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A NOTE ON THE GEOMETRIC MEANS OF AN ENTIRE FUNCTION OF ORDER ZERO

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1. INTRODUCTION

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z = re^{i\theta}$, be a power series

which converges for all z, and let

$$M(r) = \max_{|z|=r} |f(z)|,$$

$$\mu(r) = \max_{0 \leqslant n \leqslant \infty} |a_n| r^n$$

and

$$\nu(r) = max\,\{N : \mu(r) = |a_N| r^N\}.$$

The order ρ for $f(z)$ is given by

$$(1.1)\quad \lim_{r\to\infty} \sup \frac{\log \log M(r)}{\log r} = \rho,\ 0 \leqslant \rho \leqslant \infty.$$

For an entire function, other than a polynomial, of order zero, the logarithmic order ρ^* is defined [2] as :

$$(1.2)\quad \lim_{r\to\infty} \sup \frac{\log \log M(r)}{\log \log r} = \rho^*,\ 1 \leqslant \rho^* \leqslant \infty.$$

Pólya and Szegö [1] defined the geometric mean of $f(z)$, for $|z| = r$, as :

$$(1.3)\quad G(r, f) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f(r e^{i\theta})|\, d\theta\right\}.$$

Let us set the following geometric means of $f(z)$ and its n-th derivative, that is $f^{(n)}(z)$, as;

$$(1.4)\quad G(r, f^{(n)}) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f^{(n)}(r e^{i\theta})|\, d\theta\right\},$$

$$(1.5)\quad g_\delta^*(r, f) =$$

$$= \exp\left\{\frac{\delta+1}{2\pi(\log r)^{\delta+1}}\int_1^r \int_0^{2\pi} x^{-1} (\log x)^\delta \log |f(x e^{i\theta})|\, dx\, d\theta\right\}$$

and

$$(1.6)\quad g_\delta^*(r, f^{(n)}) =$$

$$= \exp\left\{\frac{\delta+1}{2\pi(\log r)^{\delta+1}}\int_0^r \int_0^{2\pi} x^{-1} (\log x)^\delta \log |f^{(n)}(x e^{i\theta})|\, dx\, d\theta\right\}$$

where $0 < \delta < \infty$.

It is known that [2] :

$$(1.7) \quad \lim_{r\to\infty} \sup \frac{\log \nu(r)}{\log \log r} = \rho^* - 1.$$

In this paper we investigate some of the growth properties of the geometric means $G(r, f)$, $G(r, f^{(n)})$, $g_\delta^*(r, f)$, and $g_\delta^*(r, f^{(n)})$.

2. MAIN RESULTS

Theorem 1. *Let $f(z)$ be an entire function, other than a polynomial, of order zero with logarithmic order ρ^*. Then*

$$(2.1) \quad \lim_{r\to\infty} \sup \frac{\log \left[r \left\{ \frac{G(r, f^{(1)})}{G(r, f)} \right] \right\}}{\log \log r} = \rho^* - 1$$

in the neighbourhood of points where $| f(z) | > M(r)\, \nu^{-1/8}$.

Proof. From (1. 4), we have

$$(2.2) \quad G(r, f^{(1)}) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log | f^{(1)}(r e^{i\theta}) | \, d\theta \right\}$$

$$= \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f^{(1)}(r e^{i\theta})}{f(r e^{i\theta})} \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log | f(re^{i\theta}) | \, d\theta \right\}.$$

Also, we have [3], in the neighbourhood of points where

$| f(z) | > M(r)\, \nu^{-1/8}$,

$$(2.3)\quad \frac{f^{(1)}(z)}{f(z)} = \left\{ 1 + h(z)\,\nu(R)^{-1/16} \right\} \frac{\nu(r)}{z}\ ;\ |h| < k\,.$$

On using (1. 3) and (2. 3) in (2. 2), we find that

$$G(r, f^{(1)}) =$$

$$= G(r, f)\, exp \left[\frac{1}{2\pi} \int_0^{2\pi} log \left\{ \left| 1 + h(z)\,\nu(R)^{-1/16} \right| \left| \frac{\nu(r)}{z} \right| \right\} d\theta \right].$$

This gives

$$(2.4)\quad G(r, f^{(1)}) > G(r, f)\, \frac{\nu(r)}{r} \left(1 - k\,\nu(R)^{-1/16}\right)$$

and

$$(2.5)\quad G(r, f^{(1)}) < G(r, f)\, \frac{\nu(r)}{r} \left(1 + k\,\nu(R)^{-1/16}\right)$$

in the neighbourhood of points where $|f(z)| > M(r)\,\nu^{-1/8}$.

Proceeding to superior limits, as $r \to \infty$, of both sides in (2. 4) and (2. 5), and using (1. 7), we get

$$\lim_{r\to\infty} \sup \frac{log\left[r \left\{ \frac{G(r, f^{(1)})}{G(r, f)} \right\} \right]}{log\ log\ r}$$

$$= \lim_{r \to \infty} \sup \frac{\log \nu(r)}{\log \log r} = \rho^* - 1.$$

This proves Theorem 1.

Corollary 1. *For the entire function $f(z)$ of logarithmic order ρ^*,*

(2. 6)
$$\lim_{r \to \infty} \sup \frac{\log \left[r \left\{ \frac{G(r, f^{(n)})}{G(r, f)} \right\}^{1/n} \right]}{\log \log r} = \rho^* - 1.$$

Writing (2. 4) and (2. 5) for the s–th derivative of $f(z)$,

we get

$$G(r, f^{(s)}) > (G(r, f^{(s-1)}) \frac{\nu(r)}{r} \; (1 - k\, \nu(R)^{-1/16})$$

and

$$G(r, f^{(s)}) < G(r, f^{(s-1)}) \frac{\nu(r)}{r} \; (1 + k\, \nu(R)^{-1/16}),$$

respectively. Taking $s = 1, 2, \ldots, n$, multiplying all the inequalities thus obtained and proceeding to limits, (2.6) follows :

Theorem 2. *For every entire function $f(z)$, other than a polynomial. of logarithmic order ρ^*,*

(2. 7)
$$\lim_{r \to \infty} \sup \frac{\log \left[r^{(\delta+1)/(\delta+2)} \left\{ \frac{g_\delta^*(r, f^{(1)})}{g_\delta^*(r, f)} \right\} \right]}{\log \log r} \leqslant \rho^* - 1$$

provided $| f(z) > | M(r) \nu^{-1/8}$.

Proof. From (1. 4) and (1. 6), we have

$$(2.\ 8)\quad \log g_\delta^*(r\ f^{(1)}) = \frac{\delta+1}{(\log r)^{\delta+1}} \int_0^r x^{-1} (\log x)^\delta \log G(x, f^{(1)})\, dx$$

$$= 0(1) + \frac{\delta+1}{(\log r)^{\delta+1}} \int_{r_0}^r x^{-1} (\log x)^\delta \log G(x, f^{(1)})\, dx,\ r > r_0 .$$

Further, from (2. 1), we have

$$(2.\ 9)\quad G(r, f^{(1)}) < r^{-1}\, G(r, f)\ (\log r)^{\rho^* + \varepsilon - 1},$$

for $r > r_0$ and $\varepsilon > 0$.

Using (2. 9) in (2. 8), we get, for any $r > r_0$ and $\varepsilon > 0$,

$$\log g_\delta^*(r, f^{(1)})$$

$$< 0\,(1) + \log g_\delta^*(r, f) - \frac{\delta+1}{\delta+1} \log r + (\rho^* + \varepsilon - 1) \log \log r.$$

Proceeding to superior limit, as $r \to \infty$, the result follows.

Corollary 2. *For an entire function with logarithmic order* ρ^* ,

$$\lim_{r\to\infty} \sup \frac{\log \left[r^{(\delta+1)/(\delta+2)} \left\{ \frac{g_\delta^*(r\ f^{(n)})}{g_\delta^*(r, f)} \right\}^{1/n} \right]}{\log\ \log\ r} \leqslant \rho^* - 1.$$

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