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ON THE GENERALIZED LAPLACE TRANSFORM OF LORENTZ SPACES

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1. INTRODUCTION

The object of the present paper is to find necessary and sufficient conditions that a function $S(x)$ on $x > 0$ be the generalized Laplace transform [3] (whose theory has been discussed in detail in a series of papers and my thesis [1]) of a function $f(y)$ in $\wedge(\delta)$ [4].
The spaces $\wedge(\delta)$ and $M(\delta)$ are defined [4] as follows.

Let $0 < \delta < 1, 0 < f \leqslant \infty$.
Let f be measurable on $(0, l,$ and in case $l = \infty$ let the set where $|f(x)| > \varepsilon$ have finite measure for each positive ε.

Define $\| f(\,.\,) \|_{\wedge(\delta)} = \delta \int_0^l x^{\delta-1} f^*(x)\, dx,$

where $F^*(x)$ is the equimeasurable rearrangement of $|f|$ in decreasing order and

$$\| f(\,.\,) \|_{\wedge(\delta)} = \sup_E \, (M(E))^{-\delta} \int_E |f(x)|\, dx, \; E \subseteq (0, l).$$

The spaces $\wedge(\delta)$ and $M(\delta)$ consist of those f for which $\| f(.) \|_{\Delta(\delta)} < \infty$, and $\| f(\,.\,) \|$ respectively.

This paper gives representation theorem for the space $\Lambda(\delta)$ while the corresponding theorem for the $M(\delta)$ appear in my another paper. Throughout this paper we denote the space $\Lambda(\delta)$ on $(0, \infty)$ by $\Lambda(\delta)$ while the corresponding norm by $\| f(\,.\,) \|$.

Also the sqace $\Lambda(\delta)$ which is L_p over $(0, l)$ will be denoted by $\Lambda(\delta, l)$.

2. The Representation Theorem

Before giving the representation theorem we prove

Lemma 2. 1. *If* $f \varepsilon \Lambda(\delta)$ *and*

(2. 1) $$S(x) = A \int_0^\infty (xy)^\beta \, {}_1F_1\,(a;\, b;\, -xy)\, f(y)\, dy, \text{ where}$$

$$a = \beta+\eta+1,\ b = \alpha+\beta+\eta+1 \text{ and } A = \frac{\Gamma(a)}{\Gamma(b)},$$

then $$\int_0^\infty x^{-\delta}\, S(x)\, dx < \infty.$$

The above condition is both necessary and sufficient if $f(y)$ is positive and decreasing.

Proof. Suppose $f \varepsilon \Lambda(\delta)$. Then

$$\int_0^\infty x^{-\delta} \,|\, S(x) \,|\, dx \leqslant A \int_0^\infty x^{-\delta}\, dx \int_0^\infty |\,(xy)^\beta \, {}_1F_1\,(a;b;-xy)\, f(y)\,|\, dy$$

$$= A \int_0^\infty y^\beta \,|\, f(y) \,|\, dy \int_0^\infty x^{\beta-\delta} \, {}_1F_1\,(a;\, b;\, -xy)\, dx$$

$$= \delta^{-1} \frac{\Gamma(\beta-\delta+1)\, \Gamma(\eta+\delta)}{\Gamma(\alpha+\eta+\delta)} \int_0^\infty y^{\delta-1} \,|\, f(y) \,|\, dy$$

$$= \delta^{-1} \frac{\Gamma(\beta-\delta+1)\,\Gamma(\eta+\delta)}{\Gamma(\alpha+\eta+\delta)} \; \| f\,(\cdot) \|_{\Lambda(\delta)} < \infty$$

This proves the first prat.

For sufficiency conditions of the second part, when f is positive and decreasing, we see that

$$\int_0^{\infty} x^{-\delta}\, |\, S\,(x)\, |\, dx = \frac{\Gamma\,(\beta-\delta+1)\;\Gamma\,(\eta+\delta)}{\Gamma\,(\alpha+\eta+\delta)} \; \| f\,(\,\cdot\,) \|_{\Lambda(\delta)}$$

and $f \varepsilon\, (\delta)$. This proves the second part of lemma.

Theorem 2. 1. *The necessary and sufficient conditions that a function* $S(x)$ *defined for* $x > 0$, $\beta \geqslant 0$, & $\eta+1 > 0$, *be the generalized Laplace transform* (2. 1) *of a function* f *in* $\Lambda\,(\delta)$ *are*

(i) S *has derivatives of all orders in* $(0, \infty)$ *and* $S^{(n)}(x) \rightarrow 0$ as $x \rightarrow \infty$ $(n=0, 1, 2, 3,\ldots)$

(ii) $\| Q_{n;}\, (S\,(x)\,) \|_{\Lambda(\delta)}$ B, *where* B *is independent of* n.

The operator $Q_{n,\,t}\,(g\,(x)\,)$ in defined [2] as follows.

Definition. An operator $Q_{n,\,t}\,(g\,(x))$ is defined for any positive number t and any positive integer n by the relation

$$W_n\,(g\,(x)\,) = (-)^n\, x^{\beta+n-1}\, (d/dx)^n\, (x^{-\beta}\, g\,(x)\,)$$

$$Q_{n,\,t}\,(g\,(x)\,) = \frac{1}{\Gamma\,(\beta+n+1-a)}\, (W_n\, g(x)\,)\; x = \frac{n}{t}$$

Proof. The condition is necessary.

Let

$$S\,(x) = A \int_0^{\infty} (x\,y)^{\beta}\; {}_1F_1\,(a;\, b;\, -x\,y)\, f\,(y) dy,\, f \varepsilon\, \Delta(\delta).$$

Again by [2]

$$Q_{n,y}(S(x)) = \frac{\Gamma(a+n)}{\Gamma(\beta+n+1-\alpha)\,\Gamma(b+n)}\left(\frac{n}{y}\right)^{\beta+n+1}$$

$$\int_0^\infty t^{\beta+n}\,{}_1F_1\left(a+n;\, b+n;\, -\frac{nt}{y}\right) f(t)dt = \int_0^\infty \Psi(t,y)\, f(t)dt,$$

where $\Psi(t,y) = \frac{\Gamma(a+n)}{\Gamma(b+n)\,\Gamma(\beta+n+1-\alpha)}\left(\frac{n}{y}\right)^{\beta+n+1}$

$t^{\beta+n}\,{}_1F_1\left(a+n;\, b+n;\, -\frac{nt}{y}\right).$

Hence [5] for each $c > 0$ $\int_0^c Q_{n,y}(S(x))^*\,dy$

$\leqslant \int_0^c f^*(y)dy$, since $\Psi(t,y)$ satisfies conditions of Theorem 3. 8. 1 of [5].

Therefore, as in [5], for any $c > 0$

$$\delta \int_0^c y^{\delta-1}\, Q_{n,t}(f(x))^*\,dy \leqslant \delta \int_0^c y^{\delta-1} f^*(y)dy\,.$$

Letting c tend to infinity we have

$$\| Q_n;(s(x))\|_{\Lambda(\delta)} \leqslant \| f(.)\|_{\Lambda(\delta)}$$

which shows that (ii) is necessary.

Proof of sufficiency of conditions. Lorentz has proved [4] that, if $g(t)$ is positive and non-increasing then

$$\int_0^\infty t^{p-1}\,|\,g(t)|^p\,dt \leqslant K_p\left\{\int_0^\infty |\,g(t)\,|\,dt\right\}^p \quad p \geqslant 1.$$

In this result if we put $p = \frac{1}{\lambda}$, $g(t) = t^{\delta-1} Q_{n,t}(S(x))^*$, we get

$$\int_0^\infty | Q_{n,y}(S(x)) |^{1/\delta} dt = \int_0^\infty \left\{ Q_{n,y}(S(x))^* \right\}^{1/\delta} dy$$

$$\leqslant K_{1/\delta} \left\{ \int_0^\infty y^{\delta-1} Q_{n,y}(S(x))^* \right\} dy$$

$$\leqslant K_{1/\delta} B^{1/\delta}$$

Hence

$$\| Q_{n_i}(S(x)) \|_{L_{1/\delta}} \leqslant B', \text{ where } B' = \left(K_{1/\delta} \right)^\delta B.$$

Now by a theorem on weak compactness of set of functions [6] it follows that $f \varepsilon L_{1/\delta}$ and an increasing unbounded $\{n_i\}$ exists such that

(i) $\| f(.) \| \leqslant B'$

(ii) $S(x) = A \int_0^\infty (xy)^\beta \, {}_1F_1(a; b; -xy)\, dy$.

(iii) For any $\chi \varepsilon L\{I-\delta\}^{-1}\}$

$$\lim_{i\to\infty} . \int_0^\infty \chi(y)\, Q_{n_i, y}[S(x)]\, dy = \int_0^\infty \chi(y) f(y)\, dy$$

If we now show that $f \varepsilon \Lambda(\delta)$ then our theorem is proved. We have, as in [5], for $\chi \varepsilon M(\delta)$

$$\int_0^\infty \chi(y)\, Q_{n,y}(S(x))\, dy \leqslant \| \chi(.) \|_{M(\delta)}$$

$$\| Q_{n,\nu} (S(x)) \|_{\Lambda(\delta)} \leqslant B \| \chi (.) \|_{M(\delta)}$$

But [4] $L((1-\delta))^{-1} \subseteq M(\delta)$. Therefore for any

$\chi \varepsilon L((1-\delta))^{-1})$, *by* (iii),

$$\left| \int_0^\infty \chi (y) f (y) dy \right| = \lim_{i \to \infty} \left| \int_0^\infty \chi (y) \, Q_{n_i, 1} (S (x)) dy \right|$$

$$\leqslant B \| \chi \|_{M(\delta)}$$

and thus [5] for any positive $L((1-\delta)^{-1})$

$$\int_0^\infty \chi (y) f^* (y) dy \leqslant B \| \chi (.) \|_{M(\delta)}$$

Now let $\chi(y) = \delta y^{\delta-1}$, $0 < \varepsilon \leqslant y \leqslant R$; $\chi (y) = 0$ otherwise. Then

$\chi \varepsilon L((1-\delta)^{-1})$ and $\| \chi (.) \| \leqslant 1$.

Hence $\delta \displaystyle\int_\varepsilon^R f^* (y) dy \leqslant B$.

Letting $\varepsilon \to 0$ & $R \to \infty$ we have

$\| f (.) \|_{\Lambda(\delta)} < \infty$ and $f \varepsilon \Lambda (\delta)$.

This concludes the proof. It may be mentioned here, in passing that results obtained here hold for more general spaces e. g. reflexive Banach space. Proof is much similar and hence ommitted.

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