Jñānābha, Vol. 12, 1982

# A STUDY OF A GENERAL INTEGRAL TRANSFORM WITH SYMMETRIC KERNEL

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( *Received : March 12, 1982; Revised : May 2, 1982* )

**Abstract**

In this paper we first establish two theorems involving integral transforms whose kernels are sufficiently general in character. Three recent theorems, which are quite general and of interest in themselves follow as special cases of our theorems. Next we evaluate an interesting integral with the application of one of the corollaries of the theorems. The findings of this paper are sufficiently general in nature and are capable of unifying a large number of similar results lying scattered in the literature and obtained from time to time by several authors. The results of this paper can also be used for evaluating new integrals.

## 1 Introduction

This work is in continuation of the earlier work by the author [4]. Herein we not only generalize one of the main results of our previous paper referred above, but in turn also unify and extend the two very general theorems obtained earlier by Buschman [1] and Gupta [8]. Several other theorems lying scattered in the literature also follow as special cases of our theorems. An interesting integral has been evaluated as application of one of the corollaries of the main theorem.

The following definitions and results will be used in the sequel.

(a) (i) **Laplace transform** :

$$L\{f(x);s\}=\int_0^\infty e^{-sx}f(x)\,dx \tag{1.1}$$

(ii) **H-function transform:**

$$H\{f(x);s\}=\int_0^\infty H_{p,q}^{m,n}\left[sx\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]f(x)\,dx \tag{1.2}$$

where the $H$- function occurring on the right hand side of (1.2) is the well- known $H$- function of Fox (See Gupta and Jain [7] ).

(b) **Three known results:**

(i) $$L\left\{x^c H_{p,q}^{m,n}\left[zx^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right];s\right\}$$

$$=s^{-c-1}H_{p,q+1}^{m+1,n}\left[zx^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(c+1,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] \tag{1.3}$$

where $\sigma>0$, Re $(s)>0$ Re $[c+1-\sigma\left(\frac{a_j-1}{\alpha_j}\right)]>0$,

$j=1,\dots,m$,

$$\mu=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j>0,$$

$$\lambda'=\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j>0,\ |\arg z|<\frac{\lambda'\pi}{2}$$

The above result is given by Gupta [5, p. 100, eq. (8)]

(ii) $$L\left\{e^{-\beta x}x^{2\nu(\lambda+1)}H_{p,q+1}^{m,n}\left[zx^{2\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(-2\nu(\lambda+1),2\nu)\end{matrix}\right.\right];s\right\}$$

$$= (s+\beta)^{-2\nu}(\lambda+1)^{-1} H_{p,q}^{m,n}\left[ z\ (s+\beta)^{-2\nu} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \tag{1.4}$$

provided that,

$$Re\,(s+\beta) > 0\,,\ \nu > 0\,,\ Re\left\{ \nu\,(\lambda+1+\frac{b_j}{\beta_j}) + \tfrac{1}{2} \right\} > 0\,,$$

$$j = 1\,, \dots, m\,,$$

$$\mu' = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{p} \alpha_j > 0\,,\ |\arg z| > \frac{\lambda' \pi}{2}\,,$$

$$\lambda' = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j - 2\nu + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j > 0.$$

This result can easily be obtained with the help of (1. 3) and using a well – known property of the Laplace transform [3, p. 129, eq. (5) ].

(iii) A special case of the result given by Mittal and Gupta [9, p. 122, eq. (2. 2) ].

$$\int_0^\infty e^{-sx}\, x^{\lambda-1} H_{p,q}^{m,n}\left[ \beta x^h \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right]$$

$$H_{p',q'}^{m',n'}\left[ \delta x^k \middle| \begin{matrix} (c_j, \gamma_j)_{1,p'} \\ (d_j, \delta_j)_{1,q'} \end{matrix} \right] dx$$

$$= s^{-\lambda} H_{1,0:p,q:p',q'}^{0,1:m,n;m',n'}\left[ \begin{matrix} \beta/s^h \\ \delta/s^k \end{matrix} \middle| \begin{matrix} (1-\lambda\,; h\,, k) : \\ - \quad : \end{matrix} \right.$$

$$\left. \begin{matrix} (a_j, \alpha_j)_{1,p}\,;\ (c_j, \gamma_j)_{1,p'} \\ (b_j, \beta_j)\,,_{1,q}\,;\ (d_j, \gamma_j)_{1,q'} \end{matrix} \right]$$

provided that $\operatorname{Re}(s) > 0$, $\operatorname{Re}[\lambda + h\left(\frac{b_j}{\beta_j}\right) + k\left(\frac{d_i}{\delta_i}\right)] > 0$

$j = 1, \dots, m$, $i = 1, \dots m'$, $h$, $k$ are positive quantities,

$$\mu_1 = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0, \mu_2 = \sum_{j=1}^{q'} \delta_j - \sum_{j=1}^{p'} \gamma_j > 0,$$

$$\lambda_1 = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j > 0, |\arg \beta| < \tfrac{1}{2} \lambda_1 \pi$$

$$\lambda_2 = \sum_{j=1}^{m'} \delta_j - \sum_{j=m'+1}^{q'} \delta_j + \sum_{j=1}^{n'} \gamma_j - \sum_{j=n'+1}^{p'} \gamma_j > 0, |\arg \delta| < \tfrac{1}{2} \lambda_2 \pi.$$

The function occurring in the r. h. s. of (1. 5) stands for the $H$-function of two variables defined by Mittal and Gupta [9., *p.* 117], and we employ the contracted notation introduced by Srivastava and panda [10, *p.* 266, eq. (1. 5), *p.* 267, eq. (1. 11) ] to represent that function.

**2. Theorem** I. *If*

$$h_1(p; (b, c)) = T_1\{k(t) F[g(t)][U(t-b) - U(t-c)]; p\}$$

$$= \int_0^\infty K_1(pt) k(t) F[g(t)][U(t-b) - U(t-c)] dt \tag{2. 1}$$

*and*

$$h_2(p; (\beta^{-1/\sigma}, \gamma^{-1/\sigma})] = T_2\{F(t^{-\sigma})[U(t-\beta^{-1/\sigma})$$

$$- U(t-\gamma^{-1/\sigma})]; p\}$$

$$= \int_0^\infty K_2(pt)\, F(t^{-\sigma})\, [U(t - \beta^{-1/\sigma}) - U(t - \gamma^{-1/\sigma})]\, dt \tag{2. 2}$$

*then*

$$h_1(p\,;(b\,,c)) = \sigma \int_0^\infty \theta(p\,,u)\, h_2(u\,;(\gamma^{-1/\sigma}\,,\beta^{-1/\sigma}))\, du \tag{2. 3}$$

*where*

$$\phi(p\,,y) = T_2\{\theta(p,u)\,;y\} \tag{2. 4}$$

*and*

$$\phi(p\,,y) = K_1\{p\, h(y^{-\sigma})\}\, k\, [h(y^{-\sigma})]\, |h'(y^{-\sigma})|\, y^{-\sigma-1} \tag{2. 5}$$

$\sigma$ is a nonzero real number, $U(t-a)$ denotes the Heaviside (Unit step) function and $K_1(pt)$ and $K_2(pt)$ are kernels of the transforms $T_1$ and $T_2$. The functions $k$, $g$ and the inverse function $h = g^{-1}$ are single valued, analytic, real on $(0\,,\infty)$ and $g$ is monotonic on the sub interval $(b\,,c)$ such that $g(b) = \beta$ and $g(c) = \gamma$ $(\beta < \gamma)$. Also the integrals involved in (2. 1) to (2. 4) are absolutely convergent,

**Proof.** On making the substitutions $t = h(x)$ and then $x = y^{-\sigma}$ in the integral occurring in the right hand side of (2. *1*) and using the relation (2. 5), the integral takes the following from :

$$\sigma \int_{\gamma^{-1/\sigma}}^{\beta^{-1/\sigma}} \phi(p\,,y)\, F(y^{-\sigma})\, dy \tag{2. 6}$$

Now, we introduce the fuuction $\theta(p\,,u)$ given by (2. 4) in the above integral (2. 6) and interchange the order of integration, which is permi–ssible under the coditions stated with the theorem. Finally, on interpr–eting the result thus obtained with the help of (2. 2), we easily arrive at the desired result (2. 3).

Following the same lines, a similar result can be obtained for the case when $\beta > \gamma$.

If in Theorem I, the function $g$ is so chosen that $g(0) = 0$ and $g(\infty) = \infty$, we arrive after a little simplification, at the following interesting form of the theorem under appropriate conditions easily obtainable from Theorem I,

*Theorem I (a) If*

$$h_1(p) = T_1\{k(t)\, F[g(t)]\,;\, p\}$$

$$= \int_0^\infty K_1(pt)\, k(t)\, F[g(t)]\, dt \tag{2. 7}$$

*and*

$$h_2(p) = T_2\{F(t^{-\sigma})\,;\, p\}$$

$$= \int_0^\infty K_2(pt)\, F(t^{-\sigma})\, dt \tag{2. 8}$$

*then*

$$h_1(p) = |\sigma| \int_0^\infty \theta(p, u)\, h_2(u)\, du \tag{2. 9}$$

*where*

$$\phi(p, y) = T_2\{\theta(p, u)\,;\, y\} \tag{2. 10}$$

*and*

$$\phi(p, y) = K_1\{p h(y^{-\sigma})\}\, k[h(y^{-\sigma})]\, |h'(y^{-\sigma})|\, y^{-\sigma-1} \tag{2. 11}$$

$\sigma$ is a non zero real number.

In Theorem I, we take $T_1$ and $T_2$ to be the $H$-function transform and the Laplace transform defined by (1.2) and (1. 1) respectively, $\sigma = 1$, $k(t) = t^{\lambda}$ and $g(t) = (t^{-1/2\nu} - \beta)^{-1}$, we get on using the resuls (1. 4) and after a little simplification, a new and interesting result given below:

*Corollary 1. If*

$$h_1\,(p\,;(0\,,\beta^{-2\nu}\,))=H\,\{\,t^{\lambda}\,f\,[\,(t^{-1/2\nu}-\beta)^{-1}\,]$$

$$[1-U\,(t-\beta^{-2\nu})\,]\,;p\,\} \tag{2. 12}$$

*and*

$$h_2\,(p)=L\,\{F\,(1/t)\,;\,p\} \tag{2. 13}$$

*then*

$$h_1\,(p\,;(0\,,\beta^{-2\nu}\,))=(2\nu)\int_0^{\infty}e^{-\beta u}\;u^{2\nu\,(\lambda+1)}$$

$$H^{m',\,n'}_{p',\,q'+1}\left[\,pu^{2\nu}\,\middle|\,\begin{matrix}(a_j\,\alpha_j)_{1,p'}\\(b_j\,,\beta_j)_{1,q'}\,(-\,2\nu\,(\lambda+1)\,,\,2\nu)\end{matrix}\right]h_2\,(u)\;du \tag{2. 14}$$

provided that $\beta>0\,,\nu>0\,,\;\mathrm{Re}\left\{\nu\;(\lambda+1+\frac{b_j}{\beta_j})+\frac{1}{2}\right\}>0,$

$$j=1\,,\,\dots\,,m'$$

$$\mu=\sum_{j=1}^{q'}\beta_j-\sum_{j=1}^{p'}\alpha_j>0\,,\;\lambda'=\sum_{j=1}^{m'}\beta_j-\sum_{j=m'+1}^{q'}\beta_j-2\nu$$

$$+\sum_{j=1}^{n'}\alpha_j-\sum_{j=n'+1}^{p'}\alpha_j>0\,,\;|\arg p\,|<\frac{\lambda'\pi}{2}$$

and the integral occurring on the r. h. s. of (2, 14) is absolutely convergent .

## 3. SPECIAL CASES

(i) If in Theorem I and I (a), we take $\sigma=-1$ and both the tran–

sforms $T_1$ and $T_2$ to be Laplace transforms, we easily get after a little simplification, the two interesting theorems obtained earlier by Buschman [ 1, p. 37 Theorem 2 and 2, p. 957, Theorem 1 ].

(ii) If in Theorem I (*a*), we take g $(t) = t$, we easily arrive at the theorem recently established by the author [ 4, Theorem II].

(iii) If in Theorem I (*a*), we take g $(t) = t$ and $k\ (t) = t^{\rho/\sigma - 1}$, we obtain a theorem obtained earlier by Gupta [8, Theorem II].

(iv) On taking both $T_1$ and $T_2$ to be Laplace transforms, $\sigma = -1$, k $(t) = t^{\nu - 1}$ and g $(t) = (at^{-1} - 1)$ in Theorem I, we get (by virtue of known results [3, p. 185, eq. (30). p. 129, eq. (5) ] ) after a little simplification a result, wich is in essence similar to other interesting result obtained recently by Buschman [1, p. 41, eq. (4. 4) in the limiting case when $b \to 0$ ].

## 4 APPLICATIONS

On making suitable choice of transforms $T_1$, $T_2$, $\sigma$, g (t), k (t) and F (t), Theorem I and I (a) are capable of yielding a number of new and known theorems given from time to time by several authors, lying scattered in the literature and as illustrated by a few examples in the earlier section.

Again with the help of Corollary I, we can evaluate certain interesting and general integrals. To illustrate, we shall evaluate an infinite integral with the application of corollary I. We take

$$F\ (1/t) = t^c\, H^{M,N}_{P,Q}\left[\ s/t \ \middle|\ \begin{matrix} (c_j\ ,\ \gamma_j)\ _{1,P} \\ (b_j\ ,\ \delta_j)\ _{1,Q} \end{matrix}\right] \tag{4. 1}$$

Then on applying the result (1. 3), we get

$$h_2\ (p) = p^{-c-1}\, H^{M+1,N}_{P,Q+1}\left[\ sp \ \middle|\ \begin{matrix} (c_j\ ,\ \gamma_j)\ _{1,P} \\ (c+1\ ,\ 1)\ ,\ (b_j\ ,\ \delta_j)\ _{1,Q} \end{matrix}\right] \tag{4. 2}$$

which is valid under the conditions stated with (1. 3).

Further on substituting this value of $h_2$ (p) in the r. h. s. of (2. 14) and evaluating it with the help of result (1. 5) we obtain

$$h_1 (p ; (0 , \beta^{-2\nu}) ) = 2_\nu \beta^{c-2\nu(\lambda+1)} H^{0, 1 : M+1, N ; m', n'}_{1, 0 ; P, Q+1; p', q'+1} \left[ \begin{matrix} s/\beta \\ p/\beta^{2\nu} \end{matrix} \middle| \begin{matrix} (1+c-2_\nu(\lambda+1) ; 1, 2_\nu) : \\ \text{———} : \end{matrix} \begin{matrix} (c_j, \gamma_j)_{1,P} ; (a_j, \alpha_j)_{1,p'} \\ (c+1, 1), (d_j, \delta_j)_{1,Q} ; (b_j, \beta_j)_{1,q} (-2_\nu(\lambda+1), 2_\nu) \end{matrix} \right] \tag{4. 3}$$

Now, on comparing the equations (2. 12) and (4. 3) and changing the variable of integration slightly in (2.12), we easily arrive at the following interesting integral ;

$$\int_0^\infty t^c (\beta+t)^{-2\nu(\lambda+1)-1} H^{M, N}_{P, Q} \left[ s/t \middle| \begin{matrix} (c_j, \gamma_j)_{1,P} \\ (b_j, \delta_j)_{1,Q} \end{matrix} \right] H^{m', n'}_{p', q'} \left[ p(\beta+t)^{-2\nu} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p'} \\ (b_j, \beta_j)_{1,q'} \end{matrix} \right] dt$$

$$= \beta^{c-2\nu(\lambda+1)} H^{0, 1 : M+1, N ; m', n'}_{1, 0 : P, Q+1 : p' q'+1} \left[ \begin{matrix} s/\beta \\ p/\beta^{2\nu} \end{matrix} \middle| \begin{matrix} (c+1-2_\nu(\lambda+1) ; 1, 2_\nu) : \\ \text{——} : \end{matrix} \begin{matrix} (c_j, \gamma_j)_{1,P} ; (a_j, \alpha_j)_{1,p'} \\ (c+1, 1), (d_j, \delta_j)_{1,Q} ; (b_j, \beta_j)_{1,q'} (-2_\nu(\lambda+1), 2_\nu) \end{matrix} \right] \tag{4. 4}$$

provided that

$$\beta > 0\ ,\ \nu > 0,\ \text{Re}\ \left( \frac{c_j - 1}{C_j} - 1 - c \right) < 0\ ,$$

$$\text{Re}\ \{ 2\nu\ (\lambda + 1 + \frac{b_i}{\beta_i}) + \frac{d_k}{\delta_k} - c \} > 0$$

$$i\ , \ldots ,\ m'\ ,\ j = 1\ , \ldots ,\ N\ ,\ k = 1\ , \ldots ,\ M\ ,$$

$$\lambda_1 = \sum_{j=1}^{m'} \beta_j - \sum_{j=m'+1}^{q'} \beta_j - 2\nu + \sum_{j=1}^{n'} \alpha_j - \sum_{j=n'+1}^{p'} \alpha_j > 0\ ,$$

$$|\arg p| < \tfrac{1}{2} \lambda_1 \pi$$

$$\lambda_2 = \sum_{j=1}^{M} \delta_j - \sum_{j=M+1}^{Q} \delta_j + \sum_{j=1}^{N} \gamma_j - \sum_{j=N+1}^{P} \gamma_j > 0\ ,\ |\arg s| < \tfrac{1}{2} \lambda_2 \pi,$$

$$\mu_1 = \sum_{j=1}^{q'} \beta_j - \sum_{j=1}^{p'} \alpha_j > 0,\ \mu_2 = \sum_{j=1}^{Q} \delta_j - \sum_{j=1}^{P} \gamma_j > 0\ ,$$

If in the result (4. 4), we take $M = Q = 2$, $N = P = 0$, $d_1 = \nu'/2$ , $d_2 = -\nu'/2$ , $\delta_1 = \delta_2 = \frac{1}{2}$ and replace $s$ by $s/2$, one of the $H$–function occurring therin reduces to the well known modified Bessel function (by virtue of the known formula [6, p. 10, eq. (1. 4) ] ) and we obtain the following interesting integral under the conditions obtained easily from those stated with (4. 4).

$$\int_0^\infty t^c\ (\beta + t)^{-2\nu(\lambda+1)-1}\ K_{\nu'}\ (s/t)\ H_{p',\ q'}^{m',\ n'} \left[ p\ (\beta + t)^{-2\nu} \middle| \begin{matrix} (a_j\ ,\ \alpha_j)_{1,p'} \\ (b_j\ ,\ \beta_j)_{1,q} \end{matrix} \right] dt$$

$$= \tfrac{1}{4}\, \beta^{c-2\nu(\lambda+1)}\, H^{0,\,1\,:\,3,\,0\,;\,m',\,n'}_{1,\,0\,:\,0,\,3\,;\,p',\,q'+1} \left[ \begin{matrix} s/2\beta \\ p/\beta^{2\nu} \end{matrix} \,\middle|\, \begin{matrix} (c+1-2\nu(\lambda+1)\,;\,1,\,2\nu) : \\ - : \\ - \quad ; \quad (a_j,\,\alpha_j)_{1,p'} \\ (c+1,\,1),\,(\pm\nu'/2,\,\tfrac{1}{2})\,;\,(b_j,\,\beta_j)_{1,q'},\,(-2\nu(\lambda+1),\,2\nu) \end{matrix} \right] \tag{4.5}$$

**Acknowledgements**

The author is grateful to Dr. K. C. Gupta for his help and valuable guidance during the preparation of this paper and to Prof. H. M. Srivastava for his useful suggestions for the improvement of this paper which led to a better presentation.

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