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A GENERALIZATION OF APPELL SETS OF POLYNOMIALS

By

A. K. AGARWAL

Department of Mathematics, R. B. S. College,

Agra-282002, U. P., India

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ABSTRACT

In this paper we give a generalization of Appell sets of polynomials which includes earlier generalizations of Appell sets given by Sheffer [2 , Rainville [1], Osegov [3], and Al-Salam and Varma [4].

1. INTRODUCTION

Let $\{\phi_n(x)\}$ be a simple set of polynomials. It is called an Appell set if

(1.1) $\phi_n'(x) = \phi_{n-1}(x), \quad (n = 0, 1, 2, \cdots,)$

or, equivalently, if

(1.2) $$\sum_{n=0}^{\infty} \phi_n(x) t^n = e^{xt} A(t), \; A(0) \neq 0,$$

where $A(t)$ is an arbitrary power series in t. Sheffer [2] generalized Appell sets by considering a linear differential operator of infinite order with constant coefficients,

(1.3) $$J(D) = \sum_{k=0}^{\infty} c_k D^{k+1}, \; c_0 \neq 0,$$

as a generalization of the differential operator D. He considered simple stes of polynomials which satisfy

$$(1.4) \quad J(D)\, \phi_n(x) = \phi_{n-1}(x),$$

and proved that polynomial sets have the property (1.4) if and only if they have a generating function of the form

$$(1.5) \quad \sum_{n=0}^{\infty} \phi_n(x)\, t^n = A(t) \exp [x H(t)],$$

where

$$(1.6) \quad H(t) = \sum_{n=0}^{\infty} h_n\, t^{n+1},\ h_0 \neq 0,$$

$$(1.7) \quad A(t) = \sum_{n=0}^{\infty} a_n\, t^n,\ a_0 \neq 0.$$

Rainville [1] extended the class of Sheffer polynomials by considering the operator

$$(1.8) \quad J(\sigma) = \sum_{k=0}^{\infty} c_k\, \sigma^{k+1}$$

in which the c_k are constants with $c_0 = 0$, and σ is defined by

$$(1.9) \quad \sigma = D \prod_{i=1}^{q} (\theta + b_i - 1),\ \theta = xD.$$

He studied the polynomial sets that satisfy

$$(1.10) \quad J(\sigma)\, \phi_n(x) = \phi_{n-1}(x),$$

and proved that simple sets of polynomials have the property (1.10) if and only if they have generating function of the form

$$(1.11) \quad \sum_{n=0}^{\infty} \phi_n(x)\, t^n = A(t)\, {}_0F_q [-;\ b_1, \ldots, b_q;\ xH(t)]$$

in which $H(t)$ and $A(t)$ have formal power series expansions given by

(1.6) and (1.7) respectively, and $J(H(t)) = H(J(t)) = t$.

Osegov [3] has generalized Appell sets in a different direction. He studied polynomial sets which have the property

$$(1.12) \quad D^m \phi_n(x) = \phi_{n-m}(x) \quad (n = m, m+1, \dots),$$

where m is a fixed positive integer.

Recently Al-Salam and Verma [4] defined a class of polynomials $S^{(m)}$ by means of the operator

$$(1.13) \quad J^m(D) = \sum_{k=0}^{\infty} a_k D^{k+m},$$

where a_k $(k \geqslant 0)$ are independent of x. They called the polynomial set $\{ \phi_n(x) \}$ in $S^{(m)}$ if there is an operator $J^m(D)$ of the form (1.13) such that

$$(1.14) \quad J^m(D)\, \phi_n(x) = \phi_{n-m}(x).$$

Obviously, $S^{(m)}$ contains the polynomials which satisfy (1.4) as well as the polynomials that satisfy (1.12).

In the present note we consider a class of polynomial sets which contains all the polynomials that satisfy (1.1), (1.4), (1.10), (1.12) and (1.14) and obtain a characterization of this class.

2. THE GENERALIZED APPELL POLYNOMIALS

Let m be a fixed positive integer and let us consider the differential operator of infinite order

$$(2.1) \quad J^m(\sigma) = \sum_{k=0}^{\infty} b_k \sigma^{k+m}, \; b_0 \neq 0,$$

where $b_k (k \geqslant 0)$ is independent of x and σ is given by (1.9) we shall call the polynomial set $\{\phi_n(x)\}$ to be in $T^{(m)}$ if there is an operator $J^m(\sigma)$

of the form (2.1) such that

(2.2) $J^m(\sigma)\ \phi_n(x) = \phi_{n-m}(x) \quad (n = m, m+1, \ldots,)$.

Obviously our class $T^{(m)}$ contains all the polynomial sets which we have indicated above.

3. A CHARACTERIZATION OF $T^{(m)}$

In this section we propose to prove the following

Theorem. Let $\{\phi_n(x)\}$ be a simple set of polynomials. A necessary and sufficient condition that $\phi_n(x)$ be in $T^{(m)}$ is that $\phi_n(x)$ possess the generating function indicated in

$$(3.1) \quad \sum_{n=0}^{\infty} \phi_n(x)\ t^n = \sum_{j=1}^{\infty} A_j(t)\ {}_0F_i\ [- ; b_1, \ldots, b_i ; xH(\varepsilon_j\ t)\ ],$$

where the ε_j are primitive m th roots of unity $H(\varepsilon_j\ t)$ and $A_j(t)$, $(j = 1, 2, \ldots, m)$ have formal power series expansions of the form (1.6) and (1.7), respectively. Also $J(H(t)) = t$

Proof. We first note that if $F(t) \sum_{k=0}^{\infty} a_k\ t^k, a_0 \neq 0$, is a formal power series, then there exists another formal power series $f(t) = \sum_{i=0}^{\infty} \beta_i t^i$, $\beta_0 \neq 0$, such that

$$F(t) = (f(t))^{\gamma} \quad (\gamma = 1, 2, \ldots).$$

Now assume that $\{\phi_n(x)\} \in T^{(m)}$ and suppose that the differential operator corresponding to $\{\phi_n(x)\}$ is

$$J^m(\sigma) = \sum_{k=0}^{\infty} a_k\ \sigma^{k+m},\ a_0 = 1.$$

If we assume that

$$J^m(t) = t^m F(t)$$

we can find $f(t)$ such that

$$J^m(t) = \{tf(t)\}^m = \{J(t)\}^m.$$

where $J(\sigma)$ is of the form (1.8). This could be possible because of the fact we have noted in the beginning. Let

$$F(x, t) = \sum_{n=0}^{\infty} \phi_n(x)\, t^n.$$

Then wc get

$$\{J^m(\sigma) - t^m\}\, F(x, t) = 0,$$

in view of (2.2).

Conseqently, if $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m$ are the m roots of unity, we get

$$\{J(\sigma) - \varepsilon_j\, t\}\, F(x, t) = 0.$$

From this it follows that

$$F(x, t) = \sum_{j=1}^{\infty} A_j(t)\, {}_0F_q\, [-;\, b_1, \ldots, b_q;\, xH(\varepsilon_j\, t)\,].$$

Next assume that $F(x, t)$ is of the form (3.1), it then follows that

$$J^m(\sigma)\, F(x, t) = t^m F(x, t) = \sum_{n=m}^{\infty} \phi_{n-m}(x)\, t^n,$$

that is

$$J^m(\sigma)\, \phi_n(x) = \phi_{n-m}(x).$$

This completes the proof of the theorem.

As a corollary to the above theorem one can easily prove the following

Theorem 2. A necessary and sufficient condition that two simple sets of polynomials $\{\phi_n(x)\}$ and $\{\Psi_n(x)\}$ belong to the same operator $J^m(\sigma)$ is that there exists a sequence of numbers a_k, independent of n, such that

$$(3.2) \quad \Psi_n(x) = \sum_{k=0}^{n} a_k \, \phi_{n-k}(x).$$

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[1] E. D. Rainville, *Special Functions,* Macmillan, New York, 1960.

[2] I. M. Sheffer, Some properties of polynomial sets of type zero, *Duke Math. J.* **5** (1939), 590–622.

[3] V. B. Osegov, Some external properties of generalized Appell polynomials, *Soviet Math.* **5** (1964), 1651–1653.

[4] W. A. Al–Salam and A. Verma, Generalized Sheffer polynomials, *Duke Math. J* **37** (1970), 361–365.