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CERTAIN RESULTS ON A MULTIPLE INTEGRAL TRANSFORM

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In this paper, we establish a uniqueness theorem and three chains for a multiple integral transform which is a special case of one of the two multidimensional integral transformations introduced and studied by H. M. Srivastava and R. Panda [8]. Since the kernel of our transform is quite general in nature, the corresponding results for the multiple Whittaker, Bessel, and Laplace transforms, and other in egral transforms of several variables, can be obtained as special cases of our main findings. The chains established earlier by S. P. Goyal [2], S. K. Bose [1], R. N. Kesarwani [4], P. N. Rathie [7], U. C. Jain [3], and otheres follow as particular cases of our results.

1. Introduction and Notations

(a) **The Multiple Integral Transform.** Here we shall study the following integral transform which is a special case of one of the two multidimensional integral transformations introduced and studied by Srivastava and Panda ([8], p. 121, Eq. (1. 15)) :

$$\phi(s_1 \ldots, s_r) = s_1 \ldots s_r \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ H^{4,0}_{2,4}\left[\eta_i (s_i\, x_i)^{\sigma_i} \,\middle|\, \begin{matrix} (a_i, \alpha_i), (b_i\, \beta_i) \\ (c_i, \gamma_i), (d_i, \delta_i), (e_i, E_i), (f_i, F_i) \end{matrix} \right] \right\} f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r \quad \ldots (1.1)$$

where $\eta_i > 0, \sigma_i > 0$ $(i = 1, \ldots, r)$, $H_{p,q}^{m,n}[x]$ is fox's H–function [6], and the multiple integral on the right–hand side of (1.1) is assumed to be absolutely convergent.

We shall represent (1.1) by the following contracted notation:

$$\phi(s_1; \ldots, s_r) = \Phi\left[f(x_1, \ldots, x_r); s_1, \ldots, s_r ; \begin{matrix} (a_i, \alpha_i), (b_i, \beta_i) \\ (c_i, \gamma_i), (d_i, \delta_i) (e_i, E_i), (f_i, F_i) \end{matrix} \right] \quad \ldots (1.2)$$

The existence conditions for the transform(1.1) are easily obtainable from the conditions of existence of the general multidimensional H-function transformation given by Srivastava and Panda ([8], pp. 119—120).

(b) **Symbols Used.**

(i) $(\triangle (m, a), \sigma)$ abbreviates the m–parameter sequence

$$\left(\frac{a}{m}, \sigma\right), \left(\frac{a+1}{m}, \sigma\right), \ldots, \left(\frac{a+m-1}{m}, \sigma\right)$$

(ii) $\{(a_i^{(n)}, \alpha_i^{(n)})\}$ abbreviates the n–parameters sequence

$$(a_i', \alpha_i'), (a_i^{(2)}, \alpha_i^{(2)}), \ldots, (a_i^{(n)}, \alpha_i^{(n)})$$

(iii) $[(\triangle (\delta_\alpha, a_i^{(n)}), \alpha_i^{(n)})]$ abbreviates the $(n-1)$– parameters sequence

$$(\triangle (1, a_i^{(2)}), \alpha_i^{(2)}), (\triangle (2, a_i^{(3)}), \alpha_i^{(3)}), \ldots, (\triangle (2^{n-2}, a_i^{(n)}), \alpha_i^{(n)}).$$

Also, when there is no chance of confusion, we have used ( ), {( )} or [( )] to denote the parameters of similar types in the H–function.

2. In this section we shall establish a uniqueness theorem for the integral transform defined by (1.1). To prove this theorem, we shall require the following result :

Lemma *If*

$$\int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{r} \{ (x_i)^{\rho_i} H_{2,4}^{4,0} \left[\eta_i (s_i x_i)^{\sigma_i} \middle| \begin{matrix} (a_i, \alpha_i), (b_i, \beta_i) \\ (c_i, \gamma_i), (d_i, \delta_i), (e_i, E_i), f_i, F_i) \end{matrix} \right] f(x_1, .., x_r)\, dx_1 .. dx_r \equiv 0, \qquad \ldots (2.1)$$

then

$$f(x_1, \ldots, x_r) \equiv 0, \qquad \ldots (2.2)$$

provided that

(i) $f(x_1, \ldots, x_r)$ *is a continuous function for* $x_i > 0$ $(i = 1, \ldots, r)$

(ii) $\eta_i > 0$, $\sigma_i > 0$, $| \arg s_i | < \dfrac{U_i}{2\sigma_i} \pi$

where

$$U_i = \gamma_i + \delta_i + E_i + F_i - \alpha_i - \beta_i, \qquad \ldots (2.3)$$

and

(iii) *the integral involved in (2.1) is assumed to converge absolutely.*

Proof. On multiplying (2.1) by

$$\prod_{i=1}^{r} \{ s_i^{\lambda_i - 1} H_{2, q_i + 4}^{q_i, 2} \left[\eta_i s_i^{\sigma_i} \left(\frac{z_i}{q_i} \right)^{u_i q_i \sigma_i} \middle| \begin{matrix} (1 - a_i - \frac{\lambda_i}{\sigma_i} a_i, a_i) \quad (1 - b_i - \frac{\lambda_i}{\sigma_i} \beta_i, \beta_i) \\ (\Delta (q_i, 0), \sigma_i u_i), (1 - c_i - \frac{\lambda_i}{\sigma_i} \gamma_i, \gamma_i), (1 - d_i - \frac{\lambda_i}{\sigma_i} \delta_i, \delta_i), (\ ), (\ ) \end{matrix} \right] \}, \ldots (2.4)$$

$$\Big[\text{provided that } q_i u_i \sigma_i - U_i > 0, | \arg z_i | < \left\{ 1 - \frac{U_i}{u_i q_i \sigma_i} \right\} \frac{\pi}{2}, \text{ and } Re(\lambda_i) + \sigma_i \mu_i > 0, \text{ where}$$

$$\mu_i = min\ Re\ (c_i/\gamma_i, d_i/\delta_i, e_i/E_i, f_i/F_i) \} (i = 1,\dots, r) \Big] \quad \dots (2\ 5)$$

integrating it with respect to s_i between 0 and ∞, changing the order of integration, evaluating the s_i — integral with the help of a known result ([6], *p.* 34, *Eq.* (2. 6. 8)), and then applying well-known properties for the *H*-function ([6], *p. 4*, Eqs. (1. 2. 1) and (1. 2. 3)), we find that

$$\int_0^\infty \cdots \int_0^\infty f(x_1, \dots, x_r) \prod_{i=1}^{r} \left\{ (x_i)^{\rho_i - \lambda_i} H^{q_i, 0}_{0, q_i} \left[\left(\frac{z_i}{q_i} \right)^{u_i q_i} x_i^{-1/u_i} \; (\triangle (q_i, 0), 1) \right] \right\} dx_1 \dots dx_r \equiv 0, \quad \dots (2.6)$$

Again using the known results ([6], *p.* 5, Eq. (1. 2. 7); *p.* 145, §46) in (2. 6), it reduces to

$$\int_0^\infty \cdots \int_0^\infty f\left(x_1^{-u_1/q_1}, \dots, x_r^{-u_r/q_r} \right) \prod_{i=1}^{r} \left\{ x_i^{(-u_i q_i - \rho_i + \lambda_i - 1)/u_i q_i} exp(-z_i x_i) \right\} . dx_1 \dots dx_r \equiv 0, \quad \dots (2.7)$$

which on using Lerch's theorem ([5], p. *339*) giving us (2. 2)

Uniqueness Theorem. *If* $f_j(x_1, \dots, x_r)$ $(j = 1, 2)$ *are continuous functions in* $x_i \geqq 0$ $(i = 1, \dots, r)$ *and*

$$\Phi\left[f_1(x_1,\ldots,x_r);\, s_1,\ldots,s_r);\ \begin{matrix}(a_i,\alpha_i),(b_i,\beta_i)\\(c_i,\gamma_i),(d_i,\delta_i),(e_i,E_i),(f_i,F_i)\end{matrix}\right]$$

$$=\Phi\left[f_2(x_1,\ldots,x_r);\, s_1,\ldots,s_r);\ \begin{matrix}(a_i,\alpha_i),(b_i,\beta_i)\\(c_i,\gamma_i),(d_i,\delta_i)(e_i,E_i),(f_i,F_i)\end{matrix}\right] \quad \ldots (2.8)$$

then

$$f_1(x_1,\ldots,x_r) \equiv f_2(x_1,\ldots,x_r) \quad \ldots (2.9)$$

provided that the Φ*– transforms of* $f_j((x_1,\ldots,x_r)$ $(j=1,2)$ *defined by (1. 1) exist.*

The proof of the uniqueness theorem is a direct consequence of the aforementioned Lemma.

Next, we establish three interesting chains for the integral transform defined by (1. 1).

3. Theorem 1. *If*

$$s_1^{\rho_1}\ldots s_r^{\rho_r}\,(s_1^{\xi_1},\ldots,s_r^{\xi_r}) = \Phi\left[h(x_1,\ldots,x_r;\, s_1,\ldots,s_r;\ \begin{matrix}(a_i,\alpha_i),(b_i,\beta_i)\\(c_i,\gamma_i),(d_i,\delta_i),(e_i,E_i),(f_i,F_i)\end{matrix}\right], \quad \ldots (3.1)$$

$$f(s_1,\ldots,s_r) = \Phi\left[x_1^{\lambda_1}\ldots x_r^{\lambda_r}\,\phi(x_1,\ldots,x_r);\, s_1,\ldots,s_r;\ \begin{matrix}(g_i,G_i),(h_i,H_i)\\(k_i,K_i),(l_i,L_i),(m_i,M_i),(n_i,N_i)\end{matrix}\right] \quad \ldots (3.2)$$

$$\phi(s_1,\ldots,s_r) = \Phi\left[x_1^{\nu_1'}\ldots x_r^{\nu_r'}\,f(x_1,\ldots,x_r);\, s_1,\ldots,s_r;\right.$$

$$\left.\begin{matrix} (a_i', \alpha_i'), (b_i', \beta_i') \\ (c_i', \gamma_i'), (d_i', \delta_i'), (e_i', E_i'), (f_i', F_i') \end{matrix}\right] \quad \ldots (3.3)$$

$$\phi_2 (s_1, \ldots, s_r) = \Phi \left[x_1^{\nu_1^{(2)}} \ldots x_r^{\nu_r^{(2)}} \phi_1 (1/x_1, \ldots, 1/x_r) ; s_1, \ldots, s_r ; \right.$$

$$\left.\begin{matrix} (a_i^{(2)}, \alpha_i^{(2)}), (b_i^{(2)}, B_i^{(2)}) \\ (c_i^{(2)}, \gamma_i^{(2)}), (d_i^{(2)}, \delta_i^{(2)}), e_i^{(2)}, E_i^{(2)}), (f_i^{(2)}, F_i^{(2)}) \end{matrix}\right] \quad \ldots (3.4)$$

... ...

... ...

and

$$\phi_n (s_1, \ldots, s_r) = \Phi \left[x_1^{\nu_1^{(n)}} \ldots x_r^{\nu_r^{(n)}} \phi_{n-1} (1/x_1, \ldots, 1/x_r); s_1, \ldots, s_r; \right.$$

$$\left.\begin{matrix} (a_i^{(n)}, \alpha_i^{(n)}), (b_i^{(n)}, B_i^{(n)}) \\ (c_i^{(n)}, \gamma_i^{(n)}), (d_i^{(n)}, \delta_i^{(n)}), (e_i^{(n)}, E_i^{(n)}) (f_i^{(n)}, F_i^{(n)}) \end{matrix}\right] \quad \ldots (3.5)$$

then

$$\phi_n (s_1, \ldots, s_r) = \prod_{i=1}^{r} \left\{ \frac{s_i^{v_i - A_n}}{\sigma_i^{n+2}} \xi_i (\eta_i)^{[(n-1)v_i - B_n - n - 2]/\sigma_i} \right\}$$

$$\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} \left\{ (x_i)^{-1} H_{4n+6,\, 2n+6}^{4,\, 4n+4} \left[\eta_i^{-(\xi_i + n - 1)/\sigma_i} (s_i\, x_i^{\xi_i})^{-1} \right| \right.$$

$$\left(1 - c_i - \frac{\gamma_i}{\sigma_i}, \frac{\xi_i \gamma_i}{\sigma_i}\right),$$

$$\left(k_i + [v_i + 1] \frac{K_i}{\sigma_i}, \frac{K_i}{\sigma_i} \right),$$

$$(\ ),(\ ),(\ ),\{(1-c_i^{(n)}-[A_n-v_i+1]\frac{\gamma_i^{(n)}}{\sigma_i},\frac{\gamma_i^{(n)}}{\sigma_i})\},$$

$$(\ ),(\ ),(\ ),\{(1-a_i^{(n)}-[A_n-v^i+1]\frac{\alpha_i^{(n)}}{\sigma_i},\frac{\alpha_i^{(n)}}{\sigma_i})\},$$

$$\{(\ )\},\{(\ )\},\{(\ )\},(g_i+[v_i+1]\frac{G_i}{\sigma_i},\frac{G_i}{\sigma_i}),(\ )\Big]\Big\}$$

$$\{(\ )\},(1-a_i-\frac{\alpha_i}{\sigma_i},\frac{\alpha_i\,\xi_i}{\sigma_i}),(\ )$$

$$h(x_1,\ldots,x_1)\,dx_1\ldots dx_r \qquad \ldots (3.6)$$

where

$$B_k=\sum_{j=1}^{k}A_j\,,\ A_k=\sum_{j=1}^{k}\gamma_i^{(j)}, \qquad \ldots (3.7)$$

and $v_i=\lambda_i-(\rho_i/\xi_i)\ (i=1,\ldots,r)$ ... (3.8)

Theorem 1 *is valid under the following (sufficient) conditions*

(i) $\min\left\{U_i, V_i, U_i^{(j)}\right\}>0,\ |\arg s_i|<\frac{\pi}{2\sigma_i}\min(U_i, V_i, U_i^{(j)}$

where U_i *is given by* (2.3),

$$V_i=K_i+L_i+M_i+N_i-G_i-H_i\ (i=1,\ldots,r), \qquad \ldots (3.9)$$

and

$$U_i^{(j)}=\gamma_i^{(j)}+\delta_i^{(j)}+E_i^{(j)}+F_i^{(j)}-\alpha_i^{(j)}-\beta_i^{(j)} \qquad \ldots (3.10)$$

$$(i=(1,\ldots,r\ ;j=1,\ldots,n)$$

(ii) $\eta_i>0,\ \sigma_i>0,\ Re\,(A_n-v_i)+$

$$\min\left[Re\left\{\frac{\sigma_i k_i+[v_i+1]K_i}{K_i},\ \frac{\sigma_i l_i+[v_i+1]L_i}{L_i}\right.\right.,$$

$$\frac{\sigma_i m_i + [v_i + 1] M_i}{M_i}, \frac{\sigma_i n_i + [v_i + 1] N_i}{N_i} \Bigg] +$$

$$\sigma_i \min \left[Re \left\{ \frac{c_i^{(n)}}{\gamma_i^{(n)}}, \frac{d_i^{(n)}}{\delta_i^{(n)}}, \frac{e_i^{(n)}}{E_i^{(n)}}, \frac{f_i^{(n)}}{F_i^{(n)}} \right\} \right] + 1 > 0 \ (i = 1, \ldots, r)$$

(iii) *the integrals involved in (3. 1) to (3. 6) are assumed to be absolutely convergent.*

Proof. Substituting the value of $\phi(x_1, \ldots, x_r)$ from (3. 1) in (3. 2), and interchanging the order of integration therein, we get

$$f(s_1, \ldots, s_r) = \prod_{i=1}^{r} \left(\frac{s_i}{\sigma_i^2 \, \eta_i^{1/\sigma_i}} \right) \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (x_i)^{-1}$$

$$\left\{ \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left((t_i)^{\lambda_i - \rho_i/\xi_i} H_{2,4}^{4,0} \left[\eta_i^{1/\sigma_i} t_i^{1/\xi_i} x_i \right| \right. \right.$$

$$\left. \begin{matrix} \left(a_i + \frac{\alpha_i}{\sigma_i}, \frac{\alpha_i}{\sigma_i} \right), (\ ) \\ \left(c_i + \frac{\gamma_i}{\sigma_i}, \frac{\gamma_i}{\sigma_i} \right), (\ ), (\ ), (\ ) \end{matrix} \right] H_{2,4}^{4,0} \left[\eta_i^{1/\sigma_i} s_i t_i \right|$$

$$\left. \left. \left. \begin{matrix} (g_i, G_i/\sigma_i), (\ ) \\ (k_i, K_i/\sigma_i), (\ ), (\ ), (\ ) \end{matrix} \right] dt_i \right) \right\} h(x_1, \ldots, x_r) \, dx_1 \ldots dx_r$$

... (3. 11)

Evaluating the t_i— integral with the help of a known result ([6], p. *34*, *Eq.* (*2. 6. 8.*)), and putting the value of $f(s_1, \ldots, s_r)$ so obtained in (3. 3), we find that

$$\phi_1(s_1, \ldots, s_r) = \prod_{i=1}^{r} \left\{ \frac{s_i \, \xi_i \, \eta_i^{-(v_i+2)}}{\sigma_i^2} \right\} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (x_i)^{-1}$$

$$\left\{ \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left(t_i^{\nu_i' - v_i} H_{6,6}^{4,4} \left[\eta_i^{(1-\xi_i)/\sigma_i} t_i x_i^{-\xi_i} \right| \right. \right.$$

$$\left(1 - c_i - \frac{\gamma_i}{\sigma_i}, \frac{\xi_i \gamma_i}{\sigma_i}\right), (\ ), (\ ), (\ ), (g_i + [v_i + 1]$$

$$\left(k_i + [v_i + 1] \frac{K_i}{\sigma_i}, \frac{K_i}{\sigma_i}\right), (\ ), \left(1 - a_i - \frac{a_i}{\sigma_i}, \frac{\xi_i\ a_i}{\sigma_i}\right),$$

$$\left.\left.\frac{G_i}{\sigma_i}, \frac{G_i}{\sigma_i}\right), \begin{matrix}(\ ) \\ (\ )\end{matrix}\right] H^{4,0}_{2,4}\left[\eta_i\ (s_i\ t_i)^{\sigma_i} \left| \begin{matrix} (a_i', \alpha_i'), (\ ) \\ (c_i', \gamma_i'), (\ ), (\ ), (\ ) \end{matrix} \right. dt_i\right)\Bigg\}$$

$$h\ (x_1, \ldots, x_r)\ dx_1 \ldots\ dx_r$$

... (3. 12)

Interchanging the order of integration, then evaluating the t_i–integral with the help of the known result ([6], p. 34, Eq (2. 6. 8)), and after a little simplification, we get

$$\phi_1\ (s_1, \ldots, s_r) = \prod_{i=1}^{r} \left\{ \frac{\xi_i}{\sigma_i^3}\ (s_i)^{-(\nu_i' - v_i)} (\eta_i)^{-(\nu_i' + 3)/\sigma_i} \right\}$$

$$\int_0^\infty \ldots \int_0^\infty h\ (x_1, \ldots, x_r)$$

$$\prod_{i=1}^{r} \left\{ x_i^{-1}\ H^{3,8}_{10,8}\left[\eta_i^{-\ i/\sigma^i}\ x_i^{-\xi_i}\ s_i^{-1} \right| \right.$$

$$\left(1 - c_i - \frac{\gamma_i}{\sigma_i}, \frac{\xi_i \gamma_i}{\sigma_i}\right), (\ ), (\ ), (\ ),$$

$$\left(k_i + [v_i + 1] \frac{K_i}{\sigma_i}, \frac{K_i}{\sigma_i}\right), (\ ), (\ ), (\ ),$$

$$(1-c_i'-[-\nu'_i+v_i+1]\ \frac{\gamma_i'}{\sigma_i},\frac{\gamma_i'}{\sigma_i}),$$

$$(1-a_i-[-\nu_i'+v_i+1]\ \frac{a_i'}{\sigma_i},\frac{\alpha_i'}{\sigma_i}),(\ ),$$

$$\left.\begin{matrix}(\ ),(\ ),(\ ),(g_i+[v_i+1]\ \frac{G_i}{\sigma_i},\frac{G_i}{\sigma_i}),(\ ) \\ (1-a_i-\frac{a_i}{\sigma_i},\frac{\xi_i\,\alpha_i}{\sigma_i}),(\ )\end{matrix}\right] dx_1 \ldots dx_r. \quad \ldots (3.\ 13)$$

Also from (3. 4), we have

$$\phi_2(s_1,\ldots,s_r)=s_1\ldots s_r\int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{r}\left\{t_i^{\nu_i^{(2)}}H_{2,4}^{4,0}\left[\eta_i(s_i\,t_i)^{\sigma_i}\left|\begin{matrix}(a_i^{(2)},\alpha_i^{(2)}),(\ ) \\ (c_i^{(2)},\gamma_i^{(2)}),(\ ),(\ ),(\ )\end{matrix}\right.\right]\right\}\phi_1(t_1^{-1},\ldots,t_r^{-1})\,dt_1\ldots dt_r.$$

$$\ldots (3.\ 14)$$

Substituting the value of $\phi_1(t_1^{-1},\ldots,t_r^{-1})$ from (3. 13) in (3. 14) and interchanging the order of integrations (which is justified under the conditions stated with the theorem), then evaluating t_i— integral and applying a well known property for Fox's H-function ([6], p. 4, Eq. (1. 2. 3)), we fiend that

$$\phi_2(s_1,\ldots,s_r)=\prod_{i=1}^{r}\left\{\frac{\xi_i}{\sigma_i^4}(s_i)^{-(\nu_i^{(2)}+\nu_i'-v_i)}\right.$$

$$\left.(\eta_i)^{-(\nu_i^{(2)}+2\nu_i'-v_i+4)/\sigma_i}\right\}$$

$$\int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{r}\left\{x_i^{-1}H_{14,10}^{4,12}\left[\eta_i^{-(\xi_i+1)/\sigma_i}\,x_i^{-\xi_i}\,s_i^{-1}\right|\right.$$

$$(1 - c_i - \frac{\gamma_i}{\sigma_i}, \frac{\xi_i \gamma_i}{\sigma_i}), (\ ), (\ ), (\ ),$$

$$(k_i + [v_i + 1] \frac{K_i}{\sigma_i}, \frac{K_i}{\sigma_i}), (\ ), (\ ), (\ ),$$

$$\{(1 - c_i^{(2)} + [A_2 + v_i + 1] \frac{\gamma_i^{(2)}}{\sigma_i}, \frac{\gamma_i^{(2)}}{\sigma_i})\}, \{(\ )\}, \{(\ )\},$$

$$\{(1 - a_i^{(2)} + [A_2 + v_i + 1] \frac{\alpha_i^{(2)}}{\sigma_i}, \frac{\alpha_i^{(2)}}{\sigma_i})\}, \{(\ )\},$$

$$\left.\left.\begin{matrix} \{(\ )\}, (g_i + [v_i + 1] \frac{G_i}{\sigma_i}, \frac{G_i}{\sigma_i}), (\ ), \\ (1 - a_i - \frac{\alpha_i}{\sigma_i}, \frac{\xi_i \alpha_i}{\sigma_i}), (\ ) \end{matrix}\right]\right\}$$

$$h(x_1, \ldots, x_r)\, dx_1 \ldots dx_r. \qquad \ldots (3.\ 15)$$

Repeating this process successively, we arrive at the result (3. 6).

Theorem 2. *If*

$$\Psi_1(s_1, \ldots, s_r) = \Phi\left[\pi^{-r/2} f(x_1, \ldots, x_r);\ s_1, \ldots, s_r;\ \begin{matrix} (a_i', \alpha_i'), (b_i', \beta_i') \\ (c_i' \gamma_i'), (d_i', \delta_i'), (e_i', E_i')\ (f_i', F_i') \end{matrix}\right]$$

$$\Psi_2(s_1, \ldots, s_r) = [\pi^{-r/2} x_1^{\rho_1} \ldots x_r^{\rho_r} \Psi_1(x_1^{-\lambda_1}, \ldots, x_r^{-\lambda_r});\ s_1, \ldots, s_r;\ \begin{matrix} (a_i^{(2)}, \alpha_i^{(2)}), (b_i^{(2)}, B_i^{(2)}) \\ (c_i^{(2)}, \gamma_i^{(2)}), (d_i^{(2)}, \delta_i^{(2)}), (e_i^{(2)}, E_i^{(2)}), f_i^{(2)}, F_i^{(2)}) \end{matrix}\Big] \quad \ldots (4.\ 2)$$

$$\Psi_3(s_1, \ldots, s_r) = \Phi\,[2^r \pi^{-r} x_1^{1/2} \ldots x_r^{1/2} \Psi_2(\tfrac{1}{4}x_1^2, \ldots, \tfrac{1}{4}x_r^2);\ s_1, \ldots, s_r;\ \begin{matrix} (a_i^{(3)}, \alpha_i^{(3)}), (b_i^{(3)}, B_i^{(3)}) \\ (c_i^{(3)}, \gamma_i^{(3)}), (d_i^{(3)}, (\delta_i^{(3)}), e_i^{(3)}, E_i^{(3)}), (f_i^{(3)} F_i^{(3)}) \end{matrix}\Big]$$

$$\ldots (4.\ 3)$$

... ...

... ...

$$\Psi_n (s_1,\ldots,s_r) = \Phi \Big[2^r \pi^{-r} x_1^{1/2} \ldots x_r^{1/2} \Psi_{n-1} (\tfrac{1}{4}x_1^2,\ldots, \tfrac{1}{4}x_r^2); s_1,\ldots,s_r; \begin{matrix} (a_i^{(n)}, \alpha_i^{(n)}), (b_i^{(n)}, \beta_i^{(n)}) \\ (c_i^{(n)}, \gamma_i^{(n)}), (d_i^{(n)}, \delta_i^{(n)}), (e_i^{(n)}, E_i^{(n)}) f_i^{(n)}, F_i^{(n)}) \end{matrix} \Big] \qquad \ldots (4.\ 4)$$

Then

$$\Psi_n (s_1^2/4,\ldots,s_r^2/4) = \prod_{i=1}^{r} \left\{ \frac{s_i^{1-\alpha (T_i+1)}}{\sigma_i^{n-1} \pi^{2a-1}} \times \right.$$

$$(\gamma_i)^{[2T_i (1 - 2a) + \sigma_i + 4 - 4a - 2n]/4\sigma_i} \; 2^{T_i (2a - 1) + 2n - 3}$$

$$\left. \prod_{j=2}^{n} 2^{(j-2)} \{\mu_i^{(j)} + w_i^{(j)} [2^{(j-3)} (1 - T_i) + \tfrac{1}{2}]/(\sigma_i - 1)\} \right\}$$

$$\int_0^\infty \cdots \int_0^\infty f(x_1^{2a},\ldots,x_r^{2a}) \prod_{i=1}^{r} \Big\{ (x_i)^{(\sigma_i a/2) + 2a - 1}$$

$$H^{8a,\,0}_{4a,\,8a} \left[\frac{\gamma_i^{1 + (2a-1)\lambda_i}}{4 (2a - 1) \lambda_i \sigma_i} \right.$$

$$\cdot \left\{ \prod_{j=2}^{n} 4 (j-2)\, 2^{j-3} \lambda_i w_i^{(j)} \right\}^{-1} \left| \begin{matrix} (a_i' - \tfrac{1}{4}\alpha_i', \alpha_i'), (\ \ ), \\ (c_i' - \tfrac{1}{4}\gamma_i', \gamma_i'), (\ \ ), \end{matrix} \right.$$

$$[(\Delta (\delta_\alpha, \frac{1}{2\sigma_i} \{ a_i^{(n)} [T_i + 1] a + \alpha_i^{(n)} + 2v_i a_i^{(n)} \}),$$

$$(\ \ ), (\ \ ), [(\Delta (\delta_\alpha, \frac{1}{2\sigma_i} \{\gamma_i^{(n)} [T_i + 1] a + \gamma_i^{(n)} + 2\sigma_i c_i^{(n)}\}),$$

$$\lambda_i\, a_i^{(n)})] , [(\ \ )] \\ + 2\,\sigma_i\, e_i^{(n)} \}) , \lambda_i\, \gamma_i^{(n)})] , [(\ \ )] , [(\ \ )] , [(\ \ )] \Big]\Big\} dx_1 \ldots dx_r ,$$

... (4 .5)

where

$$T_i = -2\lambda_i + 2\rho_i - (\lambda_i\, \sigma_i / 2),\ a = 2^{n-2}$$

$$\mu_i^{(j)} = c_i^{(j)} + d_i^{(j)} + e_i^{(j)} + f_i^{(j)} - a_i^{(j)} - b_i^{(j)}$$

and ... (4. 6)

$$w_i^{(j)} = \gamma_i^{(j)} + \delta_i^{(j)} + E_i^{(j)} + F_i^{(j)} - \alpha_i^{(j)} - \beta_i^{(j)}$$

$$(i = 1,\ldots,r\ ;\ j = 1,\ldots,n)$$

The (sufficient) conditions of validity of Theorem 2 are

(i) $U_i^{(j)} > 0$, $| < arg\ s_i | (\frac{1}{2}\sigma_i)\, U_i^{(j)}\, \pi$, $\gamma_i > 0$, $\sigma_i > 0$, $Re\,[(\lambda_i - \rho_i - 1) +$

$+ \lambda_i\, \sigma_i\, min\,[Re\,(c_i'/\gamma_i',\ d_i'/\delta_i',\ e_i'/E_i',\ f_i'/F_i')\,] > 0$,

$- Re\,[\,(T_i + 1)\, a + 1] +$

$$+ 2\alpha\, \lambda_i\, \sigma_i \min_{3 \le j \le n} \Big[Re \Big\{ \Big(\frac{c_i'}{\gamma_i'}, - \tfrac{1}{4}, \frac{d_i'}{\delta_i'} - \tfrac{1}{4}, \frac{e_i'}{E_i'}, - \tfrac{1}{4}, \frac{f_i'}{F_i'} - \tfrac{1}{4} \Big),$$

$$\frac{\gamma_i^{(j-1)} \{ (T_i + 1)\, 2^{j-2} + 2\} + 4\sigma_i\, c_i^{(j-1)}}{\sigma_i\, 2^{(j-1)} \lambda_i\, \gamma_i^{(j-1)}} ,$$

$$\frac{\delta_i^{(j-1)} \{ (T_i + 1)\, 2^{j-2} + 2\} + 4\sigma_i\, d_i^{(j-1)}}{\sigma_i\, 2^{(j-1)}\, \lambda_i\, \delta_i^{(j-1)}} , \ldots \Big\}\Big] > 0$$

(ii) *The various integrals involved in* (4. 1) *to* (4. 5) *are assumod to converge absolutely.*

Proof . From (4. 1) we have

$$\Psi_1\left(t_1^{\lambda_1},\ldots,t_r^{\lambda_r}\right) = \prod_{i=1}^{r}\left\{\frac{t_i^{\lambda_i}}{\sigma_i\ \pi^{\frac{1}{2}}}\right\}\int_0^{\infty}\cdots\int_0^{\infty} f(x_1,\ldots,x_r)$$

$$\prod_{i=1}^{r} H_{2,4}^{4,0}\left[\eta_i^{-1/\sigma_i}\ t_i^{\lambda_i}\ x_i \middle| \begin{matrix}(a_i',\alpha_i'),(b_i',\beta_i')\\ (c_i',\gamma_i'),(d_i',\delta_i'),(e_i',E_i'),(f_i',F_i')\end{matrix}\right]\Bigg\}$$

$$dx_1\ldots dx_r. \qquad \ldots (4.\ 7)$$

Also the equation (4. 2) gives after replacing t_i by t_i^{-1} and using well-known properties for Fox's H-function ([6] , p. 4, Eqs. (1. 2. 2) and (1. 2. 3)) :

$$\Psi_2(s_1,\ldots,s_r) = \prod_{i=1}^{r}\left\{\frac{s_i}{\sigma_i\ \pi^{\frac{1}{2}}}\int_0^{\infty}\cdots\int_0^{\infty}\prod_{i=1}^{r}\left\{t_i^{-\rho_i-2}\right.\right.$$

$$H_{4,2}^{0,4}\left[\eta_i^{-1/\sigma_i}\ t_i\ s_i^{-1}\middle|\begin{matrix}(1-c_i^{(2)},\gamma_i^{(2)}/\sigma_i),(\ ),(\ ),(\ )\\ (1-a_i^{(2)},\alpha_i^{(2)}/\sigma_i),(\ )\end{matrix}\right]\Bigg\}\times$$

$$\Psi_1(t_1^{\lambda_1},\ldots,t_r^{\lambda_r})\ dt_1\ldots dt_r. \qquad \ldots (4.\ 8)$$

Substituting the value of $\Psi_1(t_1^{\lambda_1},\ldots,t_r^{\lambda_r})$ from (4. 7) in (4. 8) , interchanging the order of integration, evaluating the t_i- integral and then replacing s_i by $s_i^2/4$ and x_i by x_i^2 $(i = 1,\ldots,r)$, we get after a little simplification

$$\Psi_2(s_1^2/4,\ldots,s_r^2/4) = \prod_{i=1}^{r}\left\{\frac{s_i^{-T_i}}{\sigma_i\ \pi}\ \eta_i^{(-2T_i+\sigma_i-4)/4\sigma_i}\right.$$

$$\left.2^{T_i+1}\right\}\int_0^{\infty}\cdots\int_0^{\infty} f(x_1^2,\ldots,x_r^2)\prod_{i=1}^{r} x_i^{(\sigma_i/2)+1}$$

$$H^{8,0}_{4,8}\left[\frac{\gamma_i^{\lambda_i+1}\,x_i^{2\sigma_i}\,s_i^{2\lambda_i\sigma_i}}{4^{\lambda_i\sigma_i}}\,\middle|\,\begin{matrix}(a_i'-\tfrac{1}{4}\alpha_i',\alpha_i'),(\ ),\\(c_i'-\tfrac{1}{4}\gamma_i',\gamma_i'),(\ ),\end{matrix}\right.$$

$$[(\triangle(1,\frac{1}{2\sigma_i}\{\alpha_i^{(2)}(T_i+1)+a_i^{(2)}+2\sigma_i\,a_i^{(2)}\}),\lambda_i\,a_i^{(2)})],$$

$$(\ ),(\ ),[(\triangle(1,\frac{1}{2\sigma_i}\{\gamma_i^{(2)}(T_i+2)+\gamma_i^{(2)}+2\sigma_i\,\gamma_i^{(2)}\}),$$

$$\left.\begin{matrix}[(\ )]\\ \lambda_i\,\gamma_i^{(2)}\}]\,,[(\ )]\,,[(\ )]\,,[(\ )]\end{matrix}\right]dx_1...dx_r \qquad ...(4.9)$$

Repeating the process successively with the relations (4. 3),... and using the wellknwon gamma duplication formula, we arrive at the required result (4. 5).

Theorem 3. *If*

$$\phi\,(s_1,...,s_r)=\Phi\left[\pi^{-r/2}f_1\,(x_1,...,x_r)\,;s_1,...,s_r\,;\right.$$

$$\left.\begin{matrix}(a_i',\alpha_i'),(b_i',\beta_i')\\(c_i',\gamma_i'),(d_i',\delta_i'),(e_i',E_i'),(f_i',F_i')\end{matrix}\right] \qquad ...(5.1)$$

$$s_1^{\rho_1}...s_r^{\rho_r}f\,(s_1^{-\lambda_1},...,s_r^{-\lambda_r})=\Phi\left[\pi^{-r/2}f_2\,(x_1,...,x_r);s_1,...,s_r);\right.$$

$$\left.\begin{matrix}(a_i^{(2)},\alpha_i^{(2)}),(b_i^{(2)},\beta_i^{(2)})\\(c_i^{(2)},\gamma_i^{(2)}),(d_i^{(2)},\delta_i^{(2)}),(e_i^{(2)},E_i^{(2)}),f_i^{(2)},F_i^{(2)})\end{matrix}\right] \qquad ...(5.2)$$

$$H^{8,0}_{4,8}\left[\frac{\gamma_i^{\lambda_i+1}\, x_i^{2\sigma_i}\, s_i^{2\lambda_i\sigma_i}}{4^{\lambda_i\sigma_i}}\;\middle|\; \begin{matrix}(a_i'-\tfrac{1}{4}\alpha_i',\ \alpha_i'),\ (\ \ ),\\ (c_i'-\tfrac{1}{4}\gamma_i',\ \gamma_i'),\ (\ \ ),\end{matrix}\right.$$

$$[(\Delta(1,\frac{1}{2\sigma_i}\{\alpha_i^{(2)}(T_i+1)+a_i^{(2)}+2\sigma_i a_i^{(2)}\}),\lambda_i a_i^{(2)})],$$

$$(\ \ ),(\ \ ),[(\Delta(1,\frac{1}{2\sigma_i}\{\gamma_i^{(2)}(T_i+2)+\gamma_i^{(2)}+2\sigma_i\gamma_i^{(2)}\}),$$

$$\left.\begin{matrix}[(\ \ )]\\ \lambda_i\gamma_i^{(2)})],[(\ \ )],[(\ \ )],[(\ \ )]\end{matrix}\right]dx_1\ldots dx_r \qquad \ldots (4.\ 9)$$

Repeating the process successively with the relations (4. 3),... and using the wellknwon gamma duplication formula, we arrive at the required result (4. 5).

Theorem 3. *If*

$$\phi(s_1,\ldots,s_r)=\Phi\left[\pi^{-r/2} f_1(x_1,\ldots,x_r);\ s_1,\ldots,s_r;\right.$$

$$\left.\begin{matrix}(a_i',\ \alpha_i'),\ (b_i',\ \beta_i')\\ (c_i',\ \gamma_i'),\ (d_i',\ \delta_i'),\ (e_i',\ E_i'),\ (f_i',\ F_i')\end{matrix}\right] \qquad \ldots (5.\ 1)$$

$$s_1^{\rho_1}\ldots s_r^{\rho_r} f(s_1^{-\lambda_1},\ldots,s_r^{-\lambda_r})=\Phi\left[\pi^{-r/2} f_2(x_1,\ldots,x_r);s_1,\ldots,s_r);\right.$$

$$\left.\begin{matrix}(a_i^{(2)},\ \alpha_i^{(2)}),\ (b_i^{(2)},\ \beta_i^{(2)})\\ (c_i^{(2)},\ \gamma_i^{(2)}),\ (d_i^{(2)},\ \delta_i^{(2)}),\ (e_i^{(2)},\ E_i^{(2)}),\ f_i^{(2)},\ F_i^{(2)})\end{matrix}\right] \ldots (5.\ 2)$$

$$s_1^{-\frac{1}{2}} \ldots s_r^{-\frac{1}{2}} f_2 (1/4s_1^2,\ldots,1/4s_r^2) = \Phi \left[2^r \pi^{-r} f_2 (x_1,\ldots,x_r) ;s_1,\ldots,s_r ; \begin{matrix} (a_i^{(3)}, \alpha_i^{(3)}), (b_i^{(3)}, \beta_i^{(3)}) \\ (c_i^{(3)}, \gamma_i^{(3)}), (d_i^{(3)}, \delta_i^{(3)}), (e_i^{(3)}, E_i^{(3)}), (f_i^{(3)}, F_i^{(3)}) \end{matrix} \right] \quad \ldots (5.3)$$

... ...

... ...

and

$$s_1^{-\frac{1}{2}} \ldots s_r^{-\frac{1}{2}} f_{n-1} (1/4s_1^2,\ldots,1/4s_r^2) = \Phi \left[2^r \pi^{-r} f_n (x_1,\ldots,x_r) ;s_1,\ldots,s_r ; \begin{matrix} (a_i^{(n)}, \alpha_i^{(n)}), (b_i^{(n)}, \beta_i^{(n)}) \\ (c_i^{(n)}, \gamma_i^{(n)}), (d_i^{(n)}, \delta_i^{(n)}), (e_i^{(n)}, E_i^{(n)}), (f_i^{(n)}, F_i^{(n)}) \end{matrix} \right] \quad \ldots (5.4)$$

Then

$$\phi (s_1,\ldots,s_r) = \prod_{i=1}^{r} \left\{ \frac{\lambda_i \, s_i^{(3/2)-(\rho_i/\sigma_i)}}{\pi^{2\alpha-1} \quad \sigma_i^{\alpha-1}} \times \right.$$

$$(\gamma_i)^{[3\lambda_i^2 (2\alpha - 1) - (n-1)^2\lambda_i - 2\rho_i]/2\lambda_i \sigma_i} \, 2^{-3\lambda_i (2\alpha - 1) + 2n - 3}$$

$$\left. \prod_{j=2}^{r} 2(j-2)\{\mu_i^{(j)} + w_i^{(j)} [2^{j-3} (1 - 3 \lambda_i) + \tfrac{1}{2}]/(\sigma_i - 1)\} \right\}$$

$$\int_0^\infty \ldots \int_0^\infty f_n (x_1^2/4,\ldots,x_r^2/4) \prod_{i=1}^{r} \left\{ x_i^{\alpha(3\lambda_i - 1)} \right.$$

$$H_{4\alpha, \, 8\alpha}^{8\alpha, \, 0} \left[4^{-\lambda_i \sigma_i (2\alpha - 1)} \, s_i^{\sigma_i} \, x_i^{2\lambda_i \sigma_i \alpha} \, \gamma_i^{1 + (2\alpha - 1) \lambda_i} \right.$$

$$\left[\prod_{j=2}^{n} 4(j-2)\, 2^{j-3}\, \lambda_i\, w_i^{(j)} \right]^{-1}$$

$$\left| \begin{matrix} (a_i' + \left[\frac{\rho_i}{\lambda_i} - \frac{1}{2}\right] \frac{a_i'}{\sigma_i}, \alpha_i'),\ (\ \ ),\ [(\triangle(\delta a\ \frac{1}{2\sigma_i}, \\ (c_i' + \left[\frac{\rho_i}{\lambda_i} - \frac{1}{2}\right] \frac{\gamma_i'}{\sigma_i}, \gamma_i'),(\ \ ),(\ \ ),(\ \ ),\ [(\triangle(\delta a,\ \frac{1}{2\sigma_i} \end{matrix} \right.$$

$$\left. \begin{matrix} \{a_i^{(n)} [-3\lambda_i + 1]\, a + \alpha_i^{(n)} + 2\sigma_i\, a_i^{(n)}\}),\ \lambda_i\, \alpha_i^{(n)})\,],\ [(\ \ )] \\ \{\gamma_i^{(n)} [-3\lambda_i + 1]\, a + \gamma_i^{(n)} + 2\sigma_i\, c_i^{(n)}\}),\ \lambda_i\, \gamma_i^{(n)})],\ [(\ \ )],[(\ \ )],[(\ \ )] \end{matrix} \right\}$$

$$dx_1 \ldots dx_r, \qquad \ldots (5.\ 5)$$

where $\mu_i^{(j)}$ *and* $w_i^{(j)}$ *are given by* (4. 6).

Theorem 3 is valid under the following conditions :

(i) $\gamma_i > 0$, $\sigma_i > 0$, $U_i^{(j)} > 0$, $|\arg s_i| < (\frac{1}{2}\sigma_i)\, U_i^{(j)}\, \pi$[$U_i^{(j)}$ *is given by*(3.10)]

$$Re\,(3\lambda_i - 2) + 2\lambda_i\, \sigma_i \min \left[Re\ \left\{ \left(\frac{c_i'}{\gamma_i'} + \frac{2\rho_i - \lambda_i}{2\lambda_i\, \sigma_i},\ \frac{d_i'}{\delta_i'} + \frac{2\rho_i - \lambda_i}{2\lambda_i\, \sigma_i}, \right. \right. \right.$$

$$\left. \left. \left. \ldots \right) \right\} \right] > 0 \text{ and}$$

$$Re\,[a\,(3\lambda_i - 1) - l] + 2a\, \lambda_i\, \sigma_i \min_{3 \leq j \leq n} \left[Re \left\{ \frac{c_i'}{\gamma_i'} + \right. \right.$$

$$\frac{2\rho_i - \lambda_i}{2\lambda_i\, \sigma_i},\ \frac{d_i'}{\delta_i'} + \frac{2\rho_i - \lambda_i}{2\lambda_i\, \sigma_i},\ \ldots,\ \frac{\gamma_i^{(j-1)}\,[(1 - 3\lambda_i)\, 2^{(j-2)} + 2] + 4\sigma_i\, c_i^{(j-1)}}{\sigma_i\, 2^{(j-1)}\, \lambda_i\, \gamma_i^{(j-1)}},$$

$$\left. \frac{\delta_i^{(j-1)}\,[(1 - 3\lambda_i)\, 2^{j-2} + 2] + 4\sigma_i\, d_i^{(j-1)}}{\sigma_i\, 2^{(j-1)}\, \lambda_i\, \delta_i^{(j-1)}},\ \ldots \right\} > 0$$

(ii) *The integrals involved in* (5. 1) *to* (5. 5) *are assumed to be absolutely convergent.*

Proof. To prove Theorem 3, we proceed in a manner similar to that of Theorem 2.

It may be remarked here that the aforementioned theorems are quite general in character and certain results concerning the multidimensional Whittaker, Bessel, and Laplace transforms etc. can be deduced from these theorems by suitably specializing the parameters. Also, if we take $r=1$ in Theorems *1* to *3*, we shall arrive at the chains for the integral transform with fox's H–function as the kernel. The results thus obtained include the chains established earlier by Goyal [2], Bose [1], Kesarwani [4], Rathie [7] and Jain [3] as their particular cases.

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