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A FIXED POINT THEOREM FOR ORBITALLY CONTINUOUS FUNCTION

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Recently, Jaggi [1] has given a result on fixed point theorem for orbitally continuous mapping and has shown that a mapping may be x – orbitally continuous for some $x \in X$ without being orbitally continuous. The purpose of this note is to extend the result of Jaggi by using a wider class of mapping.

Theorem. *Let K be a nonempty convex subset of a normed linear space X, let $T: K \rightarrow K$ be a mapping satisfying*

(1) $d(Tx, Ty) \leqslant h \max\{d(x, y), [d(x, Tx) + d(y, Ty)]/2,$

$[d(x, Ty) + d(y, Tx)]/2\},$

for all x, y in K and $0 \leqslant h < 1$. If there exists an x_0 in K and a λ in $(0, 1)$ such, that

(2) $\left\{T_{\lambda}^{n}(x_0)\right\}$ clusters to a point $z \in K$,

and

(3) T_λ is x_0 – orbitally continuous and asymptotically regular at x_0,

where $T_\lambda = \lambda I + (1 - \lambda) T$.

Then $\left\{ T_\lambda^n (x_0) \right\}$ converges to z and z is a unique fixed point of T in K.

Proof. By (2), there exists a subsequence $\left\{ T_\lambda^{n_i} (x_0) \right\}$ which converges to z. Since T_λ is x_0 – orbitally continuous, we have

$$(I - T_\lambda) T_\lambda^{n_i} (x_0) \to (I - T_\lambda) z.$$

Also, due to asymptotic regularity of T_λ at x_0, we observe that

$$\| (I - T_\lambda) T_\lambda^{n_i} (x_0) \| = \| T_\lambda^{n_i} (x_0) - T_\lambda^{n_i+1} (x_0) \| \to 0$$

as $i \to \infty$.

Therefore, $\| (I - T_\lambda) z \| = 0$, which gives

$T_\lambda (z) = z$ and $T (z) = z$.

for uniqueness, let z' be another fixed point of T.

Then By (1)

$$\| z - z' \| \leqslant h \max. \{ \| z - z' \|, [\| z - Tz \| + \| z' - Tz' \|] /2, [\| z - Tz' \| + \| z' - Tz \|]/2 \}.$$

$\leqslant h \{ \| z - z' \| \}$, from which we have

$\| z - z' \| = 0$, i.e., $z = z'$, as $0 \leqslant h < 1$.

Finally, consider the convergence of $\left\{ T_\lambda^n (x_0) \right\}$.

For all $x \in K$, we have

(4) $\| T_\lambda (x) - z \| = \| \lambda (x - z) + (1 - \lambda) (Tx - Tz) \|$

$\leqslant \lambda \| x - z \| + (1 - \lambda) \| Tx - Tz \|$

But by (1) we have

$$\| Tx - Tz \| \leqslant h \max \{ \| x - z \|, \underline{\| x - Tx \|}_{2},$$

$$[\| x - Tz \| + \| z - Tx \|] /2 \}$$

$$\leqslant h \max \{ \| x - z \|, [\| x - z \| + \| Tz - Tx \|]/2,$$

$$[\| x - z \| + \| Tz - Tx \|] /2 \}$$

$\leqslant h \| x - z \| + h \| Tx - Tz \|$ from wheıe we get

(5) $$\| Tx - Tz \| \leqslant \frac{h}{1-h} \| x - z \|$$

$\leqslant \| x - z \|$, sice $0 \leqslant h < 1$.

from (4) and (5), we get

$$\| T_\lambda (x) - z \| \leqslant \| x - z \| .$$

Since $x \in K$ is arbitrary, it follows that

$$\| T_\lambda^{n+1} (x_0) - z \| \leqslant \| T_\lambda^{n} (x_0) - z \|,$$

which guarantees that $\left\{ T_\lambda^{n} (x) \right\} \to z$, since $T_\lambda^{n_i} (x_0) \to z$

as $i \to \infty$, and this completes the proof.

REFERENCE

[1] D. S. Jaggi, Fixed point theorems for orbitally continuous functions. II, *Indian J. Math.* **19** (1977), 113–118.