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SOME FUNDAMENTAL PROPERTIES OF A GENERALIZED q-LAPLACE TRANSFORM

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1. **Introduction**. In the year 1949, Hahn [3] defined two q–analogues of the classical Laplace transform. In a subsequent paper, he introduced an allied transform (see [4])

$$ {}_q^aL\ [f(x)\ ;\ s] \equiv g(s) = \frac{1}{1-q} \int_0^{s^{-1}} (1-qsx)_a\, f(x)\, d(q, x) \tag{1.1}$$

$$ = s^{-1} \sum_{i=0}^{\infty} q^i\, (1-q^{i-1})_a\, f(q^i/s), \tag{1.2}$$

$Re\,(s) > 0$,

and deduced certain properties for this transform.

The object of this paper is to develop some fundamental properties of the generalized q–Laplace transform (1. 1) in the from of theorems. A few worthmentioning images for (1.1) are also enumerated.

2. **Notation and Definitions**. For $0 < q < 1$, let $(1-q^\alpha)_n \equiv$

$$(q^\alpha)_n = (1-q^\alpha)\,(1-q^{\alpha+1}) \ldots (1-q^{\alpha+n-1}),\ (q^\alpha)_0 = 1.$$

We then recall the following basic hypergeometric functions :

$$ {}_{nk}\Phi_{n-1}\left[\begin{matrix} q^{(a_{n-k})} ; x \\ q^{(b_{n-1})} \end{matrix}\right] = $$

$$ \sum_{j=0}^{\infty} \frac{(q^{a_1})_j\,(q^{a_2})_j \ldots (q^{a_{n-k}})_j\,(-1)^{jk}\,q^{\frac{1}{2}jk(j-1)}\,x^j}{(q^{b_1})_j\,(q^{b_2})_j \ldots (q^{b_{n-1}})_j\,(q)_j} $$

$k = 0, 1, 2, \ldots, n,$

$$ {}_{n}\phi_{n-1}\left(\begin{matrix} q^{(c_n)} ; x \\ q^{(d_{n-1})} \end{matrix}\right) = \sum_{i=0}^{\infty} \frac{(q^{c_1})_i\,(q^{c_2})_i \ldots (q^{c_n})_i\,x^i}{(q^{d_1})_i\,(q^{d_2})_i \ldots (q^{d_{n-1}})_i\,(q)_i}, \qquad |x| < 1, $$

$$ E_q(x) = \prod_{0}^{\infty} (1 - xq^n) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(r-1)} x^r/(q)_r , $$

$$ e_q(x) = \prod_{0}^{\infty} (1 - xq^n)^{-1} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r}, \quad |x| < 1 , $$

$$ (x - y)_\nu = x^\nu \prod_{0}^{\infty} \left[\frac{1 - (y/x)\,q^n}{1 - (y/x)\,q^{\nu-n}} \right]. $$

For all values of a and q (real or complex), we denote $\frac{1 - q^a}{1 - q}$ by [a] Obviously, as $q \to 1$, [a] $\to$ a.

By a q-binomial coefficient, we mean

$$ \left[\begin{matrix} x \\ k \end{matrix}\right] = \prod_{j=1}^{k} \left(\frac{q^{x+j-1} - 1}{q^j - 1} \right) $$

and we note that

$$ \lim_{q \to 1} \left[\begin{matrix} x \\ k \end{matrix}\right] = \left(\begin{matrix} x \\ k \end{matrix}\right). $$

We define ν , a symbol for q-difference operator, by the equation

$$ \theta f(x) \equiv \theta_x f(x) = \frac{f(qx) - f(x)}{(q - 1)\,x}. $$

A repeated application of the θ-operator n is times denoted by $\theta_x^{(n)}$ or simply by $\theta^{(n)}$.

Following Hahn [3], the q-integral of a function, under appropriate conditions, is defined as

$$\int_0^x f(y)\, d(q,y) = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k),$$

$$\int_x^{\infty} f(y)\, d(q,y) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(x q^{-k}),$$

$$\int_0^{\infty} f(y)\, d(q,y) = (1-q) \sum_{k=-\infty}^{\infty} q^k f(q^k).$$

3. Main Results. In this section, we give some theorems for the transform (1. 1).

Theorem. If ${}_q^aL\,[f(x); s] = g(s)$, *then*

(i) ${}_q^aL\,[f(px); s] = g(s/p)/p$

and

(ii) ${}_q^aL\,[f(p/x); s] = p\ {}_q^aL\,[f(1/x); sp]$.

Proof. By using the definition (1. 2), we have

$${}_q^aL\,[f(px); s] = s^{-1} \sum_{i=0}^{\infty} q^i (1-q^{i+1})_a f(pq^i/s) = g(s/p)/p,$$

and this gives Theorem 1 (i).

This theorem can also be proved by making a change of variable in the Hahn's result [3, *p. 344, Eq.* (2. 17)]

$$\overset{b}{\underset{a}{S}} f(sx)\, d(q, x) = s^{-1} \overset{sb}{\underset{sa}{S}} f(x)\, d(q, x), \left(\begin{matrix} a < 1, b < 1 \\ \text{or} \\ a > 1, b > 1 \end{matrix} \right).$$

Theorem 1 (ii) can be proved in a similar way.

Theorem 2. (The q-Differentiation Theorem).

$$ {}^{a}_{q}L \left[\theta_x^{(n)} f(x) ; s \right] = \left\{ \frac{s}{1-q} \right\}^n \frac{(1 - q^{a-n+1})_n}{(1-q)^n} .$$

$$ {}^{a-n}_{q}L\ [f(x); s] - \sum_{m=1}^{n} \frac{s^{m-1} (1 - q^{a-m+2})_{m-1}}{(1-q)^{2m-1}} f^{(n-m)}(0)$$

$$= \left[\begin{matrix} a \\ n \end{matrix} \right] \frac{s^n (1-q)_n}{(1-q)^{2n}} q^{\frac{1}{2}n(n-1)}.\ {}^{a}_{q}L \left[\frac{f(x)}{(1-sxq^a)_n} ; s \right] -$$

$$- \sum_{m=1}^{n} \frac{s^{m-1} (1 - q^{a-m+2})_{m-1}}{(1-q)^{2m-1}} f^{(n-m)}(0). \qquad (3.1)$$

Proof. We have

$$ {}^{a}_{q}L\ [\theta_x f(x); s] = \frac{1}{1-q} \overset{s^{-1}}{\underset{0}{S}} (1 - qsx)_a\, \theta_x f(x)\, d(q, x). \qquad (3.2)$$

Integrating the right-hand side of (3. 2) by parts in the light of the formula (cf. Hahn [2, *p*. 7 Eq. (2. 9)])

$$S f_1(x)\, \theta_x f_2(x)\, d(q,x) = f_1(x/q) f_2(x) - S\, \theta_x f_1(x/q) f_2(x)\, d(q,x),$$

we get

$$ {}^{a}_{q}L\ [\theta f(x); s] = \frac{1}{1-q} [\{ f(x)\, (1 - sx)_a \}_0^{1/s} - $$

$$\overset{s^{-1}}{\underset{0}{S}} \; \theta\,(1-sx)_a\, f(x)\, d\,(q, x)\,] = \frac{1}{1-q}\,\{\, s\,[a]\; {}^{a-1}_{q}L\,[f(x); s] - f(0)\,\},$$

(3. 3)

by virtue of the relation (Hahn [5, p. 70, Eq. (2. 14)])

$$\theta_x^{(r)}\,(1-kx)_s = (-k)^r\, q^{\frac{1}{2}r\,(r-1)}\,[s]\,[s-1]\ldots[s-r+1]\,(1-kxq^r)_{s-r}$$

and (1. 2).

A repeated application of (3. 3) gives

$${}^{a}_{q}L\;[\theta_x^{(2)} f(x)\,;s] = {}^{a}_{q}L\,[\,\theta f^{(1)}(x); s], \text{ where } f^{(1)}(x) = \theta_x f(x)\,,$$

$$= \frac{1}{1-q}\,\{s\,[a]\;\; {}^{a-1}_{q}L\,[\,f^{(1)}(x)\,;s\,] - f^{(1)}(0)\,\}$$

$$= \left(\frac{s}{1-q}\right)^2 [a]\,[a-1]\,{}^{a-2}_{q}\,L\,[\,f(x)\,;s\,] - \frac{f^{(1)}(0)}{1-q} - \frac{s[a]}{(1-q)^2}\,f(0).$$

Proceeding in a similar manner, we obtain

$${}^{a}_{q}L\,[\,\theta^{(3)} f(x); s] = \left(\frac{s}{1-q}\right)^3 [a]\,[a-1]\,[a-2]\,{}^{a-3}_{q}L\,[f(x)\;;s\,]$$

$$-\frac{f^{(2)}(0)}{1-q} - \frac{s\,[a]}{(1-q)^2}\,f^{(1)}(0) - \frac{s^2\,[a]\,[a-1]}{(1-q)^3}\,f(0)$$

and, in general, this leads to our required result.

Theorem 3 (The p-Integration Theorem).

If ${}^{a}_{q}L\,[f(x)\,;s\,] = g\,(s)$, *then*

$${}^{a-1}_{q}L\left[\;\overset{x}{\underset{0}{S}} f\,(t)\;d\,(q,t)\;;s\;\right] = \frac{1-q}{s\,[a]}\;g\,(s),$$

where $f(x)$ is regular and $f(0) = 0$.

Proof. Setting up the function

$$F(x) = \frac{1}{1-q} \int_0^x f(t)\, d(q, t),$$

we have

$$F^{(1)}(x) = f(x),$$

where we may suppose that $f(0) = 0$.

Thus

$${}^{a}_{q}L\,[F^{(1)}(x); s] = {}^{a}_{q}L\,[f(x); s] =$$

$$\frac{1}{1-q}\,\{ s\,[a]\ {}^{a-1}_{q}L\,[F(x); s] - F(0) \},$$

which gives, because $F(0) = 0$, that

$${}^{a-1}_{q}L\,[F(x); s] = \frac{1-q}{s\,[a]}\ {}^{a}_{q}L\,[f(x); s],$$

since $f(x)$ is regular.

This proves the theorem.

Theorem 4 (The Addition Theorem).

If ${}^{a}_{q}L\,[f_r(x); s] = g_r(s)$, then

$${}^{a}_{q}L\ \sum_{r=0}^{m} f_r(x); s\,] = \sum_{r=0}^{m} g_r(s),$$

Provided that any one of the following conditions holds:

(a) m is finite, (b) m is infinite and

(i) $\sum_{r=0}^{\infty} | f_r (xq^j) |$ is covergent for each $x_0 q^j$, x_0 being fixed, and

for every $j = 0, 1, 2, \ldots,$

(ii) $| q |^j | f_r (xq^j) | = O (| h |^j), j > j_0, | h | < 1.$

The proof of this theorem is obvious.

Theorem 5. If ${}_q^aL [f(x) ; s] = g (s)$, then

$$ {}_q^{a+1} L [f(x)/x ; s] = \frac{(1 - q^{a+1})}{1 - q} \underset{qs}{\overset{\infty}{S}} g (t) \, d(q , t) , $$

provided the generalized q–Laplace transform of $f(x)/x$ exists and the corresponding series converges absolutely.

Proof. Let us consider the q–integral

$$ \frac{1}{1-q} \underset{qs}{\overset{\infty}{S}} g (x) \, d (q , x). \tag{3. 5} $$

Making use of the definition (1. 1), (3. 5) becomes equal to

$$ \frac{1}{(1-q)^2} \sum_{qs}^{\infty} \left[\underset{0}{\overset{x^{-1}}{S}} (1 - qxy)_a f(y) \, d (q , y) \right] d (q , x). $$

As a double q–sum, it can be written as

$$ (1 - q)_a \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} q^i \frac{(1 - q^{a+1})_i}{(q)_i} f (q^{i+j-1} s^{-1}) \tag{3. 6} $$

$$ = (1 - q)_a \sum_{r=0}^{\infty} q^r f (s^{-1} q^r) \sum_{j=0}^{r} \frac{(1 - q^{a+1})_{r-j} \, q^{-j}}{(1 - q)_{r-j}} $$

$$= (1-q)_a \sum_{r=0}^{\infty} \frac{q^r (q^{a+1})_r}{(q)_r} f(s^{-1} q^r) \sum_{r=0}^{r} \frac{(q^{-r})_j \, q^{-(a+1)j}}{(q^{-a-r})_j}$$

$$= (1-q)_a \sum_{r=0}^{\infty} \frac{(q^{a+2})_r}{(q)_r} f(s^{-1} q^r)$$

$$= \frac{1}{(1-q)(1-q^{a+1})} \mathop{S}_{0}^{s^{-1}} (1-q s x)_{a-1} [f(x)/x] \, d(q, x)$$

$$= \frac{1}{(1-q^{a+1})} \; {}^{a+1}_{q}L \, [f(x)/x \, ; s] \, ,$$

by virtue of the basic analogue of the well–known Gauss theorem and (1. 2).

The interchange in the order of q–summations in (3. 6) shall be permissible provided the inner series converges absolutely in addition to the existence of the generalized q–Laplace transform of $f(x)$.
Hence the theorem is established.

A result similar to the one given in Theorem 5 can be obtained by muliplying the to sides of Theorem 1 (i) by $1/p(1-q)$ and q–integrating over the interval $(1, \infty)$ with respect to p. We then have the left–hand side of Theorem 1 (i) equal to

$$\frac{1}{1-q} \mathop{S}_{1}^{\infty} \frac{1}{p(1-q)} \left[\mathop{S}_{0}^{s^{-1}} (1-qst)_a f(pt) \, d(q, t) \right] d(q, p)$$

$$= (1-q)_a s^{-1} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(q^{a+1})_j \, q^j}{(q)_j} f(q^{j-i} s^{-1}) \qquad (3.\ 7)$$

$$= (1-q)_a \, s^{-1} \sum_{j=0}^{\infty} \frac{q_j \, (q^{a+1})_j}{(q)_j} \sum_{i=1}^{\infty} f(q^{j-i} s^{-1})$$

$$= \frac{1}{1-q} \overset{s^{-1}}{\underset{0}{S}} (1 - qst)_a \left[\frac{1}{1-q} \overset{\infty}{\underset{t}{S}} \{ f(x)/x \} \, d(q, x) \right] d(q, t)$$

$$= \left[{}_q^a L \; \frac{1}{1-q} \overset{\infty}{\underset{t}{\Sigma}} \{ f(x)/x \} \, d(q, x) \; ; s \right], \tag{3.8}$$

on interchanging the q–sum and using the definition (1. 1).

The interchange in the order of q–sums in (3. 7) is assured if the q–sum

$\sum_{j=0}^{\infty} | q^j f(q^j/s) |$ converges, i. e., the generalized q–Laplace

transform of $f(x)$ exists for all values of s and provided that $|q^{j^2} f(q^{-j}/s)| = O(|h|^j)$, for every fixed j and $|h| < 1$.

Next, the right–hand side of Theorem 1 (i) is equal to

$$\frac{1}{1-q} \overset{\infty}{\underset{1}{S}} p^{-2} \, g(ds/p) \, d(q, p)$$

$$= \sum_{j=1}^{\infty} q^j g(sq^j) = \frac{1}{s(1-q)} \overset{qs}{\underset{0}{S}} g(y) \, d(q, y). \tag{3.9}$$

Comparing (3. 8) and (3. 9), we get the result :

Theorem 6. If ${}_q^a L [f(x) \; ; s] = g(s)$, then

$${}_q^a L \left[\frac{1}{1-q} \overset{\infty}{\underset{x}{S}} [f(y)/y] \, d(q, y) \; ; s \right] = \frac{1}{s(1-q)} \overset{qs}{\underset{0}{S}} g(x) \, d(q, x),$$

provided that (i) the q–integrals involved exist, (ii) $g(s)$ exists for all s and (iii) $| q^{j^2} f(q^{-j}/s) | = O(|h|^j)$, for every fixed j and $|h| < 1$.

Yet another correspondence of the type obtained in Theorem 6 can be had by combining Theorems 3 & 5. We have, in fact, the following result :

Theorem 7. If ${}_q^aL\,[f(x)\,;\,s] = g(s)$, then

$$ {}_q^{a-1}L\left[\frac{1}{1-q}\overset{x}{\underset{0}{S}}\{f(t)/t\}\,d(q,t)\,;\,s\right] = \frac{1-q}{s}\overset{\infty}{\underset{qs}{S}} g(x)\,(q, dx), $$

provided $f(x)/x$ is supposed regular and $[f(x)/x]_{x=0} = 0$.

Proof. Let us suppose that

$$ F(x) = \frac{1}{1-q}\overset{x}{\underset{0}{S}}\frac{f(t)}{t}\,d(q,t), $$

so that $\theta F(x) \equiv F^{(1)}(x) = f(x)/x$, provided $F^{(1)}(0) = 0$.

Now applying Theorem 3 to $\overset{x}{\underset{0}{S}}[f(t)/t]\,d(q,t)$, for $n = 1$, we receive

$$ {}_q^aL\left[\frac{f(x)}{x}\,;\,s\right] = \frac{s[a]}{1-q}\,{}_q^{a-1}L\left[\frac{1}{1-q}\overset{x}{\underset{0}{S}}\frac{f(t)}{t}\,d(q,t)\,;\,s\right], $$

if $\qquad F^{(1)}(0) = 0.$

The proof is now readily completed in view of Theorems 5.

We remark in passing that Theorems 1 to 7 for the transform (1. 1) just established reduee to Rules 1 to 7 due to Abdi [*1*, pp. 392–396] for Hahn's q–Laplace transform, when $a \to \infty$.

4. APPLICATIONS. In this section, we give certain images of some functions by employing Theorem 4 for the generalized q–Laplace transform (1. 1)

$f(x)$	$g(s) = {}_q^{a}L\,[\,f(x)\,;\,s]$
1. x^{k-1}	$(1-q)_a\, s^{-k}\, {}_1\phi_0\,(q^{a+1}\,;\,q^k)$, $\mathrm{Re}(k) > 0.$
2. $x^{k-1}\, e_q\,(bx)$	$\dfrac{(1-q)_a}{s^k(1-q^k)_{a+1}}\; {}_2\phi_1\left(\begin{matrix} q^k, 0;\ b/s \\ q^{k+a+1} \end{matrix}\right)$, $\mid b \mid < \mid s \mid$, $\mid b \mid < 1/\mid x \mid$, $\mathrm{Re}\,(k) > 0.$
3. $x^{k-1}\, E_q\,(bx)$	$\dfrac{(1-q)_a}{s^k(1-q^k)_{a+1}}\, {}_1\phi_1\,[q^k\,;\,q^{k+a+1}\,;\,b/s]$, $\mathrm{Re}(k) > 0.$
4. $x^{k-1}\, {}_n\phi_{n-1}\left(\begin{matrix} q^{(c_n)}; \\ q^{(d_{n-1})} \end{matrix}\ bx\right)$	$\dfrac{(1-q)_a}{s^k(1-q^k)_{a+1}}\; {}_{n+1}\phi_n\left(\begin{matrix} q^{(c_n)}, & q^k & \\ q^{(d_{n-1})}, & q^{(k+a+1)} & ;b/s \end{matrix}\right)$ $\mid b \mid < [\,\mid s \mid, 1/\mid x \mid\,]$, $\mathrm{Re}(k) > 0.$
5. $x^{k-1}\, {}_{n-p}\phi_{n-1}\left[\begin{matrix} q^{(c_{n-p})} & ;\,bx \\ q^{(p_{n-1})} & \end{matrix}\right]$	$\dfrac{(1-q)_a}{s^k(1-q^k)_{a+1}}\; {}_{n-p+1}\phi_n\left[\begin{matrix} q^{(c_{n-p})}, & q^k & ;\ b/s \\ q^{(d_{n-1})}, & q^{k+a+1} & \end{matrix}\right]$, $\mathrm{Re}(k) > 0,\ p = 0, 1, 2, \ldots, n.$

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