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A CLASS OF UNIVALENT FUNCTIONS. III

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Abstract

There are many classes of univalent functions in the unit disk U. In this paper, we consider a class $P_p(a, b)$ of univalent functions

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

in the unit disk U satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + (1-2a)} \right| < b \qquad (z \in U)$$

for a $(0 \leqslant a < 1)$ and b $(0 < b \leqslant 1)$. It is the purpose of this paper to show a distortion theorem, coefficient estimates, an argument theorem and a radius of convexity for this class $P_p(a, b)$.

1. Introduction

Let T denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0)$$

analytic and univalent in the unit disk $U = \{ |z| < 1 \}$. A. Schild

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[5] considered a subclass of T consisting of polynomials having $|z| = 1$ as radius of univalence and some results for this class. In 1975, H. Silverman [6] studied the subclasses $S^*(a)$ and $C^*(a)$ of T, classes of starlike functions of order a and convex functions of order a, respectively. Furthermore, V. P. Gupta and P. K. Jain [2] studied the class $P^*(a, b)$ of functions $f(z) \in T$ satisfying the condition

$$(1) \qquad \left| \frac{f'(z) - 1}{f'(z) + (1 - 2a)} \right| < b \qquad (z \in U)$$

for a $(0 \leqslant a < 1)$ and b $(0 < b \leqslant 1)$. And, recently, S. Owa [3] studied the class $P(a, b)$ of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic and univalent in the unit disk U satisfying the condition (1) for a $(0 \leqslant a < 1)$, b $(0 < b \leqslant 1)$ and $z \in U$.

In this paper, we consider the analytic and univalent functions

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (P \in N)$$

in the unit disk U satisfying the condition (1) for a $(0 \leqslant a < 1)$, $b(0 < b \leqslant 1)$ and $z \in U$. And we denote the class of all such functions $f(z)$ by $P_p(a, b)$.

2. A Distortion Theorem

In the first place, we require the following lemma.

Lemma 1. *Let a function*

$$H(z) = 1 + b_p z^p + b_{p+1} z^{p+1} + \dots \qquad (p \in N)$$

be analytic in the unit disk U. Then $H(z)$ is analytic and satisfies the condition

$$\left| \frac{1 - H(z)}{(1-2a) + H(z)} \right| < b \qquad (z \in U)$$

for a $(0 \leqslant a < 1)$ and b $(0 < b \leqslant 1)$ if, and only if, there exists an analytic function $\phi(z)$ in the unit disk U such that $|\phi(z)| \leqslant b$ for $z \in U$ and

$$H(z) = \frac{1 - (1-2a)\, z^p \phi(z)}{1 + z^p \phi(z)}.$$

Proof. We employ the technique used by K. S. Padmanabhan [4]. Let a function

$$H(z) = 1 + b_p z^p + b_{p+1} z^{p+1} + \ldots \qquad (p \in N)$$

satisfy the condition

$$\left| \frac{1 - H(z)}{(1-2a) + H(z)} \right| < b \qquad (z \in U)$$

for a $(0 \leqslant a < 1)$ and b $(0 < b \leqslant 1)$. Setting

$$z^{p-1} h(z) = \frac{1 - H(z)}{(1-2a) + H(z)},$$

we note that the function $h(z)$ is analytic in the unit disk U, satisfies $|h(z)| < b$ for $z \in U$ and $h(0) = 0$. Hence by using Schwarz's lemma, we have $h(z) = z\,\phi(z)$, where $\phi(z)$ is analytic in the unit disk U and satisfies $|\phi(z)| \leqslant b$ for $z \in U$. Thus we get

$$H(z) = \frac{1 - (1-2a)\, z^{p-1} h(z)}{1 + z^{p-1} h(z)}$$

$$= \frac{1 - (1-2a)\, z^p \phi(z)}{1 + z^p \phi(z)}$$

On the other hand, if

$$H(z)\ \frac{1-(1-2a)z^{p}\phi(z)}{1+z^{p}\phi(z)}$$

and $|\phi(z)| \leqslant b$ for $z \in U$, then the function $H(z)$ is analytic in the unit disk U. Furthermore, since $|z^{p}\phi(z)| \leqslant b|z|^{p} < b$ for $z \in U$, we obtain

$$\left|\frac{1-H(z)}{(1-2a)+H(z)}\right| = |z^{p}\phi(z)| < b$$

for $z \in U$. This completes the proof of the lemma.

Theorem 1. *Let a function*

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be in the class $P_p(a, b)$. Then

$$\frac{1+b(2a-1)|z|^{p}}{1+b|z|^{p}} \leqslant |f'(z)| \leqslant \frac{1-b(2a-1)|z|^{p}}{1-b|z|^{p}},$$

$$|f(z)| \geqslant \int_{0}^{|z|} \frac{1+b(2a-1)t^{p}}{1+bt^{p}}\, dt$$

and

$$|f(z)| \leqslant \int_{0}^{|z|} \frac{1+b(2a-1)t^{p}}{1-bt^{p}}\, dt$$

for $z \in U$.

Proof. Since $f(z)$ is in the class $P_p(a, b)$, $f(z)$ satisfies the condition (*1*). Hence, by Lemma *1* we have

$$f'(z) = \frac{1-(1-2a)z^{p}\phi(z)}{1+z^{p}\phi(z)},$$

where $\phi(z)$ is an analytic function in the unit disk U and satisfies $|\phi(z)| \leqslant b$ for $z \in U$. After a simple computation, we have

$$\left| f'(z) - \frac{1 + b^2(1-2a)|z|^{2p}}{1 - b^2|z|^{2p}} \right| \leqslant \frac{2b(1-a)|z|^p}{1 - b^2|z|^{2p}}.$$

This inequality gives that

$$\frac{1 + b(2a-1)|z|^p}{1 + b|z|^p} \leqslant |f'(z)| \leqslant \frac{1 - b(2a-1)|z|^p}{1 - b|z|^p}$$

for $z \in U$.

In order to show the upper bound for $f(z)$, integrating the line segment from 0 to z, we get

$$|f(z)| = \left| \int_0^z f'(t)\, dt \right|$$

$$\leqslant \int_0^{|z|} |f'(te^{i\theta})|\, dt$$

$$\leqslant \int_0^{|z|} \frac{1 - b(2a-1)t^p}{1 - bt^p}\, dt.$$

Finally, to show the lower bound of $f(z)$, we integrate along the path L whose image is the segment $[0, f(z)]$. Consequently

$$|f(z)| = \left| \int_L f'(t)dt \right|$$

$$\geqslant \int_L |f'(t)|\, d|t|$$

$$\geqslant \int_0^{|z|} \frac{1 + b(2a-1)t^p}{1 + bt^p}\, dt.$$

We can show that all estimates of this theorem are sharp.

3. Coefficient estimates

Theorem 2. *Let a function*

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be in the class $P_p(a, b)$. *Then*

$$| a_{p+n} | \leqslant \frac{2b(1-a)}{p+n}$$

for any $n \geqslant 1$.

Proof. We use a method of R. M. Goel [1]. Since $f(z)$ belongs to the class $P_p(a, b)$, in view of Lemma 1, we obtain

$$f'(z) = \frac{1-(1-2a)\, z^{p-1} h(z)}{1+z^{p-1}h(z)},$$

where $|h(z)| < b$ for $z \in U$ and $h(0) = 0$. Hence,

$$\{ (2a-1) - f'(z) \} z^{p-1} h(z) = f'(z) - 1,$$

that is,

$$\left\{ 2(a-1) - \sum_{n=1}^{\infty} (p+n)\, a_{p+n} z^{p+n-1} \right\} z^{p-1} h(z)$$

$$= \sum_{n=1}^{\infty} (p+n)\, a_{p+n} z^{p+n-1} .$$

Hence we can write

$$\left\{ 2(a-1) - \sum_{n=1}^{k} (p+n)\, a_{p+n} z^{p+n-1} \right\} z^{p-1} h(z)$$

$$= \sum_{n=1}^{k+1} (p + n)\, a_{p+n}\, z^{p+n-1} + \sum_{n=k+1}^{\infty} c_n z^n ,$$

c_n being some complex number. Now, since $h(z)$ has modulus at most b in the unit disk U, by using Parseval's identity, we have

$$\sum_{n=1}^{k+1} (p + n)^2 \,|\, a_{p+n} \,|^2 \,|\, z \,|^{2\,(p+n-1)} + \sum_{n=k+1}^{\infty} |\, c_n \,|^2 \,|\, z \,|^{2n}$$

$$\leqslant 4b^2 (1 - a)^2 + b^2 \sum_{n=1}^{k} (p + n)^2 \,|\, a_{p+n} \,|^2 \,|\, z \,|^{2\,(p+n-1)}.$$

Hence further

$$\sum_{n=1}^{k+1} (p + n)^2 \,|\, a_{p+n} \,|^2 \leqslant 4b^2 (1 - a)^2$$

$$+ b^2 \sum_{n=1}^{k} (p + n)^2 \,|\, a_{p+n} \,|^2$$

and thus

$$(p + k + 1)^2 \,|\, a_{p+k+1} \,|^2$$

$$\leqslant 4b^2 (1 - a)^2 - (1 - b^2) \sum_{n=1}^{k} (p + n)^2 \,|\, a_{p+n} \,|^2$$

$$\leqslant 4b^2 (1 - a)^2 ,$$

because $0 < b \leqslant 1$. Consequently, we obtain

$$|\, a_{p+n} \,| \leqslant \frac{2b\,(1 - a)}{p + n}$$

for any $n \geqslant 1$.

4. An argument Theorem

Theorem 3. *Let a function*

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathcal{N})$$

be in the class $P_p(a,b)$. Then

$$|\arg\{f'(z)\}| \leqslant \sin^{-1}\left(\frac{2b(1-a)|z|^p}{1+b^2(1-2a)|z|^{2p}}\right)$$

Proof, Let $f(z) \in P_p(a, z)$. Then, in view of Lemma 1, we get

$$\left| f'(z) - \frac{1+b^2(1-2a)|z|^{2p}}{1-b^2|z|^{2p}} \right| \leqslant \frac{2z(1-a)|z|^p}{1-b^2|z|^{2p}}.$$

Accordingly we obtain

$$|\arg\{f'(z)\}| \sin^{-1}\left(\frac{2b(1-a)|z|^p}{1+b^2(1-2a)|z|^{2p}}\right).$$

5. A radius of convexity

Theorem 4. *Let a function*

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathcal{N})$$

be in the class $P_p(a, b)$ with $0 \leqslant a < 1/2$, $0 < b \leqslant 1$ and $bp \leqslant 1$.

Then the function $f(z)$ maps

$$|z| < \frac{\{(1-2a)b+2(1-a)\}}{2(1-2a)b}$$

$$-\frac{[\{(1-2a)b+2(1-a)\}^2-4(1-2a)b}{2(1-2a)b}]^{\frac{1}{2}}$$

on to a convex domain.

Proof, We used a method of N. S. Sohi [7]. Since f(z) is in the class $P_p(a, b)$, by Lemma 1, we have

$$f'(z) = \frac{1-(1-2a)z^p\phi(z)}{1+z^p\phi(z)},$$

wher $\phi(z)$ is an analytic function in the unit disk U and satisfies $|\phi(z)| \leqslant b$ for $z \in U$ On differentiating both sides of the above equality with respect to z logarithmically, we obtain

$$1+\frac{zf''(z)}{f'(z)} = 1-\frac{2(1-a)\{pz^p\phi(z)+z^{p+1}\phi'(z)\}}{\{1+z^p\phi(z)\}\{1-(1-2a)z^p\phi(z)\}}.$$

Since

$$\left|\frac{\phi'(z)}{b}\right| \leqslant \frac{1-|\phi(z)/b|^2}{1-|z|^2}$$

for the analytic function $\phi(z)$ in the unit disk U, we get

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$

$$\geqslant 1-\frac{2(1-a)|z|^p\{p|\phi(z)|+|z\phi'(z)|\}}{|\{1+z^p\phi(z)\}\{(1-2a)z^p\phi(z)\}|}$$

$$\geqslant 1-\frac{2(1-a)|z|^p\{b|z|+|\phi(z)|\}\{bp-|z\phi(z)|\}}{b(1-|z|^2)|\{1+z^p\phi(z)\}\{1-(1-2a)z^p\phi(z)\}|}.$$

Furthermore, since

$$|\{1 + z^p \phi(z)\}\{1 - (1 - 2a) z^p \phi(z)\}|$$

$$\geqslant \{1 - |z^p \phi(z)|\}\{1 - (1 - 2a)|z^p \phi(z)|\}$$

$$\geqslant \{1 - |z \phi(z)|\}\{1 - (1 - 2a)|z \phi(z)|\}$$

for $0 \leqslant a < \frac{1}{2}$ and $p \in \mathcal{N}$, we have

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\}$$

$$\geqslant 1 - \frac{2(1 - a)|z|^p \{b|z| + |\phi(z)|\}\{bp - |z \phi(z)|\}}{b(1 - |z|^2)\{1 - |z \phi(z)|\}\{1 - (1 - 2a)|z \phi(z)|\}}$$

$$\geqslant 1 - \frac{2(1 - a)|z|}{(1 - |z|)\{1 - 2a)|z \phi(z)|\}}$$

with the aid of the condition $bp \leqslant 1$. Consequently, if

$$|z| < \frac{\{(1 - 2a)b + 2(1 - a)\}}{2(1 - 2a)b}$$

$$- \left[\frac{\{(1 - 2a)b + 2(1 - a)\}^2 - 4(1 - 2a)b}{2(1 - 2a)b}\right]^{\frac{1}{2}}$$

then we have

$$Re\left\{1 + \frac{f''(z)}{f'(z)}\right\} > 0.$$

This proves the theorem.

REFERENCES

[1] R. M. Goel, A class of univalent functions whose derivatives have positive real part in the unit disc, *Nieuw Arch. Wisk.* (3) **15** (1967) 55–63.

[2] V. P. Gupta and P. K. Jain, Certain classes of univalent functions

with negative coefficients. II, *Bull. Austral. Math Soc.* **15** (1976), 467–473.

[3] S. Owa, A class of univalent functions. II, to oppear.

[4] K. S. Padmanabhan, On certain classes of starlike functions in the unit disc, *J. Indian Math. Soc.* (N. S.) **32** (1968), 89–103.

[5] A. Schild, On a class of functions schlicht in the unit circle, *Proc. Amer. Math. Soc.* **51** (1954), 115–120.

[6] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.

[7] N. S. Sohi, A class of *p*–valent analytic functions, *Indian J. Pure Appl. Math.* **10** (1979), 826–834.