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INTEGRATION OF CERTAIN PRODUCTS ASSOCIATED WITH THE *H*-FUNCTION OF SEVERAL VARIABLES

by

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Abstract

Five integrals associated with the product of the general H function of several complex variables, defined by H. M. Srivastva and R. Panda, are evaluated. These integrals provide a unification of a number of earlier results indicated in Section 3.

1. Introduction

The H- function of several complex variables $x_1, \ldots, x_r$ (defined by Srivastava and Panda [5] by means of a multiple Mellin–Barnes type integral) is represented in the following from [5, p. 168, Eq. (1.3) *et seq*]:

$$(1.1)\quad H(x_1, \ldots, x_r)$$

$$= H^{0,\, l\, :\, (m', n')\, ;\, \ldots\, ;\, (m^{(r)}, n^{(r)})}_{A,\, C\, :\, [B', D']\, ;\, \ldots\, ;\, [B^{(r)}, D^{(r)}]}$$

$$\left(\begin{matrix} [(a) ; \alpha' \ldots , \alpha^{(r)}] : [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}]; \\ [(c) : \beta' \ldots , \beta^{(r)}] : [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \; x_1, \ldots, x_r \right)$$

$$= \frac{1}{(2\pi w)^r} \int_{L_1} \cdots \int_{L_r} \Psi(s_1, \ldots, s_r) \prod_{i=1}^{r} \left\{ \phi_i(s_i)\, x_i^{s_i}\, ds_i \right\}, \; w = \sqrt{-1}.$$

For the values of the functions Φ and Ψ, the various conditions on the parameters and conditions of absolute convergence of the multiple contour integrals on the R. H. S. of (1.1),and for its particular cases, etc., we refer to Srivastava and Panda ([5] and [6]). These conditions are assumed to be satisfied by the various $H-$ functions occurring in this paper.

Here (a) denotes the set of A parameters $a_1, \dots, a_A$ and

$(b^{(i)})$ denotes the set of $B^{(i)}$ parameters $b_1^{(i)}, \dots, b^{(i)}_{B^{(i)}}$, etc.

Also,the appearance of the asterisk(*)indicates that the parameters at those places are the same as the parameters of the $H-$ function of r variables in the L. H. S. of (1. 1) at the corresponding places.

2. The Main Integrals

We evaluate the following definite integrals:

$$(2.1) \qquad \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1+kx)^{-\lambda-\mu} H(z_1X_1, \dots, z_rX_r)\,dx$$

$$= (1+k)^{-\lambda} H^{0,\, l+2\,:\,*}_{A+2,\, C+1\,:\,*} \left(\begin{matrix} P : * ; \\ Q : * ; \end{matrix} \; G_1, \dots, G_r \right)$$

where

$$P : [\, 1-\lambda : u_1, \dots, u_r\,], [\, 1-\mu : v_1, \dots, v_r\,],$$
$$[\,(a) : \alpha', \dots, \alpha^{(r)}\,],$$

$$Q : [\, 1-\lambda-\mu ; u_1+v_1, \dots, u_r+v_r\,], [\,(c) : \beta', \dots, \beta^{(r)}\,],$$

$$X_i = \frac{x^{u_i}(1-x)^{v_i}}{(1+kx)^{u_i+v_i}}, \; G_i = z_i(1+k)^{-u_i},$$

$$D_j^{(i)} = \frac{d_j^{(i)}}{\delta_j^{(i)}}, \; U_j = \left(\lambda + \sum_{i=1}^{r} u_i \, D_j^{(i)} \right),$$

$$V_j = \left(\mu + \sum_{i=1}^{r} V_i \, D_j^{(i)} \right),$$

provided that

$$min \{ \text{Re} \, (\lambda , \mu , U_j , V_j) \} > 0 \, , k > -1 ;$$

(2. 2) $$\int_0^1 \cdots \int_0^1 H \left(z_1 X_1' , \ldots , z_r X_r' \right) \prod_{i=1}^{r} \left(f (x_i) \, dx_i \right)$$

$$= A_r H \begin{matrix} 0 , l : (m' , n' + 2) ; \ldots ; (m^{(r)} , n^{(r)} + 2) \\ A , C : [B' + 2 , D' + 1] ; \ldots ; [B^{(r)} + 2 , D^{(r)} + 1] \end{matrix}$$

$$\left(\begin{matrix} * : K ; \\ * : T ; \end{matrix} \; z_1 , \ldots , z_r \right)$$

where

$$K : (1 - \lambda_1 , u_1) , (1 - \mu_1 , v_1) , [(b') : \phi']$$

$$; \ldots ; (1 - \lambda_r , u_r) , (1 - \mu_r , v_r) , [(b^{(r)}) : \phi^{(r)}],$$

$$T : (1 - \lambda_1 - \mu_1 , u_1 + v_1) , [(d') : \delta']$$

$$; \ldots ; (1 - \lambda_r - \mu_r , u_r + v_r) , [(d^{(r)}) : \delta^{(r)}],$$

$$f (x_i) = x_i^{\lambda_i - 1} (1 - x_i)^{\mu_i - 1} (1 + k_i x_i)^{-\lambda_i - \mu_i} .$$

$$X_i' = \frac{x_i^{u_i} (1 - x_i)^{v_i}}{(1 + k_i x_i)^{u_i + v_i}} , \; z_i = z_i (1 + k_i)^{u_i} ,$$

$$A_r = \prod_{i=1}^{r} (1 + k_i)^{-\lambda_i} , \; U_j' = (\lambda_i + u_i D_j^{(i)}) ,$$

$$V_j' = (\mu_i + v_i D_j^{(i)})$$

provided that $\min\{\operatorname{Re}(\lambda_i, \mu_i, U_j', V_j')\} > 0$, $k_i > -1$;

$$(2.3) \quad \int_0^1 \cdots \int_0^1 g(y) H(z_1 X_1' Y_1, \ldots, z_r X_r' Y_r)$$

$$\prod_{i=1}^{r} (f(x_i)\, dx_i)\, dy$$

$$= A_r' H^{0, l+2 : (m', n'+2); \ldots; (m^{(r)}, n^{(r)}+2)}_{A+2, C+1 : [B'+2, D'+1; \ldots; [B^{(r)}+2, D^{(r)}+1]}$$

$$\left(\begin{matrix} R : K ; \\ S : T : \end{matrix} Z_1', \ldots, Z_r' \right)$$

where

$R : [1-p : e_1, \ldots, e_r], [1-q : f_1, \ldots, f_r]$,

$[(a) : \alpha', \ldots, \alpha^{(r)}]$,

$S : [1-p-q : e_1 + f_1, \ldots, e_r + f_r], [(c) : \beta', \ldots, \beta^{(r)}]$,

$g(y) = y^{p-1}(1-y)^{q-1}(1+ky)^{-p-q}$, $A_r' = (1+k)^{-p} A_r$,

$$Y_i = \frac{y^{e_i}(1-y)^{f_i}}{(1+ky)^{e_i+f_i}}$$

$Z_i' = z_i (1+k)^{-e_i} (1+k_i)^{-u_i}$, $E_j = (p + \sum_{i=1}^{r} e_i D_j^{(i)})$

$F_j = (q + \sum_{i=1}^{r} f_i D_j^{(i)})$

$\min \operatorname{Re}(p, q, \lambda_i, \mu_i U_j', V_j', E_j, F_j) > 0$, $k > -1$, $k_i > -1$,

X_i', $f(x_i)$, A_r, K and T are defined above with (2.2).

(2.4) $$\int_0^1 \cdots \int_0^1 g(y)\, H\,(z_1 X_1'' Y_1, \ldots, z_r X_r'' Y_r) \prod_{i=1}^{r} (f(x_i)\, dx_i)\, dy$$

$$= A_r'' H \begin{matrix} 0, l+2 : (m', n+1); \ldots; (m^{(r)}, n^{(r)}+1) \\ A+2, C+1 : [B'+1, D'+1]; \ldots; \end{matrix}$$

$$[B^{(r)}+1, D^{(r)}+1]$$

$$\left(\begin{matrix} R ; K' ; \\ S : T' ; \end{matrix} \quad Z_1', \ldots, Z_r' \right)$$

where

$K' : (1-\lambda_1, u_1), [(b') : \phi'] ; \ldots ; (1-\lambda_r, u_r),$

$[(b^{(r)}) : \phi^{(r)}],$

$T' : (1-\lambda_1-\mu_1, u_1), [(d') : \delta] ; \ldots ; (1-\lambda_r-\mu_r, u_r)$

$[(d^{(r)}) : \delta^{(r)}],$

$$X_i'' = \frac{x_i^{\,u_i}}{(1+k_i x_i)^{u_i}}, \quad A_r'' = A_r' \prod_{i=1}^{r} \Gamma(\mu_i),$$

provided that $min \{ \text{Re}(p, q, \lambda_i, \mu_i, E_j, F_j, U_j') \} > 0$,
$k > -1, k_i > -1$,

(2.5) $$\int_0^1 \cdots \int_0^1 g(y)\, H(z_1 X_1''' Y_1, \ldots, z_r X_r''' Y_r)$$

$$\prod_{i=1}^{r} (f(x_i)\, dx_i)\, by,$$

$$= A_r''' H \begin{matrix} 0, l+2 ; (m', n'+1); \ldots; (m^{(r)}, n^{(r)}+1) \\ A+2, C+1 ; [B'+1, D'+1]; \ldots; \end{matrix}$$

$$[B^{(r)}+1, D^{(r)}+1]$$

$$\left(\begin{matrix} R : K'' ; \\ S ; T'' : \end{matrix} \quad G_1', \ldots, G_r' \right)$$

where

$T'' : (1 - \lambda_1 - \mu_1 , v_1) , [(d') : \delta'] ; \ldots ; (1 - \lambda_r - \mu_r , v_r),$

$[(d^{(r)}) : \delta^{(r)}] ,$

$K'' : (1 - \mu_1 , v_1) , [(b') : \phi'] ; \ldots ; (1 - \mu_r , v_r) ,$

$[(b^{(r)}) : \phi^{(r)}] ,$

$$X_i''' = \frac{(1 - x_i)^{v_i}}{(1 + k_i x_i)^{v_i}} , \; G_i' = z_i (1 + k)^{-e_i} ,$$

$$A_r''' = A_r' \prod_{i=1}^{r} \Gamma (\lambda_i)$$

provided that $min \{ \text{Re} (\lambda_i ; \mu_i , p , q , E_j , F_j , V_j') \} > 0$,

$k > - 1 , k_i > - 1$.

$\{ Here \; i = 1 , \ldots , r \; and \; j = 1 \ldots , m^{(i)} . \}$

Method of Derivation : To establish (2.1), we Substitute for the H- function of several variables occurring in the integrand of (2.1) in terms of the multiple contour integrals from (1.1), change the order of integration (which is justified under the given conditions). evaluate the x-integral by means of [1, p. 10. Eq. (11)], and interpret the resulting integrals by virtue of (1.1) to get the desired result.

The proofs of the remaining integrals can be given on similar lines.

3. Particular Cases : A large number of integrals involving various special functions of one or severals variables may be obtained by appropriate specialization of the integrals (2.1) to (2.5) but these are not recorded here for lack of space. To illustrate, if we put $k=0$ or $k_i=0$ $(i=1 , \ldots , r)$, (2.1) to (2.5) reduce redily to a number of elegant integrals given by Exton recently (see [2], Chapter 1). Also, if we put $A=C=0$ in (2.1), (2.3) to (2.5), we get quite elegant results involving product of r H- functions of one variable. In case $r=1$ or 2, a number of integrals, including those of Koul [3], and Koul and Raina [4], follow as special cases of our results.

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