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INTEGRALS INVOLVING THE H-FUNCTIONS OF SEVERAL VARIABLES. II

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Abstract

In this paper we evaluate four integrals involving the product of elementary special functions and the multivariable H–functions introduced by H M. Srivastava and R. Panda. The integrals are quite general in nature and from them a large number of new results can be obtained simply by specializing the parameters of the multivariable H–function.

1. INTRODUCTION

The multivariable H–function occurring in this paper is a special case of the H–function of several variables introduced by Srivastava and Panda [10, p. 271, Eq. (4. 1)] :

$$H_1[x_1, \ldots, x_n] \equiv H^{0,0:(\mu', \nu'); \ldots; (\mu^{(n)}, \nu^{(n)})}_{A, C:[B', D']; \ldots; [B^{(n)}, D^{(n)}]}$$

$$\left(\begin{matrix} [(a) : \theta_1', \ldots, \theta^{(n)}] : [(b') : \phi'] : \ldots ; [(b^{(n)}) : \phi^{(n)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(n)}] : [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}]; \end{matrix} \; x_1 ; \ldots ; x_n \right)$$

$$= \frac{1}{(2\pi w)^n} \int_{L_1} \cdots \int_{L_n} \Psi_1(\xi_1, \ldots, \xi_n) \prod_{k=1}^{n} \Phi_k(\xi_k)\, x_k\, d\xi_k$$

$$w = \sqrt{-1} \qquad (1.\ 1)$$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma[d_j^{(i)} - \delta_j^{(i)} \xi_i] \prod_{j=1}^{\nu^{(i)}} \Gamma[1 - b_j^{(i)} + \phi_j^{(i)} \xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}} \Gamma[1 - d_j^{(i)} + \delta_j^{(i)} \xi_i] \prod_{j=\nu^{(i)}+1}^{B^{(i)}} \Gamma[b_j^{(i)} - \phi_j^{(i)} \xi_i]} \tag{1.2}$$

$$\Psi_1(\xi_1, \dots, \xi_n) = \frac{1}{\prod_{j=1}^{A} \Gamma[a_j - \sum_{i=1}^{n} \theta_j^{(i)} \xi_i] \prod_{j=1}^{C} \Gamma[1 - c_j + \sum_{i=1}^{n} \Psi_j^{(i)} \xi_i]} \tag{1.3}$$

an empty product is interpreted as unity, the coefficients

$\theta_j^{(i)}, j = 1, \dots, A$; $\phi_j^{(i)}, j = 1, \dots, B^{(i)}$; $\Psi_j^{(i)} = 1, \dots, C$; $\delta_j^{(i)}, j = 1, \dots, D^{(i)}$ and $i = 1, \dots, n$ are positive numbers and $\mu^{(i)}, \nu^{(i)}, A, B^{(i)}, C, D^{(i)}$ are integers such that $A \geqq 0$, $0 \leqq \mu^{(i)} \leqq D^{(i)}$, $C \geqq 0$ and $0 \leqq \nu^{(i)} \leqq B^{(i)}$, $i = 1, \dots, n$.

The multiple integral in (1.1) converges absolutely ([3], [11, p. 130]). if

$$|\arg x_i| < \tfrac{1}{2} \pi \Delta_i', \; i = 1, \dots, n. \tag{1.4}$$

where

$$\Delta_i' \equiv -\sum_{j=1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \; i = 1, \dots, n. \tag{1.5}$$

2. THE MAIN INTEGRALS

First Integral

$$\int_0^\infty t^{\rho-1} \; J_\delta(at)\, J_\mu(at)\, J_\nu(2bt)\, H_1[x_1 \, t^{\rho_1}, \ldots, x_n \, t^{\rho_n}]\, dt$$

$$= \frac{a^{\delta+\mu}}{b^{\delta+\mu+\rho}\, 2^{\delta+\mu}} \sum_{r=0}^{\infty} \frac{\left(\frac{\delta+\mu+1}{2}\right)_r \left(\frac{\delta+\mu+2}{2}\right)_r \left(\frac{a^2}{b^2}\right)_r}{r!\, \Gamma(\delta+r+1)\, \Gamma(\mu+r+1)\, (\delta+\mu+1)_r} \times$$

$$\times \; H^{0,\,2\,:\,(\mu',\,\nu')\,;\,\ldots\,;\,(\mu^{(n)},\,\nu^{(n)})}_{A+3,\,C+1\,:\,[B',\,D']\,;\,\ldots\,;\,[B^{(n)},\,D^{(n)}]}$$

$$\Bigg(\left[\frac{2-\delta-\mu\pm\nu-2r-\rho}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right], \left[(a) ; \theta', \ldots, \theta^{(n)}\right]$$

$$\left[(c) : \Psi', \ldots, \Psi^{(n)}\right], \left[\frac{2-\mu+\nu-\delta}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right]:$$

$$\left[\frac{2-\delta-\mu+\nu-\rho}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right] : [(b') : \phi'] ; \ldots ; [(b^{(n)}) : \phi^{(n)}] ;$$

$$[(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ;$$

$$b^{-\rho_1} x_1, \ldots, b^{-\rho_n} x_n \Bigg) \tag{2.1}$$

provided that $\rho_i > 0 \; \forall \; i \in \{1, \ldots, n\}$.

$$0 < a < b, \; |\arg x_i| < \tfrac{1}{2} \pi \Delta_i' \; \mathrm{Re}\, (\delta+\mu+\nu+\rho \sum_{i=1}^{n} \rho_i \, a_i) > 0,$$

where $a_i \equiv \min \mathrm{Re} \left\{ \frac{d_j^{(i)}}{\delta_j^{(i)}} \right\} (j = 1, \ldots, \mu^{(i)})$ (2. 2) and the series occurring on the right hand side of (2.1) is absolutely convergent. $j_\nu(z)$ stands for the well-known Bessel function of the first kind.

Second Integral

$$\int_0^\infty t^{\rho-1} \, Cos\,(2at)\, K_\mu(t)\, k_\nu(t)\, H_1\,[x_1 t^{\rho_1}, \ldots, x_n t^{\rho_n}]\, dt$$

$$= 2^{\rho-3} \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r}}{r!} (\tfrac{1}{2})_r \, H^{0,\ 6\ :\ (\mu', \nu');\ldots;(\mu^{(n)}, \nu^{(n)})}_{A+6,\ C+3\ :\ [B', D'];\ldots;[B^{(n)}, D^{(n)}]}$$

$$\left(\begin{matrix} \left[\frac{2-\rho\pm\mu\pm\nu-2r}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right], \\ \left[(c) : \Psi', \ldots, \Psi^{(n)}\right], \left[1-\rho : \rho_1, \ldots, \rho_n\right], \end{matrix} \right.$$

$$\left[\frac{2-\rho-2r}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right], \left[\frac{1-\rho-2r}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right],$$

$$[(a) : \theta', \ldots, \theta^{(n)}] :$$

$$\left[\frac{2-\rho}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right], \left[\frac{1-\rho}{2} : \frac{\rho_1}{2}, \ldots, \frac{\rho_n}{2}\right] :$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(n)}) : \phi^{(n)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} \ 2^{\rho_1} x_1, \ldots, 2^{\rho_n} x_n \right) \tag{2.3}$$

where $\rho_i > 0 \ \forall \ i \in \{1, \ldots, n\}$. Re $(a) < 1$, $|\arg x_i| < \frac{1}{2}\pi \Delta_i'$,

Re $(\rho\pm\mu\pm\nu + \sum_{i=1}^{n} \rho_i \alpha_i) > 0$ where Δ_i' and α_i are defined in (1.5) and (2.2) respectively. The series occurring on the right hand side of (2.3) is absolutely convergent.

Also, $K_\nu(z)$ stands for the modified Bessel function of the second kind.

Third Integral

$$\int_{-\infty}^{\infty} t^{2\rho_1} \, exp\,(-t^2)\, H_{2\nu}(t)\, H_1 [x_1 t^{2\rho_1}, \ldots, x_n t^{2\rho_n}]\, dt$$

$$= \frac{\sqrt{\pi}}{2^{2\rho-2\nu}} H \begin{matrix} 0 \quad , \quad 1 \quad : (\mu', \nu') ; \ldots ; \mu^{(n)}, \nu^{(n)} \\ A+1, C+1 : [B', D'] ; \ldots ; [B^{(n)}, D^{(n)}] \end{matrix}$$

$$\left(\begin{matrix} [-2\rho : 2\rho_1, \ldots, 2\rho_n], [(a) : \theta', \ldots, \theta^{(n)}] : \\ [(c) : \Psi', \ldots, \Psi^{(n)}], [\nu-\rho : \rho_1, \ldots, \rho_n] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(n)}) : \phi^{(n)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} \; 2^{-\rho_1} x_1, \ldots, 2^{-\rho_n} x_n \right) \quad (2.4)$$

where $\rho_i > 0 \;\forall\; i \in \{1, \ldots, n\}$. $|\arg x_i| < \frac{1}{2}\pi \triangle_i'$,

$Re\,(1 + 2\rho + 2 \sum_{i=1}^{n} \rho_i \alpha_i) > 0$. Here $\triangle_i'$ and α_i are same as in

(2. 1). Also $H_{2\nu}(x)$ stands for the Hermite polynomial.

Fourth Integral

$$\int_0^{\infty} t^{\rho-1} \; exp\,(-\sigma t)\; E\,(\alpha_1, \beta_1 :: \sigma t)\; E\,(\alpha_2, \beta_2 :: \xi t)$$

$$H_1 [x_1 t^{\rho_1}, \ldots, x_n t^{\rho_n}] dt$$

$$= \sigma^{-\rho} \sum_{r=0}^{\infty} \sum_{u=0}^{\infty} \frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+u)\,\Gamma(\beta_2+r+u)}{} \left(\frac{\xi-\sigma}{\xi}\right)^u \times$$

$$\times H^{0\ ,\ 3\ :\ (\mu', \nu')\ ;\ \dots\ ;\ (\mu^{(n)}, \nu^{(n)})}_{A+3\ ,\ C+2\ :\ [B', D']\ ;\ \dots\ ;\ [B^{(n)}, D^{(n)}]}$$

$$\left(\begin{matrix} [1-\rho-\alpha_1-\alpha_2 : \rho_1, \dots, \rho_n], [1-\rho-\beta_1-\beta_2 : \rho_1, \dots, \rho_n], \\ [(c) : \Psi', \dots, \Psi^{(n)}], [1-\rho-r-\alpha_1-\beta_1-\beta_2 : \rho_1, \dots, \rho_n], \end{matrix} \right.$$

$$[1-\rho-r-\alpha_1-\beta_2 : \rho_1, \dots, \rho_n], [(a) : \theta', \dots, \theta^{(n)}] ;$$

$$1-\rho-r-u-\alpha_1-\alpha_2-\beta_2 : \rho_1, \dots, \rho_n] :$$

$$\left. \begin{matrix} [(b) : \phi'] ; \dots ; [(b^{(n)}) : \phi^{(n)}] ; \\ [(c') : \delta'] ; \dots ; [(d^{(n)}) : \delta^{(n)} ; \end{matrix} \ \frac{x_1}{(\sigma)^{\rho_1}}, \dots, \frac{x_n}{(\sigma)^{\rho_n}} \right) \tag{2. 5}$$

The integral (2. 5) is valid if $\rho_i > 0 \ \forall \ i \in \{1, \dots, n\}$.

$\text{Re}\,(\xi) > \frac{1}{2} R\,(\sigma)$, $|\arg x_i| < \frac{1}{2}\pi \Delta_i'$, $\text{Re}(\alpha_1 + \alpha_2 + \rho + \sum_{i=1}^{n} \rho_i \alpha_i) > 0$,

$R\,(\beta_1 + \beta_2 + \rho + \sum_{i=1}^{n} \rho_i \alpha_i) > 0$. Here Δ_i' and α_i are same as in (2.1)

The series occurring on the right hand side of (2. 5) is absolutely convergent.

Derivation of (2. 1). The integral (2. 1) can be established if we express the H-function of several variables in the integrand on the left of (2. 1) in terms of its Mellin-Barnes type contour integral (1. 1) interchange the order of integrations (which is justified due to the absolute convergence of the integral involved in the process) and evaluate the inner integral with the help of the following known result [5, p. 350, Eq. (7)] :

$$\int_0^{\infty} x^{\rho-1} J_\lambda(ax)\, J_\mu(ax)\, J_\nu(2bx)\, dx$$

$$= \frac{a^{\lambda+\mu} b^{-\lambda-\mu-\rho} \Gamma\left(\frac{\lambda+\mu+\nu+\rho}{2}\right)}{2^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(1-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho)} \times$$

$$\times {}_4F_3\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}, \frac{\lambda+\mu+\nu+\rho}{2}, \frac{\rho+\mu-\nu+\rho}{2} ;\right.$$

$$\left.\lambda+1, \mu+1, \lambda+\mu+1 ; \frac{a^2}{b^2}\right),$$

$$0 < a < b, \text{ Re } (\lambda+\mu+\nu+\rho) > 0 \qquad (2.\ 6)$$

Now, writing series expansion for ${}_4F_3$, changing the order of integration and summation and interpreting the result thus obtained with the help of (1. 1), we obtain the right hand side of (2. 1).

The integrals (2. 3) to (2. 5) can be established in a similar manner with the difference that we use the known results [5, p. 371, Eq, (51)], [4, p. 93, (2. 2. 7)], [4, p 90, (2. 2. 2)] respectively instead of (2. 6).

3. PARTICULAR CASES

I. If we take all of θ's, ϕ's, Ψ's, δ's equal to unity in (2. 5), we get the corresponding result for the G-function of several variables introduced by Khadia and Goyal [8].

II. On putting $n=2$ in (2. 1) we obtain an integral involving the H–function of two variables introduced by Mittal and Gupta [9] and Goyal [6], which is a special case of an integral recently established by Goyal, S. P. [7 p. 39, Fq. (4. 4)].

III. On putting $n=2$ in (2. 4) and (2. 5), our main integrals reduces to the known integrals involving the H–function of two variables, recently obtained by Chobisa [4, p. 85 Eq. (2. 1. 1)], [4, p. 89, (2. 1. 6)] respectively.

Several other integrals can also be obtained as particular cases of our main results simply by specializing the parameters of the multivariable H–function.

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