xi - z) P'(z) = \arg(-nP(z))$, so from (10) one has

(12) $$(1+\rho)|P'(z)| = |\xi - z|\,|P'(z)| \neq n|P(z)|.$$

for $\rho > k$. Hence

(13) $$(1+\rho)|P'(z)| > n|P(z)|$$

because the otherwise inequality is violated for sufficiently large ρ. Making $\rho \to k$ in (13) one has

(14) $$|P'(z)| \geqslant \frac{n}{1+k} |P(z)|.$$

And (14) implies (10).

Further, if P(z) has all its zeros in $|z| \leqslant k$, then, following Gauss–Lucas Theorem, $P^{(\nu)}(z)$; $0 \leqslant \nu \leqslant n$ has all its zeros in $|z| \leqslant k$. Thus iterated application of Theorem A establishes:

Theorem 2. *If P(z) is a ploynomial of degree n having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for* $|z| = 1$

$$(16) \quad |P^{(m)}(z)| \geqslant \frac{n(n-1)\ldots(n-m+1)}{(1+k)^m} |P(z)| .$$

There is equality in (16) for $P(z) = (z+k)^n$.

3. Now, we consider the polar derivation of higher order and present some related observations. It is not difficult to show that m th order poler derivative $P_{\xi_1 \cdots \xi_m}(z)$ of a polynomial P(z) of degree n, for any $\xi_1, \ldots, \xi_m (m \leqslant n)$ is given by:

$$(17) \quad \frac{1}{n(n-1)\ldots(n-m+1)} P_{\xi_1 \cdots \xi_m}(z)$$

$$= \frac{(\xi_1 - z)(\xi_2 - z)\ldots(\xi_m - z)}{n(n-1)\ldots(n-m+1)} P^{(m)}(z) + \ldots$$

$$+ \sum \frac{(\xi_1 - z)\ldots(\xi_\nu - z) P^{(\nu)}(z)}{n(n-1)\ldots(n-\nu+1)} + \ldots$$

$$+ \Sigma (\xi_1 - z) \frac{P'(z)}{n} + P(z).$$

where $\Sigma (\xi_1 - z) \ldots (\xi_\nu - z)$ represent the sum of the product of $(\xi_j - z)$, $j = 1, 2, \ldots, m$ taken ν at a time.

From Laguerre's Theorem, we immediately have:

Lemma 1. *If P(z) is a polynomial of degree n and has no zeros in a circular domain C, then for any* $\xi_1 \ldots \xi_m \in C$, $(m < n)$ *and* $z \in C$

(18) $P_{\xi_1 \cdots \xi_m}(z) \neq 0.$

And from DeBruijn's Theorem, we also have:

Lemma 2. *If* $P(z)$ *is a polynomial of degree* n *and* $S = \{P(z): z \in C\}$ *where* C *is a circular domain, then for any* $\xi_1, \ldots, \xi_m \in C$ $(m \leqslant n)$ *and* $z \in C$,

(19) $$\frac{1}{n(n-1)\ldots(n-m+1)} P_{\xi_1 \cdots \xi_m}(z) \in S.$$

By use of Lemmas 1 and 2, we give an alternate proof of:

Theorem B [3; p. 503]. *If* $P(z)$ *is a ploynomial of degree* n *having no zeros in* $|z| < k$, $k \geqslant 1$ *and* $|P(z)| \leqslant 1$ *for* $|z| \leqslant 1$, *then*

(20) $$|P^{(m)}(z)| \leqslant \frac{n(n-1)\ldots(n-m+1)}{1+k^m}.$$

Alternate Proof of Therorem B. In Lemma 1, let the circular domain C be the disk $|z| < k$ and $\xi_1 \ldots \xi_m$ be the m th root of $\rho^m e^{i\phi}$ where $0 < \rho < k$ and ϕ is arbitrary. So, (18) takes the form

(21) $$\rho^m e^{i\phi} \frac{P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} + \lambda(z) \neq 0$$

for $|z| \leqslant 1$. where $\lambda(z) = \dfrac{(-z)^m P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} +$

$$+ \sum \frac{(-z)^{m-1} P^{(m-1)}(z)}{n(n-1)\ldots(n-m+2)} + \ldots + \sum \left(- \frac{zP'(z)}{n}\right) + P(z).$$

In (21) we choose ϕ appropriately to get

(22) $$\rho^m \left|\frac{P^{(m)}(z)}{n(n-1)\ldots(n-m+1)}\right| \neq \lambda(z)$$

which implies that

$$(23)\quad \rho^{m}\left|\frac{P^{(m)}(z)}{n(n-1)\dots(m-m+1)}\right| < |\lambda(z)|.$$

Making $\rho \to k$, we have

$$(24)\quad k^{m}\left|\frac{P^{(m)}(z)}{n(n-1)\dots(n-m+1)}\right| \leqslant |\lambda(z)|$$

for $|z| \leqslant 1$; see [4]. Moreover, if $\xi_1, \dots, \xi_m$ are the m th root of $e^{i\phi}$ in the Lemma 2 where the circular domain C is the unit disk $|z| \leqslant 1$ and S is contained $|w| \leqslant 1$, in view of the hypothesis, we have for $|z| \leqslant 1$,

$$(25)\quad \left|\frac{e^{i\phi}\, P^{(m)}(z)}{n(n-1)\dots(n-m+1)} + \lambda(z)\right| \leqslant 1.$$

An appropriate choice of ϕ (a priori)in (25) gives

$$(26)\quad \left|\frac{P^{(m)}(z)}{n(n-1)\dots(n-m+1)}\right| + |\lambda(z)| \leqslant 1$$

for $|z| \leqslant 1$. Using (24) in (26) one gets (20). The proof is complete. Finally, we present an alternate version of Lemma 1 in the following:

Theorem 3. *Let* $P(z)$ *be a polynomial of degree* n *having no zeros in a circular domain* C *which contains all ahe zeros of a polynomial* $Q(z)$ *of degree* m, $m \leqslant n$, *then*

$$(27)\quad \sum_{\nu=0}^{m} (-1)^{m-\nu}\, \frac{Q^{(\nu)}(z)\, P^{(m-\nu)}(z)}{\nu!\, n(n-1)\dots(n-m+\nu+1)} \neq 0$$

for $z \in C$ *if* $m < n$ *and for all* z *if* $m = n$.

Proof of Theorem 3. Let $\xi_1, \dots, \xi_m \in C$ be the zeros of

$$Q(z) = \sum_{\nu=0}^{m} b_\nu z^\nu \text{ i. e. } Q(z) = (-1)^m b_m (\xi_1 - z) \ldots (\xi_m - z).$$

Without any difficulty, it can be verified that

(28) $Q^{(\nu)}(z) = (-1)^{m-\nu} b_m \nu! \, \Sigma \, (\xi_1 - z)(\xi_2 - z) \ldots (\xi_{m-\nu} - z).$

Substituting (28) in (18) we get (27) for $z \in C$ in case $0 \leqslant m < n$. For $m = n$, note that the left-hand side of (18) and, therefore, (27) is a constant for all z.

If P(z) and Q(z) are polynomials of degree n satisfying the hypothesis of Theorem 3, then (27) becomes

(29) $$\sum_{\nu=0}^{n} (-1)^{n-\nu} Q^{(\nu)}(z) P^{(n-\nu)}(z) \neq 0$$

setting $z = 0$ in (29) one gets the Grace Apolarity Theorem [2; p. 593].

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