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ON SOME CLASSES OF REGULAR UNIVALENT FUNCTIONS

by

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Abstract

Let $S\ (m, M)$ be the class of regular functions $f(z) = z + \ldots$ for which $\left|\dfrac{z f'(z)}{f(z)} - m\right| < M$, $|m-1| < M \leqslant m$ in $|z| < 1$. We now show that integral operators of the from

$$F(z) = \left[\frac{c+p}{z^c} \int_0^z t^{c-1} f(t)^p \, dt\right]^{1/p} = \ + \ldots$$

for suitable choices of the parameters c and p transforms $S(m, M)$ into $S(m, M)$. Our result generalizes a recent result (Theorem 2) of S. Miller, P. Mocanu and M. Reade with other restrictions on the parameters, we obtain transformations of the set $S^*(\alpha) \times S(m, W)$ into $S^*(\alpha)$. Here $S^*(\alpha)$ denotes, as usual, the set of all regular (normalized) univalent functions starlike of order α in $|z| \leqslant 1, 0 \leqslant \alpha < 1$.

1 Introduction

Let A be the class of functions f which are regular in the unit disc $D = \{z : |z| < 1\}$ with the normalization $f(0) = 0, f'(0) = 1$.

Further, let m and M be arbitrary fixed real numbers which satisfy the relation $(m, M) \in E$ where $E = \{ (m, M) : | m-1 | < M \leqslant m \}$.

Let $\quad S = \{f : f \in A$ and f is univalent in $D\}$,

$$S^*(\alpha) = \left\{ f : f \in A, \operatorname{Re}\left(\frac{z f'(z)}{f(z)}\right) > \alpha,\ 0 \leqslant \alpha < 1,\ z \in D \right\}$$

and

$$S(m, M) = \left\{ f : f \in A, \left| \frac{z f'(z)}{f(z)} - m \right| < M, \right.$$

$$\left. (m, M) \in E,\ z \in D \right\}$$

It is well known that $S(m, M) \subset S^*(m-M) \subset S^*(0) \subset S$, $S^*(0) \equiv S^*$.

Many authors have studied integral operators of the form

$$I(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

where c is a real (or complex) constant and f belongs to certain subclass of S (see [1] to [5], [8] to [12] and [14]).

The purpose of this paper is to extend some of these results by using a general integral operator which transforms $S(m, M)$ into $S(m, M)$ and $S^*(\alpha) \times S(m, M)$ into $S^*(\alpha)$.

2. The following two lemmas are essential to the discussion that follows. The first may be found in [13] and the second in [6].

Lemma 1. *A function $f \in S(m, M)$ if and only if there exists a function w regular in D and satisfying $w(0) = 0$, $|w(z)| < 1$ such that*

$$\frac{z f'(z)}{f(z)} = \frac{1 + a\, w(z)}{1 - b\, w(z)},\quad a = \frac{M^2 - m^2 + m}{M},\quad b = \frac{m-1}{M},\ z \in D.$$

Lemma 2. *If $w(z)$ is regular for $|z| \leqslant r < 1$, $w(0) = 0$ and $|w(z_1)|$*

$$= \max_{|z|=r} |w(z)| \text{ then } z_1 w'(z_1) = k\, w(z_1),\ k \geqslant 1.$$

3. Principal results

Theorem 1. *Suppose* $f \in S(\mathrm{m}, \mathrm{M})$, $p > 0$, $c > -\dfrac{p(1-a)}{(1+b)}$.

Then the function F defined by

$$(1) \qquad F(z)^p = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)^p\, dt$$

belongs to $S(\mathrm{m},\mathrm{M})$.

Proof : *Let us choose a function w such that*

$$(2) \qquad \frac{z F'(z)}{F(z)} = \frac{1+aw(z)}{1-bw(z)}, \quad a = \frac{M^2-m^2+m}{M}, \quad b = \frac{m-1}{M}$$

w $(0) = 0$ and w is either regular or meromorphic in D. From (1) and (2) we have

$$(3) \qquad (c+p)\left(\frac{f(z)}{F(z)}\right)^p = \frac{(c+p) + (ap-bc)\, w(z)}{1-bw(z)}.$$

Logarithmic differentiation yields

$$(4) \qquad \frac{z f'(z)}{f(z)} - m = \frac{(1-m) + (a+bm)\, w(z)}{1-bw(z)}$$

$$+ \frac{(a+b)\, zw'(z)}{[\,1-bw(z)\,]\,[\,(c+p) + (ap-bc)\, w(z)\,]}.$$

Let z_1 with $|z_1| = r_0$ be the nearest pole of $w(z)$ in D. Hence $w(z)$ is regular in $|z| < r_0 < 1$. By Lemma 2, for $|z| \leqslant r < r_0$ there is a point z_0 such that

$$(5) \qquad z_0 w'(z_0) = kw(z_0),\ k \geqslant 1.$$

From (4) and (5) we have

$$(6) \qquad \frac{z_0 f'(z_0)}{f(z_0)} - m = \frac{N(z_0)}{D(z_0)}.$$

where $N(z_0) = (1-m)(c+p) + (ap-bc)(a+bm)\, w^2(z_0)$

$+ [(c+p)(a+bm) + (ap-bc)(1-m) + k(a+b))]\, w(z_0)$

and $\quad D(z_0) = (c+p) - b(ap-bc)\, w^2(z_0) + (ap-2bc-bp)\, w(z_0)$.

Now suppose that it were possible to have

$M(r, w) = \max_{|z|=r} | w(z) | = 1$ for some $r < r_0 < 1$. At the point z_0 where this occurred we would have $| w(z_0) | = 1$. Then we have

$$(7) \qquad | N(z_0) |^2 - M^2 | D(z_0) |^2 = A + 2B\, Re\,(w(z_0)) + C\, Re\,(w^2(z_0)).$$

where $A = k(a+b) [k(a+b) + 2M(c+p) - 2Mb(ap-bc)]$,

$B = Mk(a+b) [(ap-bc) - b(c+p)]$ and $C = 0$.

Hence $A + 2B = k(a+b) [k(a+b) + 2M(1-b)(c-cb+p+pa)]$

> 0 if $c \geqslant -\dfrac{p(1+a)}{(1-b)}$.

and $\quad A - 2B = k(a+b) [k(a+b) + 2M(1+b)(c+cb+p-pa)]$

> 0 if $c \geqslant -\dfrac{p(1-a)}{(1+b)}$.

Therfore, from (6) and (7), it follows that

$$\left| \frac{z_0 f'(z_0)}{f(z_0)} - m \right| > M.$$

But this is contrary to the fact that $f \in S(m, M)$. So we can not have $M(r, w) = 1$. Since $| w(0) | = 0$, $| w(z) | \neq 1$ and $| w(z) |$ is continuous in $| z | < r_0$ so $| w(z) | < 1$ and w is regular in D. Hence from (2) and Lemma 1, $F \in S(m, M)$.

If we take $m = M$, $m \to \infty$ in Theorem 1 then the recent result [11] follows :

Corollary *Miller, Mocanu and Reade [10] if* $p > 0$, $c \geqslant 0$, $f \in S^*$ *then the function* F *defined by (1) belongs to* S^*.

THEOREM 2. *Suppose* $f \in S^*(\alpha)$, $g \in S(m, M)$, $p > 0$, $c \geqslant 0$, $(m, M) \in \{ (m, M) : | m-1 | < M \leqslant m^* \}$ *and* $m^* = min \{ m, m - 1$

$$+ \frac{1-\alpha}{2(c+p+p\alpha)} \}.$$ *Then the function* F *defined by*

$$(8) \qquad f(z)^p = \frac{c+2p}{z^{c+p}} \int_0^z t^{c-1} f(t)^p g(t)^p dt$$

belongs to $S^*(\alpha)$. In (8) all powers are principal ones.

THEOREM 3. *Suppose* $f \in S^*(\alpha)$, $g \in S(m, M)$, $p > 0$, $c \geqslant 0$,

$(m, M) \in \{ (m, M) : | m-1 | < M \leqslant m^* \}$ *and* $m^* =$

$$min \left\{ m, m - 1 + \frac{1-\alpha}{2p(c+1+\alpha)} \right\}.$$ Then the function F defined by

$$(9) \qquad F(z) = \frac{c+2}{z^{c+1}} \int_0^z t^{c-p} f(t) g(t)^p dt$$

belongs to $S^*(\alpha)$. In (9)all powers are principal ones.

The proofs of Theorems 2 and 3 are similar to that of Theorem 1 and will be omitted here. Dwivedi [4] has studied some special cases of Theorem 2 and 3, namely, $p = 1$.

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