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# AN INTEGRAL INVOLVING A GENERAL POLYNOMIAL SET

*By*


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## ABSTRACT

In the present paper we evaluate an integral of the product of two $H$–functions and the general polynomial set $\{\bar{R}_n(x, y)\}$, defined by (1) below. Some particular cases of this integral are also indicated.


## 1. INTRODUCTION

Recently, Singh and Pandey [8] studied a general polynomial set $\{\bar{R}_n(x, y)\}$ defined by means of the generating function

$$(1) \qquad \sum_{n=0}^{\infty} \bar{R}_n(x,y) t^n = Q\, J_{\gamma_1}\left( \frac{a_1 x^{m_1} y^{m_2} t^{m_3}}{(1 - v x^{-m} t^{m_4})^{\alpha}} \right)$$

$$\times H^{l_1, l_2}_{p, q+1}\left[ \frac{-\mu y^{r_1} t}{(1 - v x^{-m} t^{m_4})^{\beta}} \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, 1)\,(b_2, B_2), \ldots, (b_{q+1}, B_{q+1}) \end{matrix} \right]$$

where $\mu \neq 0, v \neq 0, \alpha \geqslant 0, m$ and $m_i$ ( $i = 1, 2, 3, 4$ ) are positive integers,

$$\sum_{1}^{p} A_j - \sum_{2}^{q+1} B_j \leqslant 1,$$

$$\sum_{1}^{l_2} A_j - \sum_{j=l_2+1}^{p} A_j + \sum_{2}^{l_1} B_j - \sum_{j=l_1+1}^{q+1} B_j \equiv \lambda' > 0,$$

$$\left| \arg \frac{-\mu y^{r_1} t}{(1-vx^{-m} t^{m_4})^{\beta}} \right| < \tfrac{1}{2}\pi\lambda',$$

and

$$(2)\ Q = \frac{\prod_{j=l_2+1}^{p} \Gamma(a_j - A_j b_1) \prod_{j=l_1+1}^{q+1} \Gamma(1-b_j+B_j b_1)\, 2^{\nu_1} \nu_1 !}{\prod_{1}^{l_2} \Gamma(1-a_j+A_j b_1) \prod_{2}^{l_1} \Gamma(b_j-B_j b_1) (-\mu y^{r_1} t)^{b_1} (\alpha_1 x^{m_1} y^{m_2} t^{m_3})^{\nu_1}}$$

The right–hand side of (1) contains the $H$–function of Fox [2], and the Bessel function of order $\nu_1$ and of the first kind. {See also [3], [4] and [11]. }

It is evident from the nature of the parameters that almost all orthogonal and other polynomials can be derived from (1), such as Hermite, Laguerre, Jacobi, Gegenbauer, Sister Celine, Bateman's, etc., and also some discrete polynomials [1. p. 225 (9), 226 (5) ]. For detailed relationships see [7] to [9]. {See also [6], [10] and [11].}

For convenience, the following notations have been used:

$(m) = 1, 2, 3, \ldots, m.$

$(a_p) = 1, 2, ., a_p.$

$$[\,(M_1(i,j))\,] = \prod_{j=1}^{l_2} \prod_{i=1}^{A_j} \left( \frac{i-a_j + A_j b_1}{A_j} \right).$$

$$[\,1-(M_2(i,j))\,] = \prod_{j=l_2+1}^{p} \prod_{i=1}^{A_j} \left(1 - \frac{i-1+a_j-A_j b_1}{A_j}\right).$$

$$[\,(\mathcal{N}_1(i,j))\,] = \prod_{j=l_1+1}^{q+1} \prod_{i=1}^{B_j} \left( \frac{i-b_1+B_j\, b_1}{B_j} \right)$$

$$[\,1-(\mathcal{N}_2(i,j))\,] = \prod_{j=2}^{l_1} \prod_{i=1}^{B_j} \left(1- \frac{i-1+b_j-B_j\, b_1}{B_j} \right)$$

$$E = U/V, \text{ where } U = (-1)^{\sum_{j=1}^{l_1} B_i} \prod_{j=1}^{p} A_j^{A_j}$$

$$V = (-1)^{\sum_{j=l_2+1}^{p} A_j} \prod_{j=2}^{q+1} B_j^{B_j}$$

$$\triangle^1{}_k\,[m_4;\,1-(M_1(i,j))-n]$$

$$= \prod_{j=1}^{l_2} \prod_{i=1}^{A_j} \prod_{l=1}^{m_4} \left( \frac{-(M_1(i,j))-n+l}{m_4} \right)_k.$$

$$\triangle^2{}_k\,[m_4;\,(M_2(i,j))-n]$$

$$= \prod_{j=l_2+1}^{p} \prod_{i=1}^{A_j} \prod_{l=1}^{m_4} \left( \frac{M_2(i,j))-n+l-1}{m_4} \right)_k.$$

$$\triangle^3{}_k\,[m_4;\,1-(\mathcal{N}_1(i,j))-n]$$

$$= \prod_{j=l_1+1}^{q+1} \prod_{i=1}^{B_j} \prod_{l=1}^{m_4} \left( \frac{-(\mathcal{N}_1(i,j))-n+l}{m_4} \right)_k.$$

$$\triangle^4{}_k\,[m_4;\,(\mathcal{N}_2(i,j))-n\,] = \prod_{j=2}^{l_1} \prod_{i=1}^{B_j} \prod_{l=1}^{m_4}$$

$$\left( \frac{(N_1(i,j)) - n + l - 1}{m_4} \right)_k .$$

## 2. Preliminaries and The Main Result

Singh and Pandey [9] studied the asymptotic behaviour of the general polynomial set $\{\bar{R}_n(x, y)\}$, for $m_4\beta = 1$. From (1) we have

$$(3) \quad \lim_{n\to\infty} \frac{(n! \,[\, N_1(i,j)\,)]_n \,[1-(N_2(i,j)\,)]_n \,\bar{R}_n(x, yn^{(w+1/m_4)/r_1}}{[\,(M_1(i,j)\,)]_n \,[1-(M_2(i,j)\,)\,]_n \,(\mu E y^{r_1})_n \; n^{(w+1/m_4)n}}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{[n/m_4]} \frac{\triangle_k[m_4; -n] \; \triangle^3_k[m_4; 1-(N_1(i,j)\,)-n\,]}{k! \; \triangle^1_k[m_4; 1-(M_1(i,j)\,)-n]}$$

$$\frac{\triangle^4_k[m_4; (N_2(i,j)\,)-n]}{\triangle^2_k[m_4; (M_2(i,j)\,)-n]}$$

$$\times \frac{v^k \, m_4^{\,m_4 wk}}{-n^{wm_4 k}} \frac{(1-\alpha-\beta b_1-n\beta)_k}{(-n)^k \,(\mu E y^{r_1})^{m_4^k} \, x^{mk}}$$

$$= \sum_{k=0}^{\infty} \frac{(\beta v)^k}{k! \,(\mu E y^{r_1})^{m_4^k} \; x^{mk}}$$

$$= e^{\beta v/(\mu E y^{r_1})^{m_4} x^m}$$

where

$$w = \sum_{j=2}^{q+1} B_j + 1 - \sum_{j=1}^{p} A_j$$

Proceeding in a similar way, we obtain the corresponding results for $m_4\beta > 0$, $m_4\beta < 0$, and $\beta = 0$.

We shall prove the following main result involving the products of two $H$-functions and the general polynomial set $\{\bar{R}_n(x, y)\}$:

$$(4) \int_0^\infty y^{\rho-1} H_{e_1, e_2}^{e_3, e_4}\left[ cy^{n_1} \middle| \begin{matrix} \{(ae_1, Ae_1)\} \\ \{(be_2, Be_2)\} \end{matrix} \right]$$

$$H_{i_1, i_2}^{i_3, i_4}\left[ \eta y^{n_1 \lambda} \middle| \begin{matrix} \{(g_{i_1}, G_{i_1})\} \\ \{(f_{i_2}, F_{i_2})\} \end{matrix} \right] \times \bar{R}_n(x, y)\, dy$$

$$= \sum_{k=0}^{[n/m_4]} \sum_{s=0}^{[n/2m_3]} \frac{[M_1(i,j)]_z\, [1-(M_2(i,j))]_z}{k!\, z!\, s!\, (\nu_1+1)_s\, [(N_1(i,j))]_z}$$

$$\frac{v^k (\mu E)^z (-1/4)^s (a_1)^{2s}}{[1-(N_2(i,j))]_z\, x^{mk-2m_1 s}} \times \frac{(\alpha+\beta b_1)_{z\beta+k}}{(\alpha+\beta b_1)_{z\beta}}$$

$$\times \int_0^\infty y^{\rho+r_1(z)+2m_2 s-1} H_{i_1, i_2}^{i_3, i_4}\left[ \eta y^{n_1 \lambda} \middle| \begin{matrix} \{(g_{i_1}, G_{i_1})\} \\ \{(f_{i_2}, F_{i_2})\} \end{matrix} \right]$$

$$H_{e_1, e_2}^{e_3, e_4}\left[ cy^{n_1} \middle| \begin{matrix} \{(ae_1, Ae_1)\} \\ \{(be_2, Be_2)\} \end{matrix} \right] dy$$

where

$$\mathrm{Re}\,(\rho + n_1 \alpha_1 + n_1 \lambda \alpha_2) > 0,$$

$$\alpha_1 = \min_{1 \leqslant h \leqslant e_3} \mathrm{Re}\,(b_h/B_h)$$

$$\alpha_2 = \min_{1 \leqslant h \leqslant i_3} \mathrm{Re}\,(f_h/F_h)$$

$$\mathrm{Re}\,(\rho + n_1 \alpha_3 + n_1 \lambda \alpha_4) < 0,$$

$$\alpha_3 = \max_{1 \leqslant h \leqslant e_4} \mathrm{Re}\left(\frac{a_h - 1}{A_h}\right)$$

$$\alpha_4 = \max_{1 \leqslant h \leqslant i_4} \mathrm{Re}\left(\frac{g_h - 1}{G_h}\right)$$

$$\sum_1^{e_1} A_j - \sum_1^{e_2} B_j \leqslant 0, \quad \sum_1^{i_1} G_j - \sum_1^{i_2} F_j \leqslant 0,$$

$$\sum_{1}^{e_4} A_j - \sum_{e_4+1}^{e_1} A_j + \sum_{1}^{e_3} B_j - \sum_{e_3+1}^{e_2} B_j \equiv \lambda_1 > 0,$$

$$\sum_{1}^{e_4} G_j - \sum_{i_4+1}^{i_4} G_j + \sum_{1}^{e_3} F_j - \sum_{i_3+1}^{i_2} F_j \equiv \lambda_2 > 0,$$

$$|\arg c| \leqslant \tfrac{1}{2}\pi\lambda_1, \ |\arg \gamma_i| \leqslant \tfrac{1}{2}\pi\lambda_2,$$

and $z = n - m_4\beta - 2m_3\beta.$

**Proof.** Using the known result [3] in the right-hand side of (4), we have

$$= \frac{1}{n_1} \sum_{k=0}^{[n/m_4]} \sum_{s=0}^{[n/2m_3]}$$

$$\frac{[(M_1(i,j))]_z \, [1-(M_2(i,j))]_z \, v^k (\mu E)^z (-1/4)^s (a_1)^{2s}}{k!\, z!\, s!\, (\nu_1+1)_s \, [(N_1(i,j))]_z \, [1-(N_2(i,j))]_z \, x^{mk-2m_1s}}$$

$$\times \frac{(a+\beta b_1)_{z\beta+k}}{(a+\beta b_1)_{z\beta}}$$

$$\times \frac{1}{c^{\rho+r_1(z)+2m_2s)/n_1}} H^{i_3+e_4,\ i_4+e_3}_{i_1+e_2,\ i_2+e_1}\left[\frac{\gamma_i}{c\lambda}\middle| \begin{matrix}\{(g_{i_4},\ G_{i_4})\},\\ \{(f_{i_3},\ F_{i_3})\},\end{matrix}\right.$$

$$\{(1-be_2 - \frac{(\rho+r_1(z)+2m_2s)}{n_3} Be_1,\ \lambda Be_2)\},$$

$$\{(1-ae_1 - \frac{(\rho+r_1(z)+2m_2s)}{n_3} Ae_1,\ \lambda Ae_2)\},$$

$$\left.\begin{matrix}(g_{i_4}+1,\ G_{i_4}+1), \ldots, (g_{i_1},\ G_{i_1})\\ (f_{i_3}+1,\ F_{i_3}+1), \ldots, (f_{i_2},\ F_{i_2})\end{matrix}\right]$$

provided that

$$\text{Re}\left(\rho+\frac{r_1(z)+m_2 s}{n_1}+f_h \mid F_h+b_i B_i\right)>0(h=1,\ldots,i_3;$$

$i=1,\ldots, e_3$ ),

$$\text{Re}\left(\rho+\frac{r_1(z)+2m_2 s}{n_1}+A_{j-1} \mid A_j+gh_1-1 \mid Gh_1\right)<0$$

$(J=1,\ldots,e_4;\ h_1=1,\ldots,i_4),\ \lambda>0,\ |\arg \eta|<\frac{1}{2}\pi\lambda_2,\ |\arg c|<\frac{1}{2}\pi\lambda_1.$

## 3. PARTICULAR CASES

On specializing the various parameters involved in integral relation (4), we obtain several known integral relations involviog the products of $H$-functions and Laguerre polynomials, $H$-fnnctions and Sister Celine polynomials, etc.

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