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# A UNIFIED PRESENTATION OF TWO GENERAL SEQUENCES OF FUNCTIONS

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## 1. INTRODUCTION

In an attempt to unify several polynomial systems, Srivastava and Singhal [4] studied the polynomials $\{ T_n^{(\alpha, \beta)} (x, a, b, c, d, p, r) \mid n=0, 1, 2, \ldots \}$

defined by

$$(1.1)\quad T_n^{(\alpha, \beta)} (x, a, b, c, d, p, r) = \frac{(ax+b)^{-\alpha} (cx+d)^{-\beta}}{n!} \times$$

$$\times \exp (px^r) D^n x \{ (ax+b)^{n-\alpha} (cx+d)^{n-\beta} \exp (-px^r) \},$$

and Srivastava and Panda [3] studied the general sequence of functions

$$\{ S_n^{(\alpha, \beta)} [x, a, b, c, d; \gamma, \varepsilon; w(x)] \mid n=0, 1, 2, \ldots \}$$

defined by

$$(1.2)\quad S_n^{(\alpha, \beta)} [x, a, b, c, d; \gamma, \varepsilon; w(x)] = \frac{(ax+b)^{-\alpha} (cx+d)^{-\beta}}{n! \; w(x)} \times$$

$$\times D^n x \{ (ax+b)^{\gamma n-\alpha} (cx+d)^{\varepsilon n-\beta} w(x)\}, Dx = d/dx$$

where $a, b, c, d, \alpha, \beta, \gamma, \varepsilon$ are constants, and $w(x)$ is independent of $n$ and differentiable an arbitrary number of times.

Recently, Singh [2] introduced a general sequence of functions defined by the following Rodrigues' formula

(1. 3) $G_n\,[p;\,h(x),\,g(x)] = e^{-pg(x)}\,D^n x\,\{\,[h(x)]^n\,e^{pg(x)}\,\}$,

where $p$ is constant and $g(x)$, $h(x)$ are suitable functions of $x$. Also Chandel and Bhargava [1] introduced a sequence of functions

$G_n\,[a,\,k;\,p,\,g(x)]$ defined by

(1. 4) $G_n\,[a,\,k;\,p,\,g(x)] = e^{-pg(x)}\,T^n_{a,k}\,\{e^{pg(x)}\}$,

where $a$, $k$, $p$ are constants, $g(x)$ is a differentiable function of $x$ and

(1. 5) $T_{a,k} = x^k\,(a+xDx),\ Dx \equiv d/dx$.

The explicit connections of (1. 1) with (1. 3), and (1. 2) with (1. 3), are respectively

(1. 6) $G_n\,[p;\,1/p\,\log_e\{\,(ax+b)^\alpha\,(x+d)^\beta\,e^{-px^r}\,\},\,(ax+b)\,(cx+d)]$

$= n!\,T_n^{(\alpha,\beta)}\,(x,\,a,\,b,\,c,\,d,\,p,\,r)$

and

(1.7) $G_n\,[p;\,1/p\,\log_e\{\,(ax+b)^\alpha\,(cx+d)^\beta\,w(x)\},\,(ax+b)^\gamma\,(cx+d)^\epsilon\,]$

$= n!\,S_n^{(\alpha,\beta)}\,[x,\,a,\,b,\,c,\,d;\,\gamma,\,\varepsilon;\,w(x)\,]$.

The object of this paper is to unify the study of the two general sequences of functions (1. 3) and (1. 4) by introducing a new sequence of functions

$\{\,G_n\,[a,\,k,\,p;\,g(x),\,h(x)]\mid n=0,\,1,\,2,\ldots\,\}$

defined by

(1. 8) $G_n\,[a,\,k,\,p;\,g(x),\,h(x)] = e^{-pg(x)}\,T^n_{a,k}\,\{\,[h(x)]^n\,e^{pg(x)}\}$

where $h(x)$, $g(x)$ are suitable functions of $x$ and $a$, $k$, $p$ are constants.

For $h(x)=1$ it reduces to (1.4) and for $k=-1$, $a=0$ it reduces to (1.3). For brevity, we shall write. $G_n\,(a, k, p; g, h)$

for $G_n\,[a, k, p; g(x), h(x)]$.

## 2. Operational Formulas

Considering

$$e^{-pg}\, T^n_{a,k}\, [h^n e^{pg} f] = e^{-pg} \sum_{s=0}^{n} \binom{n}{s} T^{n-s}_{a,k}\, (h^n e^{pg})\, T^s_{0,k} f,$$

we have

$$(2.1)\quad e^{-pg}\, T^n_{a,k}\, (h^n e^{pg} f) = \sum_{s=0}^{n} \binom{n}{s} G_{n-s}\, (a, k, p; g, h^{n/n-s})\, T^s_{0,k} f.$$

which, on simple manipulation, gives

$$(2.2)\quad e^{-pg}\, T^n_{a,k}\, (h^n e^{pg} f) = h^n \left[ T_{a,k} + x^{k-1} \left\{ \frac{nh'}{h} + pg' \right\} \right]^n f.$$

From the above two resulus we obtain

$$(2.3)\quad h^n \left[ T_{a,k} + x^{k-1} \left\{ \frac{nh'}{h} + pg' \right\} \right]^n f = \sum_{s=1}^{n} \binom{n}{s}$$

$$G_{n-s}\,(a, k, p; g, h^{n/n-s})\, T^s_{0,k} f.$$

Now replacing $g$ by $(g+s/p \log h)$ and $n$ by

$(n-s)$ in (1.8) we get

$$(2.4)\quad h^n\left[T_{a,k} + x^{k+1}\left\{\frac{nh'}{h} + pg'\right\}\right]^n f$$

$$= \sum_{s=0}^{n} \binom{n}{s} h^s G_{n-s}(a, k, p; g+s/p\ logh, h)\ T_{0,k}^{s} f,$$

which for $f=1$ reduces to

$$(2.5)\quad \left[T_{a,k} + x^{k+1}\left\{\frac{nh'}{h} + pg'\right\}\right]^n 1 = h^{-n} G_n(a, k, p; g, h).$$

Next consider

$$T^n_{a,k}\,[h^n e^{pg} f] = T^{n-1}_{a,k}\,[h^{n-1} e^{pg}\{x^{k+1}(nh'+hpg') + h\,T_{a,k}\} f]$$

where

$$f_1 = \{x^{k+1}(nh'+hpg') + h\,T_{a,k}\} f$$

and repeating the process we get

$$(2.6)\quad T^n_{a,k}\,[h^n e^{pg} f] = e^{pg} \prod_{i=0}^{n-1} \{h\,T_{a,k} + x^{k+1}(hpg' + (n-i)h'\}\ f,$$

which, by using (2. 1), gives

$$(2.7)\quad \prod_{i=1}^{n-1} \{h\,T_{a,k} + x^{k+1}(hpg' + (n-i)\,h')\,f$$

$$= \sum_{s=0}^{n} \binom{n}{s} G_{n-s}\,(a.\ k,\ p;\ g,\ h^{\frac{n}{n-s}})\quad T^s_{0,k}\ f$$

and, for $f=1$, it further gives

$$(2.8)\quad \prod_{i=0}^{n-1} \{h\, T_{a,k} + x^{k+1}(hpg' + (n-i)h'\}\, 1 = G_n\,(a, k, p; g, h).$$

Again, if we consider

$$T^n_{a,k}\,[h^n\, e^{pg} f\,] = T^{n-1}_{a,k}\,[h^{n-1}\, e^{pg}\, \{x^{k+1}\,(nh' + hpg') + h\, T_{a,k}\}\, f]$$

$$= T^{n-1}_{a,k}\,[h^{n-1}\, e^{pg}\, h^{-m}\, \{x^{k+1}\,(nh^m\, h' + h^{m+1}\, pg') + h^{m+1}\, T_{a,k}\}\, f],$$

where

$$f_1 = \{x^{k+1}\,(nh^m\, h' + h^{m-1}\, pg') + h^{m-1}\, T_{a,k}\,\}\, f,$$

then iteration leads to

$$(2.9)\quad T^n_{a,k}\,[h^n\, e^{pg} f\,] = h^{-nm}\, e^{pg}\, \times$$

$$\times \prod_{i=1}^{n-1} \{h^{m+1}\, T_{a,k} + x^{k+1}\,(h^{m+1}\, pg' + [n - (m+1)\,(i-1)]h^m\, h')\}\, f$$

which, by an appeal to (2. 1), gives

$$(2.10)\quad \prod_{i=1}^{n-1} \{h^{m+1}\, T_{a,k} + x^{k+1}\,(h^{m+1}\, pg' + [n - (m+1)\,)\,(i-1)h^m\, h')\}\, f$$

$$= h^{nm} \sum_{s=0}^{n} \binom{n}{s}\, G_{n-s}\,(a, k.\, p;\, g,\, h^{n/n-s})\, T^s_{0,k}\, f$$

and, for $f=1$,

$$(2.11)\quad \prod_{i=1}^{n-1} \{\, h^{m+1}\, T_{a,k} + x^{k+1}\,(h^{m+1}\, pg' + [n-(m+1)\,(i-1)\,]h^m\, h')\}.\ 1$$

$$= G_n\,(a,\,k,\,p;\,g,\,h).$$

Further, consider

$$T^n_{a;k}\;[h^n\, e^{pg} f\,] = T^{n-1}_{a;k}\,[h^n\, e^{pg}\, x^{-m}\left\{\frac{nh'}{h}\; x^{m+k-1}\right.$$

$$\left. +\; pg'\, x^{m+k+1} + x^m\, T_{a,k}\right\} f\,]$$

where

$$f_1 = \left(\frac{nh'}{h}\; x^{m+k-1} + pg'\, x^{m+k+1} + x^m\, T_{a,k}\right) f.$$

This, on repeating the process $n$ times, gives

$$(2.\ 12)\quad T^n_{a;k}\;[h^n\, e^{pg} f\,] = h^n\, e^{pg}\, x^{-nm}\;\times$$

$$\times\prod_{i=1}^{n}\,[x^m\, T_{a,k} + \frac{nh'}{h}\; x^{m+k-1} + pg'\, x^{m+k+1} - m\,(i-1)\; x^{m+k}\,]\, f$$

and, with the help of (2. 1), it yields

$$(2.\ 13)\quad \prod_{i=1}^{n}\,[x^m\, T_{a,k} + \frac{nh'}{h}\; x^{m+k-1} + pg'\, x^{m+k+1} - m\,(i-1)\; x^{m+k}\,]\, f$$

$$= h^{-n}\, x^{nm}\,\sum_{s=0}^{n}\binom{n}{s}\; G_{n-s}\,(a,\,k,\,p;\;g,\;\overline{h^{n-s}}^{\,n})\;T^s_{0;k}\;\;f.$$

## 3. Generating Functions

Starting from the definition (1. 8), using the properties

$$T^n_{a;k} = x^{nk}\,\prod_{j=0}^{n-1}(\delta + a + j\,k),$$

$$a^{\delta} f_1(x) f_2(x) = f_1(ax) f_2(ax);\ \delta \equiv x\frac{d}{dx}$$

and finally replacing $t$ by $t/k$, we get

$$(3.\ 1)\quad \sum_{n=0}^{\infty} G_n(a, k, p; g, h) \frac{t^n}{n!\, x^{nk} k^n} = (1-t)^{-a/k} h^n \{x(1-t)^{-1/k}\} \times$$

$$\times\ exp\,[\, p\,[g\{x(1-t)^{-1/k}\} - g(x)]].$$

If in (2. 2) we put

$$\triangle = h\, T_{a,k} + x^{k-1}(nh' + pg'h) \text{ and } f = 1$$

then

$$(3.\ 2)\quad \triangle^n (1) = G_n(a, k, p; g, h).$$

It is quite clear that

$$\triangle^{n+m} = \triangle^n \triangle^m,$$

$$\triangle^{n+m}(1) = \triangle^n \triangle^m (1)$$

so

$$(3.\ 3)\quad G_{n+m}(a, k, p; g, h) = \triangle^n G_m(a, k, p; g, h).$$

Again, putting $f = G_m(a, k, p; g, h)$ in (2. 2) and using (3. 3) and the above mentioned properties of $T_{a,k}$ and $\delta$, we get

$$(3.\ 4)\quad \sum_{n=0}^{\infty} G_{n+m}(a, k, p; g, h) \frac{t^n}{n!\, x^{kn}} = (1-tk)^{-a/k} h^n \{x(1-tk)^{-1/k}\} \times$$

$$\times\ exp\,[\, p[g\{x(1-tk)^{-1/k}\} - g(x)]\,]\, G_m(a, k, p; g, \{x(1-tk)^{-1/k}\},$$

$$h\{x(1-tk)^{-1/k}\})$$

which , on replacing $t$ by $t/k$ gives

$$(3.5)\quad \sum_{n=0}^{\infty} G_{n+m}(a, k, p; g, h) \frac{t^n}{n!\, x^{kn} k^n} = (1-t)^{-a/k} h^n\{x(1-t)^{-1/k}\} \times$$

$$\times\ exp\ [p[\ g\{x\,(1-t)^{-\frac{1}{k}}\} - g(x)]\ ]\ G_m\,(a, k.\ p;\ g, \{x(1-t)^{\ 1/k}\}.$$

$$h\,\{x\,(1-t)^{-1/k}\}\ ).$$

## 4. Some Other Relations

Consider

$$T^s_{a,k}\, G_n\,(a,\, k,\, p;\, g,\, h) = T^s_{a,k}\,\{e^{-pg}\, T^n_{a,k}\,(h^n\, e^{pg})\,\}$$

$$= \sum_{j=0}^{s} \binom{s}{j}\, T^{n+s-j}_{a,k}\,(h^n\, e^{pg})\quad T^j_{0,k}\,(e^{-pg}),$$

whence

$$(4.1)\quad T^s_{a,k}\, G_n\,(a,\, k,\, p;\, g,\, h) = \sum_{j=0}^{s} \binom{s}{j}\, G_{n+s-j}\,(a,\, k,\, p;\, g,\, \overline{h^{n+s-j}}^{\,n}\,)$$

$$\times\ G_j\,(0,\, k,\, -\, p;\, g,\, 1)$$

which, for $s=1$ gives

$$(4.2)\quad (\,T_{a,k} + x^{k+1}\, pg'\,)\ G_n\,(a,\, k,\, p;\, g,\, h) = G_{n+1}\,(a,\, k,\, p;\, g,\, \overline{h^{n+1}}^{\,n}\,).$$

Let

$$T_{a,k} + x^{k+1}\quad pg' = \theta.$$

Then

(4. 3) $\theta\, G_n (a, k, p; g, h) = G_{n+1} (a, k, p; g, h^{\frac{n}{n+1}} )$ .

An iteration of the above result yields

(4. 4) $\theta^r\, G_n (a, k, p; g, h) = G_{n+r} (a, k, p; g, h^{\frac{n}{n+r}} )$.

It is clear that

$$G_{n+r} (a, k, p; h^{\frac{n}{n+s}} ) = h^{-r}\, G_{n+r} (a, k, p; g - r/p \log h, h),$$

so

(4. 5) $\theta^r\, G_n (a, k, p; g, h) = h^{-r}\, G_{n+r} (a, k, p; g - r/p \log h, h)$;

and for $n=0$

(4. 6) $G_r (a, k, p; g - r/p \log h, h) = h^r\, G_r (a, k, p; g, h^{1/r} )$.

Using (4. 1) and (4. 5) we obtain

(4. 7) $T^s_{a;k}\, G_n (a, k, p; g, h ) = \sum_{j=0}^{s} \binom{s}{j} G_j (0, k, p; - g, 1)$

$\theta^{s-j}\, G_n (a, k, p; g, h)$.

Chandel and Bhargava [I] have used the relation

(4. 8) $\theta^n = \sum_{i=0}^{n} \binom{n}{i} G_{n-i} (a, k, p; g, 1) \ \ T^i_{0;k}$.

With the help of this result and using the elementary

formula

(4. 9) $e^{tT_{0;k}} \{ f(x) \} = f [x \{1 - tkx^k \}^{1/k} ]$,

we get

$$(4.10)\quad e^{t\theta}\; f(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} G_j\,(a, k\; p;\; g, 1)\; f\,[x\,\{1-tkx^k\}^{1/k}].$$

Now making an appeal to (1. 8) and using the property

$$e^{t\,T_{a,k}}\; e^{pg} = (1-tkx^k)^{-a/k}\quad e^{p\,[g\,\{x(1-tkx^k)^{1/k}\}]}$$

we get

$$(4.11)\quad e^{t\theta}\; f(x) = (\,1-tkx^k\,)^{-a/k}\; e^{p\,[g\,[x\,\{1-tkx^k\}^{1/k}]\,-\,g]} \times$$

$$\times f\,[x\,\{1-tkx^k\}^{-1/k}].$$

putting $f(x) = G_n\,(a, k, p;\, g, h)$ and using (4. 4) we further get

$$(4.12)\quad \sum_{j=0}^{\infty} \frac{t^j}{j!}\, G_{n+j}\,(a, k, p;\, g, h^{\frac{n}{n-j}})$$

$$= (1-tkx^k)^{-a/k}\; e^{p\,[g\,[x\,\{1-tkx^k\}^{1/k}]-g]}$$

$$\times\; G_n\,(a, k, p;\; g\,\{x\,(1-tkx^k)^{1/k},\; h\,\{x\,(1-tkx^k\,)^{1/k}\}\,),$$

while on putting $f(x)=1$ in (4. 10) we obtain

$$(4.13)\quad \sum_{j=0}^{\infty} \frac{t^j}{j!}\, G_j\,(a, k, p;\, g, 1) = (1-tkx^k)^{-a/k}$$

$$\times\; e^{p\,[g\,[x\,\{1-tkx^k\}^{1/k}]\,-\,g]}.$$

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