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BASIC PARTIAL DERIVATIVES OF BASIC HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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Abstract

Agarwal [1] obtained basic partial derivatives of basic hypergeometric functions of two variables.In this paper we obtain basic partial derivatives of some of the basic hypergeometric functions of three variables and also deduce some relations between the basic partial derivatives of these functions and their contiguous functions.

1. INTRODUCTION

Agarwal [1] obtained basic partial derivatives of basic hypergeometric functions of two variables and also deduced some relations between these derivatives and their contiguous functions. In this paper we obtain basic partial derivatives of some of the basic hypergeometric functions of three variables Φ_D defined by Andrews [2], and ${}^H\Phi_A$ and ${}^H\Phi_C$ defined by Sharan [3]. We also deduce some relations between the basic partial derivatives of these functions and their contiguous functions. A number of other relations of this type can be derived similarly for these functions as well as other basic hypergeometric functions of three variables.

2. PRELIMINARIES

The basic diefference operator $D_q f(x)$ is defined by

$$D_q\, f(x) = \frac{f(x) - f(qx)}{x}, \quad |\, q \,| < 1 \tag{2. 1}$$

This operator is the basic analogue of the differential operator $\frac{d}{dx}$.

From (2. 1) we have

$$D_q x^a = \frac{x^a - q^a x^a}{x} = (1 - q^a) x^{a-1} \qquad \text{...(2. 2)}$$

So that

$$\lim_{q \to 1} (1 - q)^{-1} D_q x^a = a x^{a-1} = \frac{d}{dx} x^a. \qquad \text{...(2. 3)}$$

For a function of three variables we shall denote the basic derivatives by the symboles δ_x , δ_y , and δ_z respectively. The basic analogue of the Lauricella function F_D is defined by [2].

$$\Phi_D [a, b_1, b_2, b_3, ; c, x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b_1)_m (b_2)_n (b_3)_p}{(1)_m (1)_n (1)_p (c)_{m+n+p}} x^m y^n z^p \qquad \text{...(2. 4)}$$

Also, Sharan [3] defined the basic analogues of the fnnctions H_A and H_C of Srivastava ([5] , [6] and [7] by

$${}^H\Phi_A [a, b, b' ; c, c' ; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (b')_{n+p}}{(1)_m (1)_n (1)_p (c)_m (c')_{n+p}} x^m y^n z^p \qquad \text{.. (2. 5)}$$

$${}^H\Phi_C [a, b, b' ; c ; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (b')_{n+p}}{(1)_m (1)_n (1)_p (c)_{m+n+p}} x^m y^n z^p \qquad \text{...(2. 6)}$$

where $(a)_m = (1-q^a) (1-q^{a+1}) \dots (1-q^{a+m-1})$.

3. Basic Partial Derivatives

We have

$$\delta_x \Phi_D = \delta_x \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m}{(1)_m (c)_m} x^m$$

$$\sum_{n,p=0}^{\infty} \frac{(a+m)_{n+p} (b_2)_n (b_3)_p}{(1)_m (1)_p (c+m)_{n+p}} y^n z^p.$$

Now using (2. 2) we get

$$\delta_x \Phi_D = \left[\frac{(1-q^a)(1-q^{b_1})(1-q)}{(1-q)(1-q^c)} \right.$$

$$\left. + \frac{(1-q^a)(1-q^{a+1})(1-q^{b_1})(1-q^{b_1+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} (1-q^2)\, x + \ldots \right]$$

$$\sum_{n,p=0}^{\infty} \frac{(b+m)_{n+p} (b_2)_n (b_3)_p}{(1)_n (1)_p (c+m)_{n+p}} y^n z^p$$

$$= \frac{(1-q^a)(1-q^{b_1})}{(1-q^c)} \left[1 + \frac{(1-q^{a+1})(1-q^{b_1+1})}{(1-q)(1-q^{c+1})} x + \ldots \right].$$

$$\sum_{n,p=0}^{\infty} \frac{(a+m)_{n+p} (b_2)_n (b_3)_p}{(1)_n (1)_p (c+m)_{n+p}} y^n z^p$$

$$= \frac{(1-q^a)(1-q^{b_1})}{(1-q^c)} \Phi_D [\, a+1, b_1+1, b_2, b_3; c+1; x, y, z \,]$$

...(3. 1)

Applying this process r times we get

$$\delta_x^r \Phi_D = \frac{(a)_r (b_1)_r}{(c)_r} \Phi_D [\, a+r, b_1+r, b_2, b_3; c+r, x, y, z]$$

...(3. 2)

Similarly we get

$$\delta_x^r \ \delta_y^s \ \delta_z^t \ \Phi_D = \frac{(a)_{r+s+t}\,(b_1)_r\,(b_2)_s\,(b_3)_t}{(c)_{r+s+t}}$$

$$\Phi_D\,[\,a+r+s+t;\ b_1+r,\ b_2+s,\ b_3+t;\ c+r+s+t;\ x\ y,\ z\,] \quad ...(3.\ 3)$$

$$\delta_x^r \ \delta_y^s\,\delta_z^t \ {}^H\Phi_A = \frac{(a)_{r+t}\,(b_1)_{r+s}\,(b)_{s+t}}{}$$

$${}^H\Phi_A\,[a+r+t;\ b+r+s,\ b'+s+t;\ c+r,\ c'+s+t;\ x, y, z] \quad ...(3.\ 4)$$

$$\delta_x^r \ \delta_y^s \ \delta_z^t \ {}^H\Phi_C = \frac{(a)_{r+t}\,(b)_{r+s}\,(b')_{s+t}}{(c)_{r+s+t}}$$

$${}^H\Phi_C\,[a+r+t;\ b+r+s,\ b'+s+t,\ c+r+s+t;\ s, y, z] \quad ...(3\ 5)$$

4. Contiguous Function Relations

It is interesting to note that there exist some relations between these partial derivatives and contiguous functions. We give here some of the relations for functions Φ_D, ${}^H\Phi_A$ and ${}^H\Phi_C$. A number of other similar relations can be derived for these functions and other basic hypergeometric functions of three variables.

We have

$$x\,\delta_x\,\Phi_D = \frac{x\,(1-q^a)\,(1-q^{b_1})}{(1-q^c)}$$

$$\Phi_D\,[a+1,\ ;\ b_1+1,\ b_2,\ b_3:\ c_1+1,\ x, y, z\,]$$

$$= \frac{(1-q^{b_1})}{q^{b_1}} \left[\ \Phi_D \quad b_1 \ + \ -\ \Phi_D\ \right] \quad ..(4\ 1)$$

$$y\,\delta_y\,\Phi_D = \frac{y\,(1-q^a)\,(1-q^{b_2})}{(1-q^c)}$$

$$\Phi_D\,[\,a+1,\ b_1,\ b_2+1,\ b_3:\ c+1;\ x, y, z\,]$$

$$= \frac{(1-q^{b_2})}{q^{b_2}} \left[\Phi_D{}^{b_2+} - \Phi_D \right] \qquad \ldots(4.\ 2)$$

$$z\ \delta_z\ \Phi_D = \frac{z\,(1-q^a)\ (1-q^{b})}{(1-q^c)}$$

$$\Phi_D\ [a+1;\ b_1,\ b_2,\ b_3+1;\ c+1;\ x,\ y,\ z]$$

$$= \frac{(1-q^{b_3})}{q^{b_3}} \left[\Phi_D{}^{b_3+} - \Phi_D \right] \qquad \ldots(4.\ 3)$$

$$x\delta_x\ {}^H\Phi_A = \frac{x\ (1-q^a)\ (1-q^b)}{(1-q^c)}$$

$${}^H\Phi_A\ [a+1;\ b+1,\ b';\ c+1,\ c';\ x,\ y,\ z\ ]$$

$$= \frac{(1-q^b)}{q^b} \left[{}^H\Phi_A{}^{b+} - {}^H\Phi_A \right] \qquad \ldots(4.\ 4)$$

$$y\delta_y\ {}^H\Phi_A = \frac{y\ (1-q^b)\ (1-q^{b'})}{(1-q^{c'})}$$

$${}^H\Phi_A\ [a,\ b+1,\ b'+1,\ c,\ c'+1;\ x,\ y,\ z\ ]$$

$$= \frac{(1-q^{b'})}{q^{b'}} \left[{}^H\Phi_A{}^{b'+} - {}^H\Phi_A \right] \qquad \ldots(4.\ 5)$$

$$z\delta_z\ {}^H\Phi_A = \frac{z\ (1-q^a)\ (1-q^{b'})}{(1-q^{c'})}\ {}^H\Phi_A\ [a+1,\ b,\ b'+1,\ c,\ c'+1;\ x,\ y,\ z]$$

$$= \frac{(1-q^a)}{q^a} \left[{}^H\Phi_A{}^{a+} - {}^H\Phi_A \right] \qquad \ldots(4.\ 6)$$

$$x\ \delta_x\ {}^H\Phi_C = \frac{x\ (1-q^a)\ (1-q^b)}{(1-q^c)}\ {}^H\Phi_C\ \left[a+1,\ b+1,\ b';\ c+1;\ x,\ y,z\right]$$

$$= \frac{(1-q^a)}{q^a} \left[{}^H\Phi_C{}^{a+} - {}^H\Phi_C \right] \qquad \ldots(4.\ 7)$$

$$y\ \delta_y\ {}^H\Phi_C = \frac{y\ (1-q^b)\ (1-q^{b'})}{(1-q^c)}\ {}^H\Phi_C\ \left[a,\ b+1,\ b'+1,\ c+1;\ x,y,x\right]$$

$$= \frac{(1-q^{b})}{q^{b}} \left[{}^{H}\Phi_C{}^{b+} - {}^{H}\Phi_C \right] \qquad \ldots(4.8)$$

$$z\, \delta_z\, {}^{H}\Phi_C = \frac{(1-q^{a})\,(1-q^{b'})}{(1-q^{c})} \; {}^{H}\Phi_C \left[a+1,\ b,\ b'+1;\ c+1;\ x, y, z \right]$$

$$= \frac{(1-q^{b'})}{q^{b'}} \left[{}^{H}\Phi_C{}^{b'+} - {}^{H}\Phi_C \right] \qquad \ldots(4.9)$$

From (3. 1)

$$x\, \delta_x\, \Phi_D = \frac{x\,(1-q^{a})\,(1-q^{b_1})}{(1-q^{c})}$$

$$\Phi_D \left[a+1,\ b_1+1,\ b_2,\ b_3;\ c+1; x,\ y,\ z \right]$$

$$= \frac{x\,(1-q^{a})\,(1-q^{b_1})}{} \left[1 + \frac{(1-q^{a+1})\,(1-q^{b_1+1})}{(1-q)\,(1-q^{c+1})}\, x \right.$$

$$+ \frac{(1-q^{a+1})\,(1-q^{a+2})}{(1-q)\,(1-q^{2})\,(1-q^{c+1})} \quad \frac{(1-q^{b_1+1})\,(1-q^{b_1+2})}{(1-q^{c+2})}\, x^{2}$$

$$\left. + \ldots \right] \cdot I$$

where

$$I = \sum_{n,\ p=0}^{\infty} \frac{(a)_{n+p}\,(b_2)_n\,(b_3)_p\, y^{n}\, z^{p}}{(q)_n\,(q)_p\,(c)_{n+p}}$$

$$x\, \delta_x\, \Phi_D = \frac{1-q^{b_1}}{q^{b_1}} \left[\frac{(1-q^{a})\,(q^{b_1}\, x}{(1-q^{c})} \right. +$$

$$\left. \frac{(1-q^{a})\,(1-q^{a+1})\,(1-q^{b_1+1})\, q^{b_1}\, x^{2}}{(1-q)\,(1-q^{c})\,(1-q^{c+1})} + \ldots \right] \cdot I$$

$$= \frac{(1-q^{b_1})}{q^{b_1}} \left[\frac{(1-q^{a})\,(1-q^{b_1+1} - 1+q^{b_1}x)}{(1-q)\,(1-q^{c})} + \right.$$

$$\left. \frac{(1-q^{a})\,(1-q^{a+1})\,(1-q^{b_1+1})(1-q^{b_1+2} - 1+q^{b_1})}{(1-q)\,(1-q^{2})\,(1-q^{c})\,(1-q^{c-1})} x^2 + \dots \right] I .$$

$$= \frac{(1-q^{b_1})}{q^{b_1}} \left[\left\{ 1 + \frac{(1-q^{2})\,(1-q^{b-1})}{(1-q)\,(1-q^{c})} x + \right. \right.$$

$$\left. \frac{(1-q^{a})\,(1-q^{a_1+1}\quad)\,(1-q^{b_1+1}\quad)\,(1-q^{b_2+2}\quad)\, x^2}{(1-q)\,(1-q^{2})\,(1-q^{c})\,(1-q^{c-1})} + \dots \right\}$$

$$- \left\{ 1 + \frac{(1-q^{a})\,(1-q^{b_1})\, x}{(1-q)\,(1-q^{c})} \right.$$

$$\left. \left. + \frac{(1-q^{a})\,(1-q^{a_1+1}\quad)\,(1-q^{b_1+1}\quad)\, x^2}{(1-q)\,(1-q^{2})\,(1-q^{c})\,(1-q^{c+1})} + \dots \right\} \right] . I.$$

$$= \frac{(1-p^{b_1})}{q^{b_1}} \left[\Phi_D \quad b_1+ \; - \; \Phi_D \right].$$

Formulas (4. 2) to (4. 9) can be proved similarly.

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