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# A STUDY OF SOME MULTIDIMENSIONAL INTEGRAL TRANSFORMS WITH APPLICATIONS

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The main content of this paper can be divided into three parts. In Section 2, we first establish some theorems involving multidimensional integral transforms. These theorems are sufficiently general in nature and are capable of giving rise to a number of further new theorems involving single, double and multidimensional integral transforms. Five recent theorems ( [1, Theorems I, II, and II (a) ], [3. Theorem 2] and [4, Theorem 3 ] ) which are of interest in themselves follow as special cases of these theorems. Next in Section 2, we evaluate a new multiple integral involving a multivariable $H$-function by applying one of our theorems.

In Section 3, we obtain multidimensional inverse Laplace transform of a function involving the product of a multivariable $H$-function and several Whittaker functions.

In Section 4, we make use of our results of Sections 2 and 3 for establishing a new and interesting theorem which interconnects the images of related functions in multidimensional Varma and Laplace transforms.

## 1. INTRODUCTION AND DEFINITIONS

Recently, Srivastava and Panda introduced and studied the multivariable $H$-function ( [ 9 ] and [ 10 ] ) and two classes of multidimensional integral transforms involving this function as kernel in a series of papers [8, Parts I and II]. Since then a number of research papers by different authors have appeared in the literature on these topics., We study here certain new aspects concerning them, which give rise to various general and useful results.

The following definitions and results will be used in the sequel :

(a) (i) **The multidimensional $H$-function Transform**

$$(1.1)\ H\{f(x_1,\ldots,x_r);\ p_1,\ldots,p_r\}$$

$$=\int_0^\infty\cdots\int_0^\infty H^{0,\,0:\,(m',\,n');\ldots;\,(m^{(r)},\,n^{(r)})}_{p,\,q\,;\,[p',\,q'];\ldots;\,[p^{(r)},\,q^{(r)}]}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}];\,[(c'):C'];\ldots;[(c^{(r)}):C^{(r)}];\\ [(b):B',\ldots,B^{(r)}]:[(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\ p_1x_1,\ldots,p_rx_r\right)$$

$$f(x_1,\ldots,x_r)\,dx_1\ldots dx_r,$$

which is a special case of one of the multidimensional integral transforms defined by Srivastava and Panda [8, Part I, p. 121, eq. (1. 15)].

The kernel of the above transform is a special case of the multivariable $H$-function defined by Srivastava and Panda [9, p. 271, eq (4. 1)]. For explanations of the various notations and conditions of existence of the multivariable $H$-function. we refer to another paper of Srivastava and Panda [10, p. 130, equations (1. 3) to (1. 10)]. Also, we shall assume that conditions correspondiog appropriately to the conditions referred to above are always satisfied by all the multivariable $H$-functions occurring in this paper.

(ii) **The multidimensional Varma transform**

$$(1.2)\quad V\{f(x_1,\ldots,x_r)\ k_1,\ldots,k_r,\ m_1,\ldots,m_r\ ;\ p_1\ldots,p_r\}$$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r) \prod_{i=1}^{r} [(p_i x_i)^{m_i - \frac{1}{2}} W_{k_i, m_i}$$

$$(p_i x_i) e^{-\frac{1}{2} p_i x_i} ] \, dx_1 \ldots dx_r$$

(iii) **The multidimensional Laplace transform**

(1. 3) $L\{f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r) \, e^{-(p_1 x_1 + \cdots + p_r x_r)} \, dx_1 \ldots dx_r$$

In each of the definitions (1. 1), (1. 2) and (1. 3), it is assumed that the multiple integrals involved converge absolutely.

It may be noted that multidimensional integral transforms defined by (1. 2) and (1. 3) are special cases of the multidimensional $H$-function transform difined by (1. 1). For various other special cases of the transform defined by (1. 1) we refer to the basic paper of Srivastava and Panda [8].

(b) **The generalized Parseval-Goldstein formula**

(1. 4) $$\int_0^\infty \cdots \int_0^\infty f_1(x_1, \ldots, x_r) \, \phi_2(x_1, \ldots, x_r) \, dx_1 \ldots dx_r$$

$$= \int_0^\infty \cdots \int_0^\infty f_2(x_1, \ldots, x_r) \, \phi_1(x_1, \ldots, x_r) \, dx_1 \ldots dx_r$$

where

(1 5) $\phi_1(p_1, \ldots, p_r) = T\{f_1(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

and

(1. 6) $\phi_2(p_1, \ldots, p_r) = T\{f_2(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

and the various integrals involved in (1. 4), (1. 5) and (1. 6) are absolutely convergnt.

## 2. Main Results and Applications

**Theorem 1.** *If*

(2. 1) $h_1(p_1,\dots,p_r) = T_1\{g(x_1,\dots,x_r)\, f(x_1,\dots,x_r)\,;\, p_1,\dots,p_r\}$

$$= \int_0^\infty \dots \int_0^\infty k_1(p_1 x_1,\dots,p_r x_r)\, g(x_1,\dots,x_r)\, f(x_1,\dots,x_r)\, dx_1 \dots dx_r$$

*and*

(2. 2) $h_2(p_1,\dots,p_r) = T_2\{f(x_1^{-\sigma_1},\dots,x_r^{-\sigma_r})\,;\, p_1,\dots,p_r\}$

$$= \int_0^\infty \dots \int_0^\infty k_2(p_1 x_1,\dots,p_r x_r)\, f(x_1^{-\sigma_1},\dots,x_r^{-\sigma_r})\, dx_1,\dots,dx_r$$

*then*

(2. 3) $h_1(p_1,\dots,p_r) = \sigma_1 \dots \sigma_r \int_0^\infty \dots \int_0^\infty h_2(p_1,\dots,p_r)\, \phi(x_1,\dots,x_r,$

$p_1,\dots,p_r)\, dx_1 \dots dx_r$

*where*

(2. 4) $p_1^{-\sigma_1-1} \dots p_r^{-\sigma_r-1}\, g(p_1^{-\sigma_1},\dots,p_r^{-\sigma_r})\, k_1(a_1 p_1^{-\sigma_1},\dots,a_1\ p_r^{-\sigma_r})$

$$= T_2\{\phi(x_1,\dots,x_r,\alpha_1,\dots,a_r)\,;\, p_1,\dots,p_r\}$$

$\sigma_1,\dots,\sigma_r$ *are non-zero real numbers of the same sign of* $\alpha_1,\dots,a_r$ *is independent* of $p_1,\dots p_r$ *and all the multiple integrals involved in* (2.1) *to* (2.4) *are assumed to be absolutely convergent.*

**Proof**. Applying the formula given by (1. 4) to the pairs (2. 2) and (2 4), we get

(2. 5) $\int_0^\infty \dots \int_0^\infty x_1^{-\sigma_1-1},\dots,x_r^{-\sigma_r-1}\, g(x_1^{-\sigma_1},\dots,x_r^{-\sigma_r})$

$$. k_1(a_1 x_1^{-\sigma_1}, \ldots, a_r x_r^{-\sigma_r}). f(x_1^{-\sigma_1}, \ldots, x_r^{-\sigma_r})\, dx_1 \ldots dx_r$$

$$= \int_0^\infty \cdots \int_0^\infty h_2 (x_1, \ldots, x_r)\, \phi(x_1, \ldots, x_r, p_1, \ldots, p_r)\, dx_1 \ldots dx_r.$$

Now changing the variables slightly on the right-hand side of the above equation and interpreting the result thus obtained in terms of (2. 1) we easily arrive at the desired result (2. 3) after replacing $a_1, \ldots, a_r$ by $p_1, \ldots, p_r$, respectively.

If, in the above theorem, we take $T_2$ as the multidimensional Laplace transform defined by (1. 3) we obtain following theorem after a little simplification.

**Theorem I (a).** *If*

(2. 6) $h_1 (p_1, \ldots, p_r) = T_1 \{ f(x_1, \ldots, x_r)\, g(x_1, \ldots, x_r) ; p_1, \ldots, p_r \}$

*and*

(2. 7) $h_2 (p_1, \ldots, p_r) = L \{ f(x_1^{-\sigma_1}, \ldots, x_r^{-\sigma_r}) ; p_1, \ldots, p_r \}$

*then*

(2. 8) $h_1 (p_1, \ldots, p_r) = \sigma_1 \ldots \sigma_r \int_0^\infty \cdots \int_0^\infty h_2 (x_1 + p_1, \ldots, x_r + p_r)$

$$\phi (x_1, \ldots, x_r, p_1, \ldots, p_r)\, dx_1 \ldots dx_r$$

*where*

(2. 9) $e^{a_1 p_1 + \ldots + a_r p_r}\, p_1^{-\sigma_1 - 1}\, g(p_1^{-\sigma_1}, \ldots, p_r^{-\sigma_r})$

$$k_1 (a_1 p_1^{-\sigma_1}, \ldots, a_r p_r^{-\sigma_r})$$

$$= L\{ \phi (x_1, \ldots, x_r, a_1, \ldots, a_r); p_1, \ldots, p_r \}$$

Re $(p_i) > 0$, $(i = 1, \ldots, r)$, $\sigma_1, \ldots, \sigma_r$ *are non-zero real numbers of the same sign, each of* $a_1, \ldots, a_r$ *is independent of* $p_1, \ldots, p_r$ *and all the* *integrals involved in (2.6) to (2.9) are assumed to be absolutely convergent.*

If, in the above theorem, we take $T_1$ to be multidimensional Laplace transform defined by (1.3), $\sigma_1 = \ldots = \sigma_r = 1$ and

$$(2.10)\quad g(x_1, \ldots, x_r) = x_1^{-\rho_1} \ldots x_r^{-\rho_r}\, H^{0,0:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p,q:[p',q'];\ldots;[p^{(r)},q^{(r)}]} \begin{pmatrix} a_1 x_1^{\lambda_1} \\ \vdots \\ a_r x_r^{\lambda_r} \end{pmatrix}$$

we get, by an appeal to the known result given by Srivastava and Panda [8, Part I, p. 126, eq. (3.8)] and after little simplification, the following interesting theorem :

**Theorem I (b)** *If*

$$(2.11)\quad h_1(p_1, \ldots, p_r) = L \left\{ x_1^{-\rho_1} \ldots x_r^{-\rho_r}\, H^{0,0:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p,q:[p',q'];\ldots;[p^{(r)},q^{(r)}]} \begin{pmatrix} a_1 x_1^{\lambda_1} \\ \vdots \\ a_r x_r^{\lambda_r} \end{pmatrix} f(x_1, \ldots, x_r) ; p_1, \ldots, p_r \right\}$$

*and*

$$(2.12)\quad h_2(p_1, \ldots, p_r) = L\{ f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$$

*then*

$$(2.13)\quad h_1(p_1, \ldots, p_r) = \int_0^\infty \ldots \int_0^\infty H^{0,0:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p,q:[p'+1,q'];\ldots;[p^{(r)}+1,q^{(r)}]} \begin{pmatrix} \ldots : \ldots, (\rho_1, \lambda_1);\ldots;(\rho_r, \lambda_r); & a_1 x_1^{-\lambda_1}, \ldots, a_r x_r^{-\lambda_r} \\ \ldots : \ldots; \qquad \ldots; & \end{pmatrix}$$

$$h_2(x_1 + p_1, \ldots, x_r + p_r)\, x_1^{\rho_1 - 1} \ldots x_r^{\rho_r - 1}\, dx_1 \ldots dx_r$$

*Here the remaining parameters of the multivariable H-function occurring in (2. 11) are the same as those of multivariable H-function occurring in (1. 1). Again three dots '...' appearing at various places on the right–hand side of (2. 13) indicate that the parameters in these places are exactly those of the multivariable H- function occurring in (1. 1). Also* $\lambda_i > 0$, $\text{Re}\,(p_i) > 0$,

$$\text{Re}\ \left\{p_i - \lambda_i\left(\frac{C_j^{(r)} - 1}{C_j^{(i)}}\right)\right\} > 0,\ j = 1, \ldots, m^{(i)},\ i = 1, \ldots, r \text{ and the}$$

*various integrals involved in (2. 11) to (2. 13) are assumed to be absolutely convergent.*

**Special Cases**

(i) On taking $g(x_1, \ldots, x_r) = x_1^{(\rho_1/c_1)-1} \ldots x_r^{(\rho_r/c_r)-1}$ in Theorem I, we easily arrive at a theorem obtained recently by Gupta [ 3, Theorem 2].

(ii) Theorem I and I (b) on proper specializations yield analogous theorem for two-dimensional transforms given recently by Garg [I, p. 442 Theorem I] and Gupta [4, p. 194, Theorem 3].

(iii) Again, Theorem I and I (a) on proper specializations give rise to other interesting theorems for one–dimensional transforms given recently by Grrg [1, pp. 442—443, Theorems II and II(a)]

**An application of Theorem I**(b) Now we shall obtain a very general and interesting integral with the application of Theorem I (b).

Thus, if in Theorem I (b), we take

$$(2.\ 14)\ f(x_1, \ldots, x_r) = \prod_{i=1}^{r}\left\{x_i^{c_i - 1}\ H_{P^{(i)},\ Q^{(i)}}^{M^{(i)},\ 0}\left[b_i\, x_i^{\sigma_i}\ \middle|\ \begin{matrix}(g_j^{(i)}, G_j^{(i)})_{1, P^{(i)}}\\(k_j^{(i)}, H_j^{(i)})_{1, Q^{(i)}}\end{matrix}\right]\right\}$$

where the $H$-functions occurring on the right-hand side are the well known (Fox's) $H$-functions.

Then, on applying a result obtained by Srivasatva and panda [8, p. 126, eq (3. 8)], we get

$$(2.15)\quad h_2(p_1,\dots,p_r)=\prod_{i=1}^{r}\left\{p_i^{-c_i}\, H_{P^{(i)}+1,\,Q^{(i)}}^{M^{(i)},\,1}\left[b_i\, p_i^{-\sigma_i}\,\middle|\,\begin{matrix}(1-c_i,\sigma_i),\,(g_j^{(i)},G_j^{(i)})_{1,\,P^{(i)}}\\(h_j^{(i)},H_j^{(i)})_{1,\,Q^{(i)}}\end{matrix}\right]\right\}$$

provided that $\mathrm{Re}\,(p_i)>0,\ \sigma_i>\ ,\ \mathrm{Re}\,[c_i+\sigma_i\,\frac{h_j}{H_j^{(i)}}]>0$

$j=1,\dots,M^{(i)},\ (i=1,\dots,r),$

$$|\arg b_i|<\tfrac{1}{2}U_i\pi,\ U_i=\sum_{j=1}^{M^{(i)}}H_j^{(i)}-\sum_{j=M^{(i)}+1}^{Q^{(i)}}H_j^{(i)}$$

$$-\sum_{j=1}^{P^{(i)}}G_j^{(i)}>0,\ i=1,\dots,r$$

Further from (2. 11) we get

$$(2.16)\ h_1(p_1,\dots,p_r)=$$

$$\int_0^\infty\cdots\int_0^\infty e^{-(p_1x_1+\dots+p_rx_r)}\,x_1^{c_1-\rho_1-1}\dots x_r^{c_r-\rho_r-1}$$

$$H_{p,\,q\,:\,[p',\,q'];\dots;[p^{(r)},\,q^{(r)}]}^{0,\,0\,:\,(m',\,n');\dots;(m^{(r)},\,n^{(r)})}\begin{pmatrix}a_1x_1^{\lambda_1}\\ \vdots\\ a_rx_r^{\lambda_r}\end{pmatrix}$$

$$.\prod_{i=1}^{r}\left\{H_{P^{(i)},\,Q^{(i)}}^{M^{(i)},\,0}\left[b_i\,x^{\sigma_i}\,\middle|\,\begin{matrix}(g_j^{(i)},G_j^{(i)})_{1,\,P^{(i)}}\\(h_j^{(i)},H_j^{(i)})_{1,\,Q^{(i)}}\end{matrix}\right]\right\}dx_1\dots dx_r$$

To evaluate the above integral, we expand $e^{-(p_1 x_1 + \dots + p_r x_r)}$ occurring on the right–hand side of (2. 16) as a multiple series, interchange the orders of integrations and summations, and evaluate the multiple integral thus obtained with the help of a recent result given by Gupta and Bhatt [5, eq. (2. 1) ], to get

$$(2.\ 17)\quad h_1(p_1, \dots, p_r) = \frac{(b_1)^{(c_1-\rho_1)/\sigma_1} \dots (b_r)^{(c_r-\rho_r)/\sigma_r}}{\sigma_1 \dots \sigma_r}$$

$$\sum_{N_1, \dots, N_r=0}^{\infty} (-1)^{N_1 + \dots + N_r} \frac{(p_1 b_1^{(1/\sigma_1)})^{N_1} \dots (p_r b_r^{(1/\sigma_r)})^{N_r}}{(N_1)! \dots (N_r)!}$$

$$H^{0,\,0:\,(m',\, n'+M');\dots;(m^{(r)},\, n^{(r)}+M^{(r)})}_{p,\,q:\,[p'+Q',\, q+P'];\dots;[p^{(r)}+Q^{(r)},\, q^{(r)}+P^{r}]}$$

$$\left(\begin{matrix} a_1\, b_1^{(-\lambda_1/\sigma_1)} \\ \vdots \\ a_r\, b_r^{(-\lambda_r/\sigma_r)} \end{matrix} \left| \begin{matrix} K : I_1 ; \dots ; I_r \\ \\ L : J_1 ; \quad ; J_r \end{matrix} \right.\right)$$

where

$$K = [\ (a) : A', \dots, A^{(r)}\ ]$$

$$L = [\ (b) ; B', \dots, B^{(r)}\ ]$$

$$I_i = (c_j^{(i)}, C_j^{(i)})\ j=1, \dots, n^{(i)}, (1-h_j^{(i)} - (\frac{c_1-\rho_i+N_i}{\sigma_i})H_j^{(i)},$$

$$\frac{\lambda_i}{\sigma_i} H_j^{(i)}),\ j=1, \dots, Q^{(i)}$$

$$(c_j^{(i)}, C_j^{(i)})\ j=n^{(i)}+1, \dots, g^{(i)}.$$

$$J_i = (d_j^{(i)}, D_j^{(i)})\ j=1, \dots, q^{(i)}, (1-g_j^{(i)} - (\frac{c_i-\rho_i+N_i}{\sigma_i})\ G_j^{(i)},$$

$$\frac{\lambda_i}{\rho_i} G_j^{(i)})\quad j=1, \dots, P^{(i)}\ (i=1, \dots, r)$$

provided that

$\text{Re}\,(p_i) > 0,\ \lambda_i > 0,\ \sigma_i > 0,$

$$\text{Re}\left\{c_i - \rho_i + \lambda_i \frac{d_j^{(i)}}{D_j^{(i)}} + \sigma_i \frac{h_k^{(i)}}{H_k^{(i)}}\right\} > 0,\ j=1,\ldots,m^{(i)},$$

$k=1,\ldots,M^{(i)}$

$$|\arg b_i| < \tfrac{1}{2} U_i \pi,\ U_i = \sum_{j=1}^{M^{(i)}} H_j^{(i)} - \sum_{j=M^{(i)}+1}^{Q^{(i)}} H_j^{(i)}$$

$$-\sum_{j=1}^{P^{(i)}} G_j^{(i)} > 0,\ i=1,\ldots,r.$$

Now, substituting the values of $h_2(p_1,\ldots,p_r)$ and $h_1(p_1,\ldots,p_r)$ from (2. 15) and (2. 17) in (2. 13) we arrive at the following multiple integral.

$$(2.\,18)\quad \int_0^\infty \cdots \int_0^\infty x_1^{\rho_1-1} \ldots x_r^{\rho_r-1} (x_1+p_1)^{-c_1} \ldots (x_r+p_r)^{-c_r}$$

$$\prod_{i=1}^{r} \left\{ H^{M^{(i)},\,1}_{P^{(i)}+1,\,Q^{(i)}} \left[ b_i(p_i+x_i)^{-\sigma_i} \left| \begin{matrix} (1-c_i,\sigma_i),\ (g_j^{(i)}, G_j^{(i)})_{1,P^{(i)}} \\ (h_j^{(i)}, H_j^{(i)})_{1,Q^{(i)}} \end{matrix} \right. \right] \right\}$$

$$\cdot\, H^{0,\,0:\,(m',\,n');\ldots;(m^{(r)},\,n^{(r)})}_{p,\,q:\,[p'+1,\,q'];\ldots;[p^{(r)}+1,\,q^{(r)}]}$$

$$\left( \begin{matrix} \ldots : \ldots, (\rho_1, \lambda_1);\ldots;(\rho_r, \lambda_r); \\ \ldots : \quad \ldots; \quad \ldots; \end{matrix} \quad a_1 x_1^{-\lambda_1},\ldots, a_r x_r^{-\lambda_r} \right) dx_1 \ldots dx_r$$

$$= \frac{b_1^{(c_1-\rho_1)/\sigma_1} \ldots b_r^{(c_r-\rho_r)/\sigma_r}}{\sigma_1 \ldots \sigma_r} \sum_{N_1,\ldots,N_r=0}^{\infty} (-1)^{N_1+\ldots+N_r}$$

$$\frac{(p_1 b_1^{1/\sigma_1})^{N_1} \ldots (p_r b_r^{1/\sigma_r})^{N_r}}{(N_1)! \ldots (N_r)!}$$

$$\cdot\, H^{0,0\,(m', n'+M');\ldots;(m^{(r)}, n^{(r)}+M^{(r)})}_{p,q:[p'+Q', q'+P'];\ldots;[p^{(r)}+Q^{(r)}, q^{(r)}+P^{(r)}]}$$

$$\left(\begin{matrix} a_1\, b_1^{(-\lambda_1/\sigma_1)} \\ \ldots \\ a_r\, b_r^{(-\lambda_r/\sigma_r)} \end{matrix}\left|\begin{matrix} K : I_1 ;\ldots; I_r \\ \\ L : J_1 ;\ldots; J_r \end{matrix}\right.\right)$$

where the symbols K, L, $I_i$ and $J_i$ stand for the quantities given with eqation (2. 17),

$$\text{Re}\,(p_i) > 0,\ \lambda_i > 0,$$

$$\text{Re}\left\{ c_i - \rho_i + \lambda_i \frac{d_j^{(i)}}{C_j^{(i)}} + \sigma_i \frac{h_k^{(i)}}{H_k^{(i)}} \right\} > 0,\ j=1,\ldots,m^{(i)},\ k=1,\ldots,M^{(i)}$$

$$\text{Re}\left\{ \rho_i - \lambda_i \left( \frac{c_j^{(i)}-1}{D_j^{(i)}} \right) \right\} > 0,\ j=1,\ldots,n^{(i)}$$

$$|\arg b_i| < \tfrac{1}{2} U_i \pi,\ U_i = \sum_{j=1}^{M^{(i)}} H_j^{(i)} - \sum_{j=M^{(i)}+1}^{Q^{(i)}} H_j^{(i)}$$

$$- \sum_{j=1}^{P^{(i)}} G_j^{(i)} > 0,\ i=1,\ldots,r.$$

The multivariable $H$-function occurring on the right-hand side of (2. 18) satisfies conditions corresponding appropriately to those referred to in the paper of Srivastava and Panda [10, p. 130, Equations (1. 6) (1. 7) ] and the multiple series is assumed to be absolutely convergent.

### 3. Multidimensional Inverse Laplace Transform of

$$H\left[ a_1\, p_1^{-\lambda_1},\ldots, a_r\, p_r^{-\lambda_r} \right] \prod_{i=1}^{r} \left\{ p_i^{-\rho_i-1}\ exp\,(z_i\, p_i) \right.$$

$$\left. W_{k_i,\, m_i}\,(2z_i\, p_i) \right\}$$

In this section, we give multidimensional inverse Laplace transform of a function involving product of a multivariable $H$-function and $r$ Whittaker functions. This easily follows from multidimensional analogue of a theorem given by Gupta and Handa [6, p. 148, Theorem] on using two known results ( [17, p. 368, eq. (3. 2) ] and [5, eq. (2. 1) ] ) and on proceeding along the lines of the paper of Gupta and Handa referred above after a little simplification.

$$(3.\ 1)\quad L^{-1}\left[\prod_{i=1}^{r}\left\{p_i^{-\rho_i-1}\ exp\,(z_i\,p_i)\,W_{k_i,\,m_i}\,(2z_i\,p_i)\right\}\right.$$

$$H^{0,\,0:\,(m',\,n')\,;\ldots;\,(m^{(r)},\,n^{(r)})}_{p,\,q:\,[p',\,q']\,;\ldots;\,[p^{(r)},\,q^{(r)}]}$$

$$\left.\left(\begin{matrix}[\,(a):\,A',\ldots,A^{(r)}]:\,[\,(c'):\,C'\,]\,;\ldots;\,[c^{(r)}):\,C^{(r)}]:\\ [\,(b):\,B',\ldots,B^{(r)}]:\,[(d'):\,D'\,]\,;\ldots;\,[\,d^{(r)}]:\,D^{(r)}];\end{matrix}\ a_1p_1^{-\lambda_1},\ldots,a_rp_r^{-\lambda_r}\right)\right]$$

$$=\prod_{i=1}^{r}\left\{(2z_i)^{\rho_i}\ x_i^{\rho_i-k_i}\ (1+x_i)^{\rho_i+k_i}\right\}\sum_{N_1,\ldots,N_r=0}^{\infty}(-1)^{N_1+\cdots+N_r}$$

$$\frac{(z_1)^{N_1}\ldots(z_r)^{N_r}}{(N_1)!\ldots(N_r)!}\ H^{0,\,0:\,(m',\,n'+2)\,;\ldots;\,(m^{(r)},\,n^{(r)}+2)}_{p,\,q:\,[p'+2,q'+3]\,;\ldots;\,p^{(r)}+2,q^{(r)}+3]}$$

$$\left(\begin{matrix}M:\,P_1\,;\ldots;\,P_r\,;\\ N:\,Q_1\,;\ldots;\,Q_r\,;\end{matrix}\ a_1\,\{2x_1\,z_1\,(1+x_1)\,\}^{\lambda_r},\ldots,a_r\,\{2x_rz_r\,(1+x_r)\}^{\lambda_r}\right)$$

where

$$M=[\,(a):\,A',\ldots,A^{(r)}\,]$$

$$N=[\,(b):\,B',\ldots,B^{(r)}\,]$$

$$P_i=(\tfrac{1}{2}\pm m_i-\sigma_i-N_i,\,\lambda_i),\,[\,(c')::\,C'\,]$$

$$Q_i = [\,(d'):D'\,],\ (\tfrac{1}{2} \pm m_i - \sigma_i,\ \lambda_i),\ (k_i - \sigma_i - N_i,\ \lambda_i)$$

$$\text{Re}\,(p_i) > 0,\ \text{Re}(z_i) > 0,\ \text{Re}[\rho_i - k_i + \lambda_i\ \frac{d_j^{(i)}}{D_j^{(i)}} + 1\,] > 0\ j=1,\ldots,m^{(i)}$$

$i=1,\ldots,r,$

the series involved in (3.1) is absolutely convergent and the multivariable $H$-functions occurring in (3.1) satisfy the conditions corresponding appropriately to the conditions mentioned by Srivastava and Panda [10, p. 130, eq. (1.6) and (1.7)].

A two-dimensional analogue of the above result was obtained earlier by Gupta and Handa [6, p. 156, eq. (4.6)].

**An Important Special Case of (3.1)**

If we put $p=q=0$ in (3.1), each multivariable $H$-function occurring therein reduces to the product of $r$ Fox's $H$-functions. Again, since each of the resulting series of Fox's $H$-function on the right hand side of (3.1) can be further represented in terms of the $H$-function of two variables by means of a known formula given by Goyal [2, p. 119, eq. (1.2)], we arrive at the following inverse Laplace transform of product of several Wittaker and Fox's $H$-functions in a neat and compact form:

$$(3.2)\quad L^{-1}\left[\prod_{i=1}^{r}\left\{ p_i^{-\rho_i-1}\ \exp(z_i p_i)\ W_{k_i,\,m_i}(2z_i p_i)\ H_{p^{(i)},\,q^{(i)}}^{m^{(i)},\,n^{(i)}}\left( a_i p_i^{-\lambda_i}\ \middle|\ \begin{matrix}(c_j^{(i)},\,C_j^{(i)})_{1,\,p^{(i)}}\\ (d_j^{(i)},\,D_j^{(i)})_{1,\,q^{(i)}}\end{matrix}\right)\right\}\right]$$

$$=\prod_{i=1}^{r}\left\{ (2z_i)^{\rho_i}\ x_i^{\rho_i-k_i}\ (1+x_i)^{\rho_i+k_i}\ H_{2,\,1:\,0,\,1;\,p^{(i)},\,q^{(i)}+2}^{0,\,2:\,1,\,0:\,m^{(i)},\,n^{(i)}}\right.$$

$$\left[\begin{array}{c} x_i \\ a_i \{2x_i z_i \ (1+x_i)\}^{\lambda_i} \end{array}\middle| \begin{array}{l} (\tfrac{1}{2} \pm m_i - \rho_i;\ 1, \lambda_i):\ -; \\ (k_i - \rho_i;\ 1, \lambda_i):\ (0, 1); \end{array}\right.$$

$$\left.\begin{array}{l} (c_j^{(i)}, C_j^{(i)})_{1,\ p^{(i)}} \\ (d_j^{(i)} D_j^{(i)})_{1,\ q^{(i)}},\ (\tfrac{1}{2} \pm m_i - \rho_i,\ \lambda_i) \end{array}\right]\Bigg\}$$

under the conditions easily obtainable from the conditions stated with result (3. 1).

The one-dimensional analogue of the above result is also of interest in itself and may find applications.

## 4 APPLICATIONS

If, in Theorem I($a$), we take $T_1$ to be the multidimensional Varma transform difined by (1. 2), $\rho_1 = \ldots = \rho_r = -1$, and

$$(4.\ 1)\quad g(x_1, \ldots, x_r) = \prod_{i=1}^{r} \left\{ x_i^{-\rho_i - m_i - \frac{1}{2}}\ H_{p^{(i)},\ q^{(i)}}^{m^{(i)},\ n^{(i)}} \left[ a_i\ x_i^{\lambda_i} \middle| \begin{array}{l} (c_j^{(i)},\ C_j^{(i)})_{1,\ p^{(i)}} \\ (d_j^{(i)},\ D_j^{(i)})_{1,\ q^{(i)}} \end{array} \right] \right\}$$

then on using the result (3. 2) and after a little simplification, we arrive at

**Theorem 3**. *If*

$$(4.\ 2)\quad H_1(p_1, \ldots, p_r) = V \left\{ \prod_{i=1}^{r} \left[ x_i^{-\rho_i - m_i - \frac{1}{2}}\ H_{p^{(i)},\ q^{(i)}}^{m^{(i)},\ n^{(i)}} (a_i x_i^{-\lambda_i}) \right] \right.$$

$$\left. f(x_1, \ldots, x_r);\ k_1, \ldots, k_r,\ m_1, \ldots, m_r;\ p_1, \ldots, p_r \right\}$$

*and*

(4. 3) $h_2(p_1, \ldots, p_r) = L\{f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

*then*

$$(4.4)\quad h_1(p_1, \ldots, p_r) = p_1^{\rho_1+m_1-\frac{1}{2}} \ldots p_r^{\rho_r+m_r-\frac{1}{2}} \int_0^\infty \ldots \int_0^\infty h_2(x_1+p_1, \ldots, x_r+p_r) \prod_{i=1}^{r} \Big\{ x_i^{\rho_i-k_i} (1+x_i)^{\rho_i+k_i} H^{0,2:1,0;m^{(i)},n^{(i)}}_{2,1:0,1;p^{(i)},q^{(i)}+2} \left( \begin{matrix} x_i \\ a_i\{p_i x_i (1+x_i)\}^{\lambda_i} \end{matrix} \middle| \begin{matrix} (\frac{1}{2}\pm m_i-\rho_i; 1, \lambda_i) : - ; (c_j^{(i)}, C_j^{(i)})_{1,p^{(i)}} \\ (k_i-\rho_i; 1, \lambda_i) : (0, 1), (d_j^{(i)}, D_j^{(i)})_{1,q^{(i)}}, (\frac{1}{2}\pm m_i-\rho_i, \lambda_i) \end{matrix} \right) \Big\} dx_1 \ldots dx_r$$

*where*

$$\text{Re} > 0, \text{Re}\left( \rho_i - k_i + \lambda_i \frac{d_j^{(i)}}{D_j^{(i)}} + 1 \right) > 0, j=1,\ldots m^{(i)}, i=1,\ldots r,$$

$$|\arg a_i| < \tfrac{1}{2} A_i \pi, A_i = \sum_{j=1}^{m^{(i)}} D_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} D_j^{(i)}$$

$$+ \sum_{j=1}^{n^{(i)}} C_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} C_j^{(i)} > 0 \; i=1, \ldots, r,$$

*the* H– *function of two variables occurring on the right–hand side of* (4. 4) *satisfies the conditions corresponding appropriately to those given by Goyal* [2, *p. 119*, (*i*) to (*vi*)]. *Also the integrals in* (4. 2) *to* (4. 4) *are assumed to be absolutely convergent.*

**Special Cases** Theorem 3 is also sufficiently general in nature and is capable of yielding a large number of simpler results involing one–and multidimensional integral transforms. Thus on reducing Fox's H—functions involved in (4.2) to well-known Meijer's *G*-functions, the one dimensional analogue of the above theorem yields a recent result given by Garg [**1**, p. 443, Theorem II (b)].

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