Jñānābha, Vol. 13, 1983

ON A SEQUENCE OF FOURIER COEFFICIENTS

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(*Received : October 21, 1981; Revised : July 26, 1982*)

ABSTRACT

In the present paper the author obtains a result for the $(N, p_n)\ C_1$ –summability of a sequence of Fourier coefficients which generalizes earlier results of Mohanty and Nanda [3], Varshney [9], and Sharma ([7], [8]).

1. Let $\Sigma\, a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{P_n\}$ be a sequence o*f* constants, real or complex, and led us write

$$P_n = p_o + p_1 + p_2 + \cdots + p_n,\ (p_{-1} = P_{-1} = 0).$$

The sequence – to – sequence transformation

$$(1.1) \qquad t_n = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k}\, S_k,\ (P_n \neq 0)$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the sequence of constants $\{p_n\}$. We say that the series Σa_n or the seqence $\{S_n\}$ is summable by Nörlund means [10] or summable (N, p_n) to the sum S if $\lim_{n\to\infty} t_n = S$.

The conditions of regularity of the method of summability (N, p_n) defined by (1.1) are

$$(1.2) \qquad \lim_{n\to\infty} \left\{ \frac{p_n}{P_n} \right\} = 0$$

By using a Tauberian theorem, he deduced Mohanty and Nanda's result as a corollary of Theorem B.

Sharma [7] noticed that the condition (3. 3) of the Theorem B implies conditions (3. 4) and (3. 5) and thus under a less stringent condition than that of Varshney's theorem, he established the following theorem :

Theorem C. *if*

$$(3.4) \quad \int_0^t |\Psi(u)|\, du = o(t),\ t \to 0$$

and

$$(3.5) \quad \int_t^\delta \frac{|\Psi(u)|}{u} \log\left(\frac{1}{u}\right) du = o\left(\log\frac{1}{t}\right),\ 0 < \delta < 1,$$

then the sequence $\{n\, B_n(x)\}$ is summable $\left(N, \frac{1}{n+1}\right) C_1$ to the value $1/\pi$.

Replacing the sequence $\left\{\frac{1}{n+1}\right\}$ by a more general sequence $\{p_n\}$, Sharma [8] again generalized Theorem B and Theorem C in the following form :

Theorem D. Let $\{p_n\}$ be a monotonic non-decreasing sequence of real positive constants such that

$$(3.6) \quad P_n = p_0 + p_1 + p_2 + \dots + p_n \to \infty \text{ as } n \to \infty,$$
$$\log n = O(P_n) \text{ as } n \to \infty,$$

and

$$(3.7) \quad \int_0^t |\Psi(u)|\, du = o\left(\frac{t}{P_\tau}\right),\ \text{as } t \to 0,$$

where $\tau = [1/t]$, the integral part of $1/t$, then the sequence $\{n\ B_n(x)\}$ is summable $(N, p_n)\ C_1$ to the value $1/\pi$.

SECTION I

The object of the present paper is to improve Theorem D by putting, instead of conditions (3. 6) and (3. 7), more general conditions by introducing a functional factor,such that when it is indentically equal to unity, we get Theorem D. Precisely, we prove

Theorem 1. *Let (N, p_n) be a regular Nörlund method defined by a sequence $\{p_n\}$ of real, positive, monotonic non-increasing coefficients such that*

(3.8) $\quad P_n = p_0 + p_1 + p_2 + \cdots + p_n \to \infty,\ n \to \infty,$

(3.9) $\quad \lambda(n) \log n = O(P_n),\ n \to \infty$

where $\lambda(t)$ is a positive, monotonic non-decreasing function.

If

(3.10) $$\Psi(t) \equiv \int_0^t |\Psi(u)|\, du = o\left(\frac{\lambda(1/t.\ t}{P_\tau}\right),\ as\ t \to 0,$$

then the sequence $\{n\ B_n(x)\}$ is summable $(N, p_n)\ C_1$ to the value $1/\pi$.

4. We need following lemmas to prove our theorem.

Lemma 1 [2].

(i) *If $\{p_n\}$ is non-negative and non-increasing, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant n$ and any n*

$$\left| \sum_{k=a}^{b} p_k\ e^{i(n-k)t} \right| \leqslant A\ P_\tau$$

where A is an absolute constant.

(ii) $1/t\ p_\tau \leqslant P_\tau$.

Lemma 2. *If $0 \leqslant t \leqslant 1/n$, then*

$$| Q_n(t) | \equiv \left| \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left(\frac{Sin\ kt}{k\ t^2} - \frac{Cos\ kt}{t} \right) \right| = O(n).$$

Proof. $$| Q_n(t) | = O \left[\left| \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k} (k^2\ t) \right| \right]$$

$$= O \left[\frac{n}{P_n} \sum_{k=1}^{n} | p_{n-k} | \right]$$

$$= O(n).$$

Lemma 3. *If* $0 < t \leqslant \pi$, *then*

$$| Q_n(t) | = O \left[\frac{P_\tau}{t\ P_n} \right].$$

Proof. By Lemma 1 (i) we have

$$| Q_n(t) | = O \left[\frac{1}{P_n} \left| \sum_{k=1}^{n} p_{n-k} \frac{Sin\ kt}{kt^2} \right| \right] + O \left[\frac{P_\tau}{t\ P_n} \right].$$

Now applying Abel's transformation on the first expression on the right-hand side and keeping in view that

$$\sum_{k=1}^{n} \left| \frac{Sin\ kt}{k} \right| \leqslant \tfrac{1}{2} \pi + 1 \quad [9]$$

it is easy to see that

$$| Q_n(t) | = O \left[\frac{1}{t\ P_n} \sum_{k=0}^{\tau} p_k \right] + O \left[\frac{P_{n-1}}{t^2\ P_n} \right]$$

$$+ O \left[\frac{P_{\tau+1}}{t^2\ P_n} \right] + O \left[\frac{1}{t^2\ P_n} \sum_{k=\tau+1}^{n-2} | \triangle\ P_k | \right]$$

$$+ O \left[\frac{P_\tau}{t\ P_n} \right]$$

$$= O\left[\frac{P_r}{t\,P_n}\right], \text{ by Lemma 1 (ii).}$$

5. Proof of Theorem 1

If we denote the (C, 1) transform of the sequence $\{n\,B_n\,(x)\}$ by t_n, we have (cf [3])

$$t_n - 1/\pi = 1/\pi\int_0^\pi \Psi\,(t)\left[\frac{Sin\;nt}{nt^2} - \frac{Cos\;nt}{t}\right]dt + o\,(1),$$

as the second integral becomes $o\,(1)$, by the Riemann–Lebesgue theorem .

On account of the regularity of method of summability under consideration, we have to show that, under our assumptions,

$$I \equiv \int_0^\pi \frac{\Psi\,(t)}{\pi\,P_n}\sum_{k=n}^{n} p_{n-k}\left[\frac{Sin\;kt}{kt^2} - \frac{Cos\;kt}{t}\right]dt = o\,(1)$$

$$as\; n \to \infty .$$

Let $$I = \int_0^\pi \Psi\,(t)\;Q_n\,(t)\;dt$$

where $$Q_n\,(t) = \frac{1}{\pi\,P_n}\sum_{k=1}^{n} p_{n-k}\left[\frac{Sin\;kt}{kt^2} - \frac{Cos\;kt}{t}\right].$$

Now we set

$$I = \left(\int_o^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right)\Psi\,(t)\;Q_n\,(t)\;dt$$

$$= I_1 + I_2 + I_3,\; say,\; where\; 0 < \delta < \pi.$$

Using Lemma 2 and condition (3. 9) we get

$$I_1 = O\left[\int_0^{1/n} |\,\Psi\,(t)\,|\;dt\right]$$

$$= O\left[n\ o\left(\frac{\lambda(n)}{n P_n}\right)\right]$$

$$= o(1), \text{ as } n \to \infty .$$

Also using Lemma 3, we have

$$I_2 = O\left[\frac{1}{P_n}\int_{1/n}^{\delta} |\Psi(t)|\ \frac{P_\tau}{t}\, dt\right].$$

Now integrating by parts and observing that P_τ is a step-function, we have (*cf.* [4], *p. 42*, foot-note)

$$I_2 = O\left(\frac{1}{P_n}\right) + O\left(\frac{\lambda(n)}{P_n}\right) + O\left(\frac{1}{P_n}\int_{1/n}^{\delta}\frac{\lambda(1/t)}{t}\, dt\right)$$
$$+ o(1)$$

$$= o(1(+ O\left(\frac{1}{P_n}\int_{1/n}^{\delta}\frac{\lambda(1/t)}{t}\, dt\right), \text{ as } n \to \infty,$$

$$= o(1) + O\left(\frac{\lambda(n)}{P_n}\int_{1/n}^{\delta} 1/t\, dt\right),$$ since $\lambda(t)$ is monotonic non-decreasing

$$= o(1) + o\left(\frac{\lambda(n)\log n}{P_n}\right)$$

$$= o(1), \text{ as } n \to \infty,$$ by (3. 9).

Finally, by virtue of Riemann–Lebesgue theorem and regularity of summability method under consideration, we

have

$$I_3 = o(1), \text{ as } n \to \infty.$$

This completes the proof of Theorem 1.

SECTION 2

6. Pandey [5] has shown that the condition(6.1)is less stringent than the conditions of Theorem B and Theorem C in Section 1. Using (6. 1) we, in this section, propose to establish a result for the (N, p_n) C_1-summability of the sequence $\{n\ B_n\ (x)\}$.

We prove :

Theorem 2. *If*

$$(6.\ 1)\quad \int_t^{\delta} \frac{|\ \Psi\ (u)\ |}{u}\ P_{[1/u]}\ du = o\ [P_{[1/t]}],$$

$0 < \delta < \pi$, as $t \to +0$, then the sequence$\{n\ B_n\ (x)\}$ is summable $(N,p_n)C_1$ to the value $1/\pi$, where $\{p_n\}$ is the same as in Theorem 1.

For proving Theorem 2 we will need the following lemma alongwith the three lemmas of Section 1.

Lemma 4. The condition (6.1) implies that

$$\int_0^t |\ \Psi\ (u)\ |\ du = o\ (t),\ as\ t \to 0.$$

This lemma can be proved following the lines of Pandey [5].

7. PROOF OF THEOREM 2

From Section 1, we have

$$I = \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right)$$

$$= I_1 + I_2 + I_3,\ say,$$

where $\Psi\ (t)$ and $Q_n\ (t)$ are the same as in Theorem 1.

Now
$$I_1 = O\left[n \int_0^{1/n} | \Psi(t) | \, dt \right] \text{ by Lemma 2.}$$

$$= O\,[\, n\, o\,(1/n)\,], \text{ by Lemma 4.}$$

$$= o\,(1), \text{ as } n \to \infty.$$

Also we have

$$I_2 = O\left[\frac{1}{P_n} \int_{1/n}^{\delta} \left(| \Psi(t) | \quad \frac{P_t}{t} \right) dt \right], \text{ by Lemma 3.}$$

$$= O\left(\frac{1}{P_n} \, o\,(P_n) \right), \text{ by } (6.1)$$

$$= o\,(1), \text{ as } n \to \infty.$$

Finally, $I_3 = o\,(1)$, *as* $n \to \infty$, since the method of summability is regular and the Riemann–Lebesgue theorem holds.

This proves Theorem 2 completely.

ACKNOWLEDGEMENTS

The author is grateful to Prof. H.M. Srivastava of the University of Victoria, Canada, for his valuable suggestions and comments which have improved the presentation of the paper.

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