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ON THE GRAVITATIONAL INSTABILITY THROUGH POROUS MEDIUM OF ASTROFHYSICAL PLASMAS

by

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Abstract

The gravitational instability of a finitely conducting hydromagnetic composite rotating plasma through porous medium is considered to include the effects due to finite Larmor radius, Hall currents and collisions with neutrals. The particular cases of the above effects on the waves propagated along and perpendicular to the magnetic field are discussed. It is found that Jeans' criterion remains unchanged in the presence of fintte conductivity, medium porosity, rotation, Hall currents, finite Larmor radius and collisions with neutrals.

I Introduction

The problem of gravitational instability of an infinite homogeneous medium was first considered by Jeans [3]. A comprehensive account of the various investigations, both in hydrodynamics and hydromagnetics, of the problem of the gravitational instability has been given by Chandrasekhar[2].He has shown that Jeans' criterion remains unaffected by the separate or simultaneous presence of uniform rotation and a uniform magnetic field. Pacholczyk and Stodolkiewicz [5] had studied the effect of finite conductivity on the gravitational instability of an interstellar medium such as an HI region using the approximation of zero Larmor radius.

Rosenbluth, Krall and Rostoker [7], Roberts and Taylor [6] and Jukes [4] have pointed out the stabilizing influence of the finite Larmor radius effects on a plasma. All these investigations have been carried out for a fully ionized plasma and the medium has been considered to be non-porous. Quite frequently the plasma is not fully ionized and may be permeated with neutral atoms. As a reasonably simple approximation the plasma may be idealized as a mixture of a hydromagnetic (ionized) component and a neutral component,the two interacting through mutual collisional (frictional) effects. Applications of the results of flow through a porous medium in the presence of a magnetic field are in the study of the stability of convective flow in the geothermal region and in the study of earth's core where the earth's mantle which consists of conducting fluid behaves like a porous medium. Bhatia and Gupta [1] have considered the finite Larmor radius effects on gravitational instability of a composite plasma. The gravitational instability of a finitely conducting plasma to include the effects due to rotation, finite Larmor radius and Hall currents has been studied by Sharma and Prakash [8].

It may therfore be of importance and is the object of the present paper to study the effects of finite conductivity, rotation, Hall currents, finite Larmor radius and the frictional effects with neutrals on the gravitational instability through porous medium of interplanetary and interstellar plasmas.

2. Perturbation Equations

Consider an infinite homogeneous composite plasma consisting of a finitely conducting hydromagnetic fluid of density ρ and a neutral gas of density ρ_d, which is uniformly rotating with velocity $\vec{\Omega}\,(0, 0, \Omega)$ and acted on by a uniformly vertical magnetic field $\mathbf{H}(0, 0, H)$ and gravity force $\mathbf{g}\,(0, 0, -g)$.

This composite plasma layer is assumed to be flowing through a porous medium. We make the assumptions that both the ionized fluid and the neutral gas behave like continuum fluids and the effects on neutral component resulting from the presence of a magnetic field and the fields of gravity and pressure are neglected. Then the linearized perturbation equations governing the motion of the composite plasma through porous medium are

$$\rho \frac{\partial \mathbf{V}}{\partial t} = - \nabla \delta p - \nabla \overleftrightarrow{P} + \rho \nabla \delta U - \frac{\rho \nu}{k_1} \mathbf{V} + \rho_d \nu_c (\mathbf{V}_d - \mathbf{V})$$

$$+ 2\rho \left(\mathbf{V} \times \vec{\Omega} \right) + \frac{1}{4\pi} \left(\nabla \times \mathbf{h} \right) \times \mathbf{H}, \tag{1}$$

$$\frac{\partial \mathbf{V}_d}{\partial t} = - \nu_c (\mathbf{V}_d - V), \tag{2}$$

$$\varepsilon \frac{\partial}{\partial t} \delta \rho = - \rho \nabla . \mathbf{V}, \tag{3}$$

$$\delta p = c^2 \delta \rho, \tag{4}$$

$$\nabla^2 \delta U = - 4 \pi G \delta \rho, \tag{5}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{H}) + \eta \nabla^2 \mathbf{h}$$

$$- \left(\frac{c}{4\pi N e} \right) \nabla \times [(\nabla \times \mathbf{h}) \times \mathbf{H}], \tag{6}$$

$$\nabla . \mathbf{h} = 0, \tag{7}$$

where $\delta\rho$, δp, δU, $\mathbf{h}$ (h_x, h_y, h_z) and $\mathbf{V}$ (u, v, w) denote respectively the perturbations in density ρ, pressure p, gravitational potential U, magnetic field $\mathbf{H}$ and velocity of the ionized (hydromagnetic) fluid. $\mathbf{V}_d$, ν_c, $\overleftrightarrow{P}$, ν, k_1, ε, c, G, η, N and e stand for velocity of the neutrals, mutual collisional (frictional) effects between the two components, pressure tensor, kinematic viscosity, of hydromagnetic fluid, medium permeability, medium porosity, velocity of sound in the medium, gravitational constant, electrical resistivity, electron number density and charge of an electron respectively. For the vertical magnetic field along the z-axis, the components of pressure tensor $\overleftrightarrow{P}$, taking into account the finite ion gyration radius, are (Roberts and Taylor 1962) :

$$P_{xx} = -\rho\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\; P_{xy} = P_{yx} = \rho\nu_0\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),$$

$$P_{xz} = P_{zx} = -2\rho\nu_0\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\; P_{yy} = \rho\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \quad (8)$$

$$P_{yz} = P_{zy} = 2\rho\nu_0\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\; P_{zz} = 0.$$

Here $\rho\nu_0 = NT/4\omega_H$, where ω_H is the ion-gyration frequency while N and T denote the number density and ion temperature respectively.

3. Dispersion Relation and Discussion

Analyzing in terms of normal modes, we seek solutions whose dependence on the space and time coordinates is of the form

$$exp\; i\,(k_x x + k_z z + \sigma t), \quad (9)$$

where σ is the growth rate of the perturbation and k_x , k_z are the wave numbers of the perturbation along the x- and z- axes.

For perturbations of the Form (9), eqs. (1)- (8) give

$$\sigma' u = -\left(\frac{k_z}{k^2}\right)\Omega_j^2 s + [i\nu_0 (k_x^2 + 2k_z^2) - 2i\Omega\;]v$$

$$+\frac{H}{4\pi\rho}(k_z h_x - k_x h_z), \quad (10)$$

$$\sigma' v = [2i\Omega - i\nu_0 (k_x^2 + 2k_z^2)]u$$

$$-2i\nu_0 k_x k_z w + \frac{H}{4\pi\rho} k_z h_y, \quad (11)$$

$$\sigma' w = -\left(\frac{k_z}{k^2}\right)\Omega_j^2 s + 2i\nu_0 k_x k_z v, \quad (12)$$

$$(i\sigma + \eta k^2)h_z = ik_z Hu - \left(\frac{cH}{4\pi Ne}\right)k_z^2 h_y, \quad (13)$$

$$(i \sigma + \eta k^2) h_y = i k_z Hv - \left(\frac{cH}{4\pi N e} \right) (- k_z^2 h_x + k_x k_z h_z), \quad (14)$$

$$(i \sigma + \eta k^2) h_z = - i k_x Hu + \left(\frac{cH}{4\pi N e} \right) k_x k_z h_y. \quad (15)$$

Here $S \left(= \frac{\delta\rho}{\rho} \right)$ denotes the condensation in the medium,

$$k^2 = k_x^2 + k_z^2 , \; \Omega_j^2 = c^2 k^2 - 4\pi G\rho , \; \alpha_0 = \frac{\rho_d}{\rho} \text{ and}$$

$$\sigma' = \sigma \left(1 + \frac{\alpha_0 \nu_c}{i \sigma + \nu_c} - \frac{i \nu_1}{\sigma k_1} \right).$$

Taking divergence of eq. (1) and using (2) — (8). we obtain

$$(\varepsilon \sigma \sigma' - \Omega_j^2) s + i k_x \{ \nu_0 (k_x^2 + 4k_z^2) - 2 \Omega \} v$$

$$+ \frac{Hk_x}{4\pi\rho} (k_z h_x - k_x h_z) = 0. \quad (16)$$

Equtions (10) — (16) can be written in the determinantal form

$$| X | \; | Y | = 0, \quad (17)$$

where $| X |$ is a seventh order determinant and $| Y |$ is a single column vector whose elements are $u, v, w, s, h_x, h_y, h_z$.

The vanishing of the determinant $| X |$ leads to the dispersion relation which is quite complex in its present form. we shall therefore discuss it in two modes of wave propagation : (i) longitudinal propagation (k_x=0) (ii) transverse propagation (k_z=0).

Mode I. Longitudinal Propagation ($k_x = 0 , k_z = k$)

For perturbations along the direction of the magnetic field, the vanishing of determinant $| X |$ gives

$$
\begin{vmatrix}
\sigma' & 2i\Omega - 2i\nu_0 k^2 & 0 & 0 & -\dfrac{kH}{4\pi\rho} & 0 & 0 \\
2i\Omega - 2i\nu_0 k^2 & -\sigma' & 0 & 0 & 0 & \dfrac{kH}{4\pi\rho} & 0 \\
0 & 0 & -\sigma' & \dfrac{\Omega_j^2}{k} & 0 & 0 & 0 \\
-ikH & 0 & 0 & 0 & i\sigma + \eta k^2 & \left(\dfrac{cH}{4\pi Ne}\right)k^2 & 0 \\
0 & -ikH & 0 & 0 & -\left(\dfrac{cH}{4\pi Ne}\right)k^2 & i\sigma + \eta k^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\sigma + \eta k^2 \\
0 & 0 & 0 & (\varepsilon\sigma\sigma' - \Omega_j^2) & 0 & 0 & -\dfrac{Hk^2}{4\pi\rho}
\end{vmatrix} = 0. \qquad (18)
$$

Equation (18) can be simplified to give

$$\sigma'(\varepsilon\sigma\sigma' - \Omega_j^2)(i\sigma + \eta k^2)[W^6 - A_5 W^5 + A_4 W^4$$

$$- A_3 W^3 + A_2 W^2 - A_1 W + A_0 = 0 \qquad (19)$$

where we have written $iW = \sigma$, $V^2 = \dfrac{H^2}{4\pi\rho}$ and

$$A_5 = 2\left(\eta k^2 + \frac{}{1 + a_0}\nu_c + \frac{\nu}{k_1}\right),$$

$$A_4 = 4(\nu_0 k^2 - \Omega)^2 + 2k^2 V^2 + \left(\nu_c \frac{}{1 + a_0} + \frac{\nu}{k_1}\right)^2$$

$$+ \frac{2\nu\nu_c}{k_1} + 4\eta k^2\left(\nu_c \frac{}{1 + a_0} + \frac{\nu}{k_1}\right) + \eta^2 k^4 + \left(\frac{cH}{4\pi Ne}\right)^2 k^4,$$

$$A_3 = 8\,(\nu_0 k^2 - \Omega)^2\,(\eta k^2 + \nu_c) + 2\,k^2 V^2$$

$$\left(\eta k^2 + \frac{\nu}{k_1} + \frac{\nu_c}{2 + \alpha_0}\right) + \frac{2\,\nu\,\nu_c}{k_1}\left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right)$$

$$+ 2\,\eta\,k^2 \left\{ \left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right)^2 + \frac{2\,\nu\,\nu_c}{k_1} \right\}$$

$$+ 2\,k^4 \left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right) \left\{ \eta^2 + \left(\frac{c\,H}{4\pi N e}\right)^2 \right\},$$

$$A_2 = 4\,(\nu_0 k^2 - \Omega)^2\,(\eta^2 k^4 + 4\,\nu_c\,\eta\,k^2 + \nu_c^2)$$

$$+ k^4 \left[V^2 - (2\,\nu_0 k^2 - 2\Omega)\,\frac{c\,H}{4\pi N e} \right]^2,$$

$$+ 2\,k^2 V^2 \left\{ \nu_c \left(\frac{\nu}{k_1} + \eta\,k^2\right) + (\nu_c + \eta\,k^2) \right.$$

$$\left. \left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right) \right\} + \frac{4\,\eta\,k^2\,\nu\,\nu_c}{k_1}\left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right)$$

$$+ \left(\frac{\nu\,\nu_c}{k_1}\right)^2 + k^4 \left\{ \left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_1}\right)^2 + \frac{2\,\nu\,\nu_c}{k_1} \right\}$$

$$\left\{ \eta^2 + \left(\frac{c\,H}{4\pi N e}\right)^2 \right\},$$

$$A_1 = 8\,(\nu_0 k^2 - \Omega)^2\,(\eta k^2 + \nu_c)\,\eta\,k^2\,\nu_c + 2\,k^4\,\nu_c$$

$$\left[V^2 - (2\,\nu_0 k^2 - 2\,\Omega) \left(\frac{c\,H}{4\pi N e}\right) \right]^2$$

$$+ 2\,k^2 V^2 \left\{ \frac{\nu\,\nu_c}{k_1}\,(\nu_c + \eta\,k^2) + \nu_c\,\eta\,k^2 \left(\frac{\nu_c}{1 + \alpha_0} + \frac{\nu}{k_2}\right) \right\}$$

$$+ 2\,\eta\,k^2 \left(\frac{\nu\,\nu_c}{k_1}\right)^2$$

$$+ \frac{\nu \, \nu_c \, k^4}{k_1} \left(\nu_c \overline{1 + a_0} + \frac{\nu}{k_1} \right) \left\{ \eta^2 + \left(\frac{c\, H}{4\pi N e} \right)^2 \right\},$$

$$A_0 = 4 \, (\nu_0 \, k^2 - \Omega \,)^2 \, \eta^2 \, k^4 \, \nu_c + k^4 \, \nu_c^2 \Big[V^2 - (\, 2 \nu_0 \, k^2 - 2\Omega \,)$$

$$\frac{c\, H}{4\pi N e} \Big]^2 + \frac{2 \, k^4 \, V^2 \, \nu \, \nu_c^2 \, \eta}{k_1}$$

$$+ \frac{\nu^2 \, \nu_c^2 \, k^4}{k_1^2} \left\{ \eta^2 + \left(\frac{c\, H}{4\pi N e} \right)^2 \right\}. \tag{20}$$

The first factor of Eq. (19), on putting $iW = \sigma$, becomes

$$W^2 - \left(\nu_c \, \overline{1 + \alpha_0} + \frac{\nu}{k_1} \right) W + \frac{\nu \, \nu_c}{k_1} = 0, \tag{21}$$

which allows both the values of W to be positive. So the real parts of both the roots of $i\sigma$ are negative which means that the system is stable. The second factor of Eq. (19), on putting $iW = \sigma$, becomes

$$\varepsilon \, W^3 - \varepsilon \left(\nu_c \overline{1 + \alpha_0} + \frac{\nu}{k_1} \right) W^2 + \left(\frac{\varepsilon \, \nu \, \nu_c}{k_1} + \Omega_j^2 \right) W$$

$$- \, \Omega_j^2 \, \nu_c = 0 \, . \tag{22}$$

When $\Omega_j^2 < 0$, at least one root of the eq. (22) is negative and so at least one root of $i\sigma$ is positive implying thereby that the plasma is unstable. Jeans' criterion for gravitational instability thus remains unchanged in the presence of finite conductivity, medium porosity, rotation, Hall currents, finite Larmor radius and collisions with neutrals.

The third factor of eq. (19) gives

$$\sigma = i \, \eta \, k^2 \, , \tag{23}$$

which corresponds to a viscous type of damped mode modified by finite conductivity.

The last factor of eq. (19) has alternately positive and negative coefficients. The real part of W is therefore positive and the composite plasma is stable.

Mode II. Transverse Propagation $(k_z = 0 , k_x = k)$

For this mode of propagation, the vanishing of the determinant $| X |$ gives

$$\begin{vmatrix} \sigma' & 2\,i\,\Omega - i\,\nu_0\,k^2 & 0 & \dfrac{\Omega_j^2}{k} & 0 & 0 & \dfrac{k\,H}{4\,\pi\,\rho} \\ 2\,i\,\Omega - i\,\nu_0\,k^2 & -\,\sigma' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\,\sigma + \eta_i\,k^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\dfrac{c\,H}{4\pi Ne}\right)k^2 & i\,\sigma + \eta_i\,k^2 & 0 \\ i\,k\,H & 0 & 0 & 0 & 0 & 0 & i\,\sigma + \eta_i\,k^2 \\ 0 & i\,k\,(\nu_0\,k^2 - 2\,\Omega) & 0 & \varepsilon\,\sigma\,\sigma' - \Omega_j^2 & 0 & 0 & \dfrac{H\,k^2}{4\pi\rho} \end{vmatrix} = 0. \qquad (24)$$

on simiplification, eq. (24) gives

$$\sigma' \,(\,i\,\sigma + \eta_i\,k^2 \,)^2 \,[\, B_6\,\sigma\,W^6 - B_5\,W^5 + B_4\,W^4 - B_3\,W^3$$

$$+ B_2 W^2 - B_1 W + B_0] = 0 , \quad (25)$$

where we have written $i W = \sigma$ and

$$B_6 = \varepsilon ,$$

$$B_5 = \varepsilon \left[2 \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right) + \eta k^2 \right],$$

$$B_4 = \varepsilon \left\{ \left(\nu_c \sqrt{1 + \alpha_o} + \frac{\nu}{k_1} \right)^2 + \frac{2 \nu \nu_c}{k_1} \right\}$$

$$+ 2 \varepsilon \eta k^2 \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right)$$

$$+ \varepsilon k^2 V^2 + \varepsilon (\nu_0 k^2 - 2 \Omega)^2 + \Omega_j^2 ,$$

$$B_3 = \varepsilon k^2 V^2 \left(\nu_c \sqrt{2 + \alpha_0} + \frac{\nu}{k_1} \right) + \frac{2 \varepsilon \nu \nu_c}{k_1} \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right)$$

$$+ \varepsilon (\nu_0 k^2 - 2 \Omega)^2 (2 \nu_c + \eta k^2) \quad (26)$$

$$+ \varepsilon \eta k^2 \left\{ \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right)^2 + \frac{2 \nu \nu_c}{k_1} \right\}$$

$$+ \Omega_j^2 \left(\nu_c \sqrt{2 + \alpha_0} + \eta k^2 + \frac{\nu}{k_1} \right),$$

$$B_2 = \varepsilon k^2 V^2 \left\{ \frac{\nu \nu_c}{k_1} + \nu_c \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right) \right\}$$

$$+ \varepsilon \left(\frac{\nu \nu_c}{k_1} \right)^2 + 2 \varepsilon \eta k^2 \frac{\nu \nu_c}{k_1} \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right)$$

$$+ \varepsilon (\nu_0 k^2 - 2\Omega)^2 (\nu_c + 2\eta k^2) \nu_c + \Omega_j^2 \left\{ \nu_c \eta k^2 + \frac{\nu \nu_c}{k_1} \right.$$

$$\left. + (\nu_c + \eta k^2) \left(\nu_c \sqrt{1 + \alpha_0} + \frac{\nu}{k_1} \right) \right\},$$

$$B_1 = \varepsilon k^2 V^2 \frac{\nu \nu_c^2}{k_1} + \varepsilon \eta k^2 \left(\frac{\nu \nu_c}{k_1} \right)^2 + \varepsilon (\nu_0 k^2 - 2\Omega)^2 \eta \nu_c^2 k^2$$

$$+ \Omega_j^2 \left[\nu_c \eta\, k^2 \left(\nu_c \overline{1+\alpha_0} + \frac{\nu}{k_1} \right) + \frac{\nu\nu_c}{k_1} \left(\nu_c + \eta\, k^2 \right) \right]$$

$$B_0 = \frac{\nu\nu_c^2\, \eta\, k^2}{k_1}\, \Omega_j^2 \,.$$

The first factor of eq. (25), on putting $iW = \sigma$, becomes eq. (21) and so the system is stable. The second factor of eq. (25) corresponds to a viscous type of damped mode modified by finite conductivity.

Consider now the last factor of eq. (25). *If* $\Omega_j^2 < 0$, the product of the roots of W is negative and so the last factor of eq. (25) has at least one negative real root. The considered plasma is therefore unstable when $\Omega_j^2 < 0$ *i. e.* when Jeans' criterion is satisfied.

We therefore conclude that Jeans' criterion remains unchanged in the presence of finite conductivity, medium porosity, rotation, Hall currents, finite Larmor radius effects and the frictional effects with neutrals for both longitudinal and transverse propagations,

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